GEOMETRY OF JETS AND FIELDS
BANACH CENTER PUBLICATIONS, VOLUME 110
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2016

COSYMPLECTIC-NIJENHUIS STRUCTURES ON LIE GROUPOIDS

AÏSSA WADE
Department of Mathematics, The Pennsylvania State University
University Park, PA 16802, USA
E-mail: wade@math.psu.edu

Dedicated to Professor Janusz Grabowski on the occasion of his 60th birthday

Abstract. This paper introduces cosymplectic-Nijenhuis structures on smooth manifolds and proposes alternative odd-dimensional counterparts of symplectic-Nijenhuis groupoids, called cosymplectic-Nijenhuis groupoids. We discuss the correspondence between cosymplectic groupoids and integrable coPoisson manifolds. Moreover, we investigate the integrability problem for coPoisson manifolds equipped with a compatible Nijenhuis operator. As a result, we obtain a one-to-one correspondence between cosymplectic-Nijenhuis groupoids and integrable coPoisson–Nijenhuis manifolds.

1. Introduction. It is well known that there is a correspondence between contact structures and symplectic structures via the symplectization scheme. Furthermore, contact manifolds are often considered as odd-dimensional analogues of symplectic manifolds. An alternative odd-dimensional counterpart of a symplectic structure is the notion of a cosymplectic structure. Recall that a *cosymplectic structure* on a $(2n+1)$-dimensional manifold M is defined by a pair (ω, η) consisting of a closed 2-form ω and a closed 1-form η such that $\omega^n \wedge \eta$ is a volume element on M.

Cosymplectic manifolds were initiated by Libermann in the late 1950's [24]. They were further studied by Lichnerowicz in his papers [26, 27] in which he called them *canonical manifolds*. In [31], Marle discussed Lie group actions on a cosymplectic manifold. The reader is referred to Blair's paper [5] for another concept of a cosymplectic structure which is different but related to Libermann's notion. Precisely, a cosymplectic manifold in the sense of Blair is a cosymplectic manifold in the sense of Libermann together with

2010 *Mathematics Subject Classification*: Primary 53D17; Secondary 17B62, 57R17.
Key words and phrases: Poisson structure, Lie algebroid, Lie groupoid, cosymplectic structure, Poisson–Nijenhuis Lie algebroid.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc110-0-19

a compatible normal almost contact metric structure. In this paper, we only consider cosymplectic structures as defined in [24].

On the one hand, cosymplectic manifolds are crucial objects in the geometric description of time dependent mechanics. They play an important role in the study of regular Lagrangian systems and Hamiltonian systems [12, 8]. In addition cosymplectic manifolds naturally appear in classical relativistic theories. For instance, it was observed in [20] that the geometric object underlying Galilei's phase space is a cosymplectic manifold. Recent works [4, 3, 6, 23, 18] resuscitate interest in the subject. On the other hand, Poisson–Nijenhuis structures on manifolds were introduced by Magri and Morosi [30] in their study of integrable Hamiltonian systems and they attracted the attention of many mathematicians due to their close relationship with integrable systems [13, 11, 10, 2, 22, 33]. Poisson–Nijenhuis structures were extended to arbitrary Lie algebroids and Leibniz algebroids by Grabowski and Urbański in [17]. For further generalizations to Courant algebroids, see e.g. [1, 21].

The purpose of this note is to introduce the study of Lie groupoids $\mathcal{G} \rightrightarrows M$ equipped with a triple $(\omega, \eta, \mathcal{N})$ consisting of a multiplicative cosymplectic structure (ω, η) and a compatible multiplicative Nijenhuis $\mathcal{N}$. In Section 2, we review Lie algebroids, Lie groupoids and Poisson–Nijenhuis Lie algebroids. Section 3 discusses cosymplectic manifolds and introduces cosymplectic-Nijenhuis structures. Section 4 reviews cosymplectic groupoids and their unit spaces called co-Poisson manifolds. Finally, in Section 6, we show that there is a one-to-one correspondence between cosymplectic-Nijenhuis groupoids and coPoisson–Nijenhuis manifolds.

2. Preliminaries

2.1. Lie algebroids. A Lie algebroid over a smooth manifold M is a vector bundle $A \to M$ together with a Lie bracket $[\ ,\ ]$ on the space $\Gamma(A)$ of smooth sections of A and a bundle map $\varrho : A \to TM$, called the anchor map, whose extension to smooth sections of A and TM (also denoted by ϱ) satisfies the relation:

$$[X, fY] = f[X, Y] + \left(\mathcal{L}_{\varrho(X)} f\right) Y, \quad \forall X, Y, \in \Gamma(A),\ f \in C^\infty(M).$$

Examples of Lie algebroids include: finite-dimensional Lie algebras, the tangent bundle TM of every smooth manifold M as well as the cotangent Lie algebroid of every Poisson manifold. It is known that any Lie algebroid $(A, [\ ,\ ]_A, \varrho_A)$ induces a Gerstenhaber algebra $(\Gamma(\wedge^\bullet A^*), \wedge, [\ ,\ ]_A)$ with a differential operator

$$d_A : \Gamma(\wedge^k A^*) \to \Gamma(\wedge^{k+1} A^*)$$

defined by:

$$\begin{aligned} d_A\alpha(X_0, X_1, \cdots, X_k) = &\sum_{i=0}^{k} (-1)^i (\varrho_A X_i)\alpha(X_0, \ldots, \widehat{X_i}, \ldots, X_k) \\ &+ \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j]_A, X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k), \end{aligned}$$

for all $\alpha \in \Gamma(\wedge^k A^*)$, $X_i \in \Gamma(A)$.

In particular, given a Poisson manifold (M,π), there is a canonical Lie algebroid structure on its cotangent bundle T^*M with the anchor map $\pi^\sharp : T^*M \to TM$ and whose Lie bracket is given by:

$$\{\alpha,\beta\}_\pi = \mathcal{L}_{\pi^\sharp\alpha}\beta - \mathcal{L}_{\pi^\sharp\beta}\alpha - d(\pi(\alpha,\beta)), \quad \forall \alpha,\beta \in \Omega^1(M),$$

where

$$\langle \beta, \pi^\sharp\alpha\rangle = \pi(\alpha,\beta), \quad \forall \alpha,\beta \in \Omega^1(M).$$

The operator $d_{T^*M} = \delta_\pi$ is defined on the space $\mathfrak{X}^\bullet(M)$ of all multi-vector fields on M as follows:

$$\delta_\pi(Q) = [\pi, Q], \quad \forall Q \in \mathfrak{X}^\bullet(M), \tag{1}$$

is a differential operator, i.e. $\delta_\pi^2 = 0$. The corresponding cohomology complex is known as the Lichnerowicz–Poisson cohomology [25]. This cohomology complex plays an important role in deformation theory and quantization of Poisson manifolds. A *Poisson vector field* on a Poisson manifold (M,π) is a vector field X on M satisfying $\delta_\pi(X) = [\pi, X] = 0$. Poisson vector fields are also called *infinitesimal automorphisms of the Poisson structure.* Thus, elements of the first Lichnerowicz–Poisson cohomology group $H^1_\pi(M)$ are just classes of Poisson vector fields (modulo Hamiltonian vector fields).

2.2. Matched pair of Lie algebroids. First, recall that a matched pair of Lie algebroids is a pair (A,B) of Lie algebroids over the same base manifold M whose direct sum $A \oplus B$ is equipped with a Lie algebroid structure for which both A and B are Lie subalgebroids [34]. The associated bicrossed product is denoted by $A \bowtie B$.

2.3. Lie groupoids. A *Lie groupoid* over a smooth manifold M (see [29]) is given by a smooth manifold G together with two surjective submersions $s,t : G \to M$ (called the source map and the target map), a smooth associative multiplication $m : G_2 \to G$, a unit section $\epsilon : M \to G$ and an inversion map $i : G \to G$, where $G_2 = \{(g,h) \in G \times G \mid s(g) = t(h)\}$ is the set of composable pairs and the following properties are satisfied:

(i) $s(m(g,h)) = s(h)$ and $t(m(g,h)) = t(g)$, $\forall\, (g,h) \in G_2$,
(ii) $m(g, m(h,k)) = m(m(g,h),k)$, $\forall\, g,h,k \in G$ with $(g,h) \in G_2$ and $(h,k) \in G_2$,
(iii) $s(\epsilon(x)) = x$ and $t(\epsilon(x)) = x$, $\forall x \in M$,
(iv) $m(g, \epsilon(s(g))) = g$ and $m(\epsilon(t(g)), g) = g$, $\forall\, g \in G$,
(v) $m(g, \iota(g)) = \epsilon(t(g))$ and $m(\iota(g), g) = \epsilon(s(g))$, $\forall g \in G$.

Here, the base manifold M, all source and target fibers are supposed to be Hausdorff but G is not necessarily Hausdorff. We will often identify M with $\epsilon(M)$.

2.4. Multiplicative k-forms. A real-valued function λ defined on a given Lie groupoid $G \overset{s}{\underset{t}{\rightrightarrows}} M$ is *multiplicative* if:

$$\lambda \circ m(g_1, g_2) = \lambda(g_1) + \lambda(g_2), \quad \forall\ (g_1,g_2) \in G_2.$$

A differential k-form ω on G is multiplicative if: $m^*\omega = pr_1^*\omega + pr_2^*\omega$, where m is the multiplication of the Lie groupoid and pr_i denotes the projection of the space G_2 of composable pairs onto the ith component.

2.5. Multiplicative vector fields. A multiplicative vector field on a groupoid G is a pair (Z, Z_0) of vector fields with $Z \in \mathfrak{X}(G)$, $Z_0 \in \mathfrak{X}(M)$ such that $Z : G \to TG$ is a morphism of groupoid over $Z_0 : M \to TM$. It essentially has the property that the flow (ϕ_t^Z) of Z is a local Lie groupoid morphism over the flow $(\phi_t^{Z_0})$ of Z_0. In fact, Z_0 is just the push-forward of Z along the source map. Applying the tangent functor to the operations in $G \overset{s}{\underset{t}{\rightrightarrows}} M$ yields the tangent prolongation groupoid $TG \rightrightarrows TM$. So, we will simply say that Z is multiplicative.

2.6. Lie algebroids in the presence of a 1-cocycle. Let A be a Lie algebroid over M together with a 1-cocycle $\phi \in \Gamma(A^*)$. The pull-back of A to $M \times \mathbb{R}$ admits a Lie algebroid structure over $M \times \mathbb{R}$, denoted by $(A \times_\phi \mathbb{R}, [\![\cdot,\cdot]\!]^\phi, \rho^\phi)$, where the smooth sections of $A \times_\phi \mathbb{R}$ are of the form $\bar{X}(x,\tau) = X_\tau(x)$, with $X_\tau \in \Gamma(A)$ for all $\tau \in \mathbb{R}$, and

$$\begin{aligned} [\![\bar{X}, \bar{Y}]\!]^\phi(x,\tau) &= [\![X_\tau, Y_\tau]\!](x) - \phi(X_\tau)(x)\frac{\partial \bar{Y}}{\partial \tau} + \phi(Y_\tau)(x)\frac{\partial \bar{X}}{\partial \tau}, \\ \rho^\phi(\bar{X})(x,\tau) &= \rho(X_\tau)(x) - \phi(X_\tau)(x)\frac{\partial}{\partial \tau}, \end{aligned} \tag{2}$$

where $\frac{\partial \bar{X}}{\partial \tau} \in \Gamma(A \times_\phi \mathbb{R})$ denotes the derivative of $\bar{X}$ with respect to τ.

Given a Lie groupoid $G \overset{s}{\underset{t}{\rightrightarrows}} M$ together with a multiplicative function σ. One can define a right action of G on the canonical projection $p_1 : M \times \mathbb{R} \to M$ as follows:

$$(x,\tau)\cdot g = (s(g), \sigma(g)+\tau), \text{ for } (x,\tau,g) \in M \times \mathbb{R} \times G, \text{ with } t(g) = x.$$

We get the corresponding action groupoid $G \times \mathbb{R} \rightrightarrows M \times \mathbb{R}$, denoted by $G \times_\sigma \mathbb{R}$, with structural functions:

$$\begin{aligned} s_\sigma(g,\tau) = (s(g),\tau), \qquad & t_\sigma(g', \tau'+, \sigma(g')) = (t(g'), \tau'), \\ m_\sigma((g,\tau),(g',\tau')) = (gg',\tau'), \; & \text{if } s_\sigma(g,\tau) = t_\sigma(g',\tau'). \end{aligned} \tag{3}$$

Let AG be the Lie algebroid of G. The multiplicative function σ induces a 1-cocycle α on AG given by

$$\langle \alpha_x, \; X_x\rangle = (X \cdot \sigma)_x, \quad \text{for } x \in M \text{ and } X \in \Gamma(AG). \tag{4}$$

The Lie algebroid of $G \times_\sigma \mathbb{R}$ can be identified with $AG \times_\alpha \mathbb{R}$. Conversely, one has the following (see Proposition 3.5 from [9]):

Proposition 2.1 ([9]). *Let A be a Lie algebroid over M, $\alpha \in \Gamma(A^*)$ a 1-cocycle and $A \times_\alpha \mathbb{R}$ the Lie algebroid given by Equation (2). If $G(A)$ (resp., $G(A \times_\alpha \mathbb{R})$) is the Weinstein groupoid of A (resp., $A \times_\alpha \mathbb{R}$) and σ is the multiplicative function associated with α, then $G(A \times_\alpha \mathbb{R}) \cong G(A) \times_\sigma \mathbb{R}$. Moreover, A is integrable if and only $A \times_\alpha \mathbb{R}$ is integrable.*

2.7. Poisson–Nijenhuis Lie algebroids. Let $(A, [\;,\;]_A, \varrho_A)$ be a Lie algebroid equipped with a vector bundle map $\mathcal{N} : A \to A$. The torsion of $\mathcal{N}$ is defined by

$$T_\mathcal{N}(X,Y) := [\mathcal{N}X, \mathcal{N}Y]_A - \mathcal{N}[X,Y]_\mathcal{N}, \quad \forall X, Y \in \Gamma(A), \tag{5}$$

where

$$[X,Y]_\mathcal{N} := [\mathcal{N}X, Y]_A + [X, \mathcal{N}Y]_A - \mathcal{N}[X,Y]_A, \quad \forall X, Y \in \Gamma(A).$$

A vector bundle map $\mathcal{N} : A \to A$ is called a *Nijenhuis operator on* A if $T_{\mathcal{N}} = 0$. In this case, one gets a new Lie algebroid structure $A_{\mathcal{N}} = (A,\ [\ ,\ ]_{\mathcal{N}},\ \varrho_A \circ \mathcal{N})$. Moreover $\mathcal{N} : A_{\mathcal{N}} \to A$ is a Lie algebroid morphism.

A Nijenhuis operator $\mathcal{N}$ on A induces a degree 1 derivation $d_{\mathcal{N}} : \Gamma(\wedge^{\bullet} A) \to \Gamma(\wedge^{\bullet} A)$ defined by:

$$\begin{aligned} d_{\mathcal{N}}\alpha(X_0, X_1, \cdots, X_k) &= \sum_{i=0}^{k}(-1)^i(\mathcal{N}X_i)\alpha(X_0, \ldots, \widehat{X_i}, \ldots, X_k) \\ &\quad + \sum_{i<j}(-1)^{i+j}\alpha([X_i, X_j]_A, X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k), \end{aligned}$$

for all $\alpha \in \Gamma(\wedge^k A)$, $X_i \in \Gamma(A)$.

A *Poisson structure on a Lie algebroid* A is given by a section $P \in \Gamma(\wedge^2 A)$ such that $[P, P]_A = 0$. Such a Poisson structure P induces a Lie algebroid structure on the dual A^* of A whose Lie bracket is defined by:

$$[\alpha, \beta]_P = \mathcal{L}^A_{P^\sharp\alpha}\beta - \mathcal{L}^A_{P^\sharp\beta}\alpha - d^A P(\alpha, \beta),$$

where $\alpha, \beta \in \Gamma(A^*)$, d^A is the de Rham differential of A and $\mathcal{L}^A_X = d^A \circ \iota_X + \iota_X \circ d^A$ for any $X \in \Gamma(A)$. This generalizes the classical case where $A = TM$.

A Poisson structure P on a Lie algebroid A is said to be *compatible* with a Nijenhuis operator $\mathcal{N} : A \to A$ if the following relations are satisfied:

$$\mathcal{N}P^\sharp = P^\sharp\mathcal{N} \quad \text{and} \quad C(P, \mathcal{N}) = [\alpha, \beta]_{\mathcal{N}P} - [\alpha, \beta]^{\mathcal{N}^*}_P = 0,$$

where $[\ ,\ ]_{\mathcal{N}P}$ is the bracket defined by $\mathcal{N}P$ and $[\ ,\ ]^{\mathcal{N}^*}_P$ is the Lie bracket obtained from the Lie bracket $[\ ,\ ]_P$ by deformation along the dual map $\mathcal{N}^* : A^* \to A^*$, i.e.,

$$[\alpha, \beta]^{\mathcal{N}^*}_P = [\mathcal{N}^*\alpha, \beta]_P + [\alpha, \mathcal{N}^*\beta]_P - \mathcal{N}^*[\alpha, \beta]_P.$$

The term $C(P, \mathcal{N})$ is called the *Magri–Morosi concomitant* of P and $\mathcal{N}$.

DEFINITION 2.2 ([17, 21]). A *Poisson–Nijenhuis Lie algebroid* is given by a triple $(A, P, \mathcal{N})$ consisting of a Lie algebroid A endowed with a Poisson structure P and a Nijenhuis operator $\mathcal{N} : A \to A$ compatible with P.

In particular, Poisson–Nijenhuis structures on M correspond to Poisson–Nijenhuis Lie algebroid structures on the standard Lie algebroid TM, and symplectic-Nijenhuis structures on M are special Poisson–Nijenhuis structures $(P, \mathcal{N})$ on TM for which P is non-degenerate. If $(P, \mathcal{N})$ is a Poisson–Nijenhuis structure on A, then the triple $(\Gamma(\wedge^{\bullet} A), [\ ,\ ]_A, d_{\mathcal{N}})$ becomes a differential graded Lie algebra.

3. Cosymplectic manifolds

3.1. Cosymplectic structures. An almost cosymplectic structure on a $(2n+1)$-dimensional manifold M is given by a pair (ω, η) consisting of a 1-form η and a 2-form ω on M such that $\eta \wedge \omega^n$ is a volume form. If in addition, both η and ω are closed differential forms then we say that (ω, η) is a cosymplectic structure on M (see [24]).

Clearly for any almost cosymplectic manifold (M, ω, η), its 2-form ω induces an almost symplectic structure on $\ker\eta$ since the relation $\eta \wedge \omega^n \neq 0$ on M implies that ω has maximal rank on $\ker\eta$. Furthermore, there exists a unique vector field $\mathbf{R}$, called the Reeb vector field, which is completely determined by:

$$\iota_{\mathbf{R}}\eta = 1, \quad \iota_{\mathbf{R}}\omega = 0.$$

One has an isomorphism $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ defined by

$$\flat(X) = \iota_X\omega + \eta(X)\eta.$$

This gives:

$$\mathbf{R} = \flat^{-1}(\eta).$$

Associated to each function $f \in C^\infty(M)$, there is a vector field X_f which is called the Hamiltonian vector field of f. It is defined by:

$$X_f = \flat^{-1}(df - (R\cdot f)\eta).$$

One gets:

$$\eta(X_f) = 0 \quad \text{and} \quad \iota_{X_f}\omega = df - (\mathbf{R}\cdot f))\eta.$$

It is known that any cosymplectic structure (η,ω) on M determines a Poisson bracket on M given by:

$$\{f,g\} = \omega(X_f, X_g),$$

for all f and $g \in C^\infty(M)$. Trivial examples of cosymplectic manifolds result from the product of any $2n$-dimensional symplectic manifold by $\mathbb{R}$ or the circle S^1. A non-trivial example of cosymplectic structure was constructed in [32].

Let (M,ω,η) be a cosymplectic manifold and π be its corresponding Poisson bivector field. Then its Reeb vector field $\mathbf{R}$ is a Poisson vector field of (M,π). On $M\times\mathbb{R}$, the bivector field

$$P = \pi + \frac{\partial}{\partial\tau}\wedge\mathbf{R} \tag{6}$$

has a maximal rank and it defines a symplectic structure on $M\times\mathbb{R}$ which is called the symplectization of the cosymplectic structure. In other words, the inverse of Π is the symplectic 2-form:

$$\Omega = (pr_1)^*\omega + d\tau\wedge(pr_1)^*\eta, \tag{7}$$

where $pr_1 : M\times\mathbb{R}\to M$ is the canonical projection onto the first factor.

3.2. Examples: cosymplectic Lie groups. A cosymplectic Lie group is a Lie group G of dimension $2n+1$ equipped with a left invariant closed 2-form ω^+ and a left invariant closed 1-form η^+ such that $\eta^+\wedge(\omega^+)^n \neq 0$ pointwise on G.

For example, consider the 3-dimensional Heisenberg Lie group H_3. We denote by $\mathcal{H}$ its Lie algebra and by (e_1,e_2,e_3) the canonical basis of $\mathcal{H}$ and (e_1^*,e_2^*,e_3^*) its dual basis. The only nonzero bracket of $\mathcal{H}$ is:

$$[e_1,e_2] = e_3.$$

Let η^+ be the left invariant form whose value at the identity element equals e_1^* together with the scalar non-degenerate 2-cocycle on $\mathcal{H}$ given by: $\omega = e_2^*\wedge e_3^*$. Then (ω^+,η^+) defines a cosymplectic structure on H_3. Consider $G = H_3\times\mathbb{R}^{*+}$, that is, the direct product of the 3-dimensional Heisenberg Lie group with the 1-dimensional Lie group whose product is defined by:

$$(x,y,z,t)(x',y',z',t') = (x+x', y+y', z+z'+xy', tt').$$

The corresponding symplectic 2-form on G is given by:

$$\overline{\omega}^{+} = \frac{1}{t}dx \wedge dt + dy \wedge dz.$$

3.3. Another perspective: symplectic mapping torus. First, we will recall the definition of a mapping torus. Consider a topological space S and a homeomorphism $\varphi\colon S \to S$. The quotient space is called a *mapping torus* and it is denoted by:

$$S_\varphi = \frac{S \times [0,1]}{(x,0) \sim (\varphi(x),1)}.$$

The projection onto the second factor endows S_φ with the structure of a fibre bundle with base S^1 and fibre S. Similarly, starting from a smooth manifold S and a diffeomorphism $\varphi\colon S \to S$, one can construct a smooth mapping torus S_φ. When S is a smooth even-dimensional manifold endowed with a symplectic form ω_s and φ is a symplectomorphism of (S, ω_s) then S_φ is called a *symplectic mapping torus.* The following result was proved in [23]:

Theorem 3.1 ([23]). *A closed manifold M admits a cosymplectic structure if and only if it is a symplectic mapping torus $M = S_\varphi$, for some symplectic manifold (S, ω_s) and some symplectomorphism $\varphi : S \to S$.*

3.4. Cosymplectic-Nijenhuis manifolds

Definition 3.2. A cosymplectic-Nijenhuis manifold is a cosymplectic manifold (M, ω, η) endowed with a Nijenhuis operator $\mathcal{N} : TM \times \mathbb{R} \to TM \times \mathbb{R}$ which is compatible with the Poisson structure $P \in \Gamma(TM \times \mathbb{R})$ associated with the cosymplectic structure and given in (6).

It is clear that given any cosymplectic-Nijenhuis manifold $(M, \omega, \eta, \mathcal{N})$, there is an associated Poisson–Nijenhuis $(M \times \mathbb{R}, P, \overline{\mathcal{N}})$. When $\mathcal{N} = (N, Z, \alpha, g)$, that is,

$$\mathcal{N}(X,f) = (NX + fZ, \alpha(X) + fg) \quad \forall\, X \in \mathfrak{X}(M), f \in C^\infty(M)$$

then

$$\overline{\mathcal{N}} = N + d\tau \otimes Z + \alpha \otimes \frac{\partial}{\partial t} + g d\tau \otimes \frac{\partial}{\partial \tau},$$

where τ is the standard coordinate in $\mathbb{R}$. Conversely, if $(\omega_s, \mathcal{N}_s)$ is a symplectic-Nijenhuis structure on S then $(S\times\mathbb{R},\ pr_1^*\omega_s,\ d\tau,\ \mathcal{N}_x + d\tau\otimes\partial_\tau)$ is a cosymplectic-Nijenhuis manifold. Recall the following result:

Proposition 3.3 ([35]). *A vector bundle endomorphism $\mathcal{N} : TM \times \mathbb{R} \to TM \times \mathbb{R}$ given by the quadruple (N, Z, α, g) defines a Nijenhuis operator (i.e. $T_{\mathcal{N}} = 0$) if and only if the* *following conditions are satisfied for all* $X, Y \in \mathfrak{X}(M)$*:*

$$[NX, NY] - N[X,Y]_N = \big(d\alpha(X,Y)\big)Z); \tag{8}$$

$$d\alpha(NX,Y) + d\alpha(X,NY) - d(N^*\alpha)(X,Y) = gd\alpha(X,Y); \tag{9}$$

$$(\mathcal{L}_Z N)X = -dg(X)Z; \quad \textit{and} \quad N^*(dg) = gdg + \mathcal{L}_Z\alpha, \tag{10}$$

where N^* *is the dual map of* N*.*

Proposition 3.4. *Let* (ω_1, η_1) *be a cosymplectic structure on* M *together with its associated Poisson tensor* π_1 *and Reeb vector field* $\mathbf{R}_1$*. Let* (π_0, X_0) *be a pair consisting of Poisson structure* π_0 *and a vector field* X_0 *on* M *such that:* $X_0 + \mathbf{R}_1$ *defines a 1-cocycle relative to the total cohomology* $d_{\pi_0} + d_{\pi_1}$ *of the double complex associated with* $(\pi_0,\ \pi_1)$ *and* $[\pi_1, \pi_0] = 0$*. Then it determines a cosymplectic-Nijenhuis structure* $(\omega_1, \eta_1, \mathcal{N})$ *on* M*.*

Proof. Consider

$$\pi = \pi_1 + \pi_0, \quad E = \mathbf{R}_1 + X_0 \quad \text{and} \quad P = \pi + \frac{\partial}{\partial\tau} \wedge E.$$

Clearly, P defines a Poisson tensor on $TM \times \mathbb{R}$ if and only if:

$$[\pi_1, \pi_0] = 0 \quad \text{and} \quad [\pi_1, X_0] + [\pi_0, \mathbf{R}_1] + [\pi_0, X_0] = 0. \tag{11}$$

Suppose (11) holds true and let $\mathcal{N} = P_0 \circ P_1^{-1} : TM \times \mathbb{R} \to TM \times \mathbb{R}$, where

$$P_0 = \pi_0 + \frac{\partial}{\partial\tau} \wedge X_0 \quad \text{and} \quad P_1 = \pi_1 + \frac{\partial}{\partial\tau} \wedge \mathbf{R}_1.$$

Then $(\omega_1, \eta_1, \mathcal{N})$ is a cosymplectic-Nijenhuis structure on M. ∎

The above proposition allows one to construct examples of cosymplectic-Nijenhuis manifolds. Here is a simple example.

Example. Let $M = \mathbb{R}^3$ with its standard coordinates (x, y, z) and the cosymplectic structure (ω, η) defined by:

$$\omega = dx_2 \wedge dx_3, \quad \eta = dx_1.$$

Consider the following tensors:

$$\pi = \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3}, \quad \mathbf{R} = \frac{\partial}{\partial x_1}, \quad \pi_0 = f(x_1, x_3)\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}, \quad X_0 = \pi^\sharp dg,$$

where $f = f(x_1, x_3)$ is independent of x_2 and $g = g(x_3)$ is a function of x_3 only. Let

$$P_0 = \pi_0 + \frac{\partial}{\partial\tau} \wedge X_0 \quad \text{and} \quad P_1 = \pi + \frac{\partial}{\partial\tau} \wedge \mathbf{R}.$$

Then $\mathcal{N} = P_0 \circ P_1^{-1}$ is a Nijenhuis operator and $(\omega, \eta, \mathcal{N})$ is a cosymplectic-Nijenhuis structure on M.

Another simple example is the above Heisenberg group H_3 (see Section 3.2). Denote by (e_1, e_2, e_3) the canonical basis of the Lie algebra $\mathcal{H}$ of H_3 and let (e_1^*, e_2^*, e_3^*) be its dual basis. Recall that the cosymplectic structure on H_3 is determined by $\omega = e_2^* \times e_3^*$ and $\eta = e_1$. Pick $J = e^3 \otimes e_3$. Then the triple of left invariant tensors (ω^+, η^+, J^+) induced by (ω, η, J) defines a cosymplectic-Nijenhuis structure on H_3.

4. Cosymplectic groupoids

Definition 4.1. A *cosymplectic groupoid* is a Lie groupoid $G \overset{s}{\underset{t}{\rightrightarrows}} M$ endowed with a cosymplectic structure (ω, η) such that both ω and η are multiplicative.

In this section, we will describe the infinitesimal objects corresponding to cosymplectic groupoids.

4.1. Bicrossed product structures on the first jet bundle J^1M. Any Poisson vector field E on a Poisson manifold (M, π) determines a Lie algebroid whose underlying vector bundle is the first jet bundle J^1M. More precisely, the anchor map of this Lie algebroid structure on J^1M is given by:

$$\varrho_{(\pi,E)}(\alpha, f) = \pi^\sharp\alpha + fE, \quad \forall\alpha \in \Omega^1(M),\ f \in C^\infty(M) \tag{12}$$

and its Lie bracket is defined by:

$$\begin{aligned} \{(\alpha,0),(\beta,0)\}_{(\pi,E)} &= (\{\alpha,\beta\}_\pi, 0),\\ \{(\alpha,0),(0,f)\}_{(\pi,E)} &= (-f\mathcal{L}_E\alpha, \mathcal{L}_{\pi\alpha}f),\\ \{(0,f),(0,g)\}_{(\pi,E)} &= (0, f\mathcal{L}_E g - g\mathcal{L}_E f), \end{aligned} \tag{13}$$

for all $(\alpha, 0)$, $(\beta, 0) \in \Gamma(T^*M \oplus \mathbb{R})$ and for all $f, g \in C^\infty(M)$. Obviously, T^*M and the line bundle $M \rtimes \mathbb{R}$ are Lie subalgebroids of J^1M. Hence, this Lie algebroid is a matched pair of these two Lie algebroids, in the sense of Mokri and Lu [34, 28]. Denote by $M \rtimes_E \mathbb{R}$ the trivial line bundle equipped with the Lie algebroid structure induced by E. The following result holds [14]:

Proposition 4.2 ([14]). *Let (M, π) be a Poisson manifold. Any Poisson vector field E on (M, π) determines a matched pair of Lie algebroids $(T^*M, M \rtimes_E \mathbb{R})$. Moreover, up to isomorphism, the bicrossed product Lie algebroid structure on J^1M depends only on the cohomology class $[E] \in H^1_\pi(M)$.*

Special case: the modular class of a Poisson manifold. The modular class of a Poisson manifold (M, π) is a first cohomology class $[E] \in H^1_\pi(M)$ which measures the obstruction to the existence of a volume form (or, in general, a half density) invariant under all Hamiltonian flows (see [37, 16]). When the modular class is non-trivial it determines a non-trivializable Lie algebroid structure on the first jet bundle J^1M.

4.2. Unit spaces of cosymplectic groupoids. We have the following result:

Proposition 4.3 ([15]). *Let $G \overset{s}{\underset{t}{\rightrightarrows}} M$ be a Lie groupoid equipped with a multiplicative smooth function σ. Then there is a one-to-one correspondence between cosymplectic groupoid structures on G and symplectic structures on $G \times_\sigma \mathbb{R}$ preserved by $\frac{\partial}{\partial\tau}$, where τ is the standard coordinate on $\mathbb{R}$.*

Proof. Clearly, any multiplicative symplectic 2-form Ω on $G \times_\sigma \mathbb{R}$ which is preserved by $\frac{\partial}{\partial\tau}$, (i.e. $\mathcal{L}_{\frac{\partial}{\partial\tau}}\Omega = 0$) gives rise to a multiplicative cosymplectic structure on G. Conversely, as we've seen above, any cosymplectic structure on G induces a symplectic form Ω on $G \times_\sigma \mathbb{R}$:

$$\Omega = (pr_1)^*\omega + d\tau \wedge (pr_1)^*\eta,$$

where $pr_1 : M \times \mathbb{R} \to M$ is the canonical projection onto the first factor. To show that Ω is multiplicative, we proceed as follows. First, we identify the set $(G \times_\sigma \mathbb{R})_2$ of composable pairs of arrows of $G \times_\sigma \mathbb{R}$ with $G_2 \times \mathbb{R}$ as follows:

$$((g, \tau - \sigma(g')), (g', \tau)) \equiv ((g, g'), \tau),$$

where G_2 is the set of composable pairs of arrows of G. The projections maps of $(G \times \mathbb{R})_2$ onto $G \times \mathbb{R}$ become $pr_i(g_1, g_2, \tau) = (g_i, \tau)$, $i = 1, 2$ and the multiplication is given by:

$$m((g, g'), \tau) = (gg', \tau).$$

Using these identifications, the fact that ω and η are multiplicative, and a simple calculation, one gets that $m^*\Omega = \mathrm{pr}_1^*\Omega + \mathrm{pr}_2^*\Omega$. Thus $(G \times \mathbb{R}, \Omega)$ is a symplectic groupoid over $M \times \mathbb{R}$. ∎

Observe that a cosymplectic groupoid is always equipped with a Poisson structure coming from the cosymplectic structure but this Poisson structure is not multiplicative, in general. Indeed if M is a single point then G is a Lie group and the multiplicativity would imply that the Poisson structure has zero rank at the identity element. In this case G must be a one-dimensional Lie group.

A *coPoisson structure* on a smooth manifold M is given by a pair $(\pi, [E])$ consisting of a Poisson structure π on M and a Poisson cohomology class $[E] \in H^1_\pi(M)$. This notion is only slightly different from the definition of Janyška and Modugno [20] since we consider a whole cohomology class instead of a specific Poisson vector.

Definition 4.4. A coPoisson structure $(\pi, [E])$ on M is integrable if its associated Lie algebroid $(J^1 M, \{\ ,\ \}_{(\pi,E)}, \varrho_{(\pi,E)})$ can be integrated into a Lie groupoid.

The following result was proved in [14]:

Lemma 4.5 ([14]). *Every integrable coPoisson manifold $(M, \pi, [E])$ is the unit space of a unique source-connected and source-simply-connected cosymplectic groupoid (G, ω, η).*

5. Cosymplectic-Nijenhuis groupoids

Definition 5.1. A *cosymplectic-Nijenhuis groupoid* is a cosymplectic groupoid (G, ω, η) equipped with a multiplicative tensor $\mathcal{N} : TG \times \mathbb{R} \to TG \times \mathbb{R}$ such that $(\omega, \eta, \mathcal{N})$ is a cosymplectic-Nijenhuis structure on G.

Definition 5.2. A coPoisson–Nijenhuis manifold is a coPoisson manifold $(M, \pi, [E])$ endowed with a Nijenhuis operator $\mathcal{N}$ on $TM \times \mathbb{R}$ such that $(\pi + \frac{\partial}{\partial t} \wedge E, \mathcal{N})$ is a Poisson–Nijenhuis Lie algebroid structure on $TM \times \mathbb{R}$.

Proposition 5.3. *The unit space of a cosymplectic-Nijenhuis groupoid is a coPoisson–Nijenhuis manifold.*

To prove this proposition, we use the following lemma:

Lemma 5.4. *Let $(G, \omega, \eta, \mathcal{N})$ be a cosymplectic-Nijenhuis groupoid together with a multiplicative function σ. Then $G \times_\sigma \mathbb{R}$ is a symplectic-Nijenhuis Lie groupoid.*

Proof. Assume that $(\omega, \eta, \mathcal{N})$ is a cosymplectic-Nijenhuis structure on G and all three tensors are multiplicative. Let $\Omega = (pr_1)^*\omega + d\tau \wedge (pr_1)^*\eta$. As we have seen in the proof of Proposition 4.3, $(G \times_\sigma \mathbb{R}, \Omega)$ becomes a symplectic groupoid. We denote by Π the associated non-degenerate multiplicative Poisson bivector field on $G \times_\sigma \mathbb{R}$. Obviously, the Nijenhuis operator $\mathcal{N}$ on $TG \times \mathbb{R}$ can be extended by linearity to $T(G \times \mathbb{R})$. Let us denote by $\overline{\mathcal{N}}$ the extended operator. Since $(TG \times \mathbb{R}, \Pi, \mathcal{N})$ is a Poisson–Nijenhuis Lie algebroid and $\overline{\mathcal{N}}$ is $\mathbb{R}$-linear, one gets $C(\Pi, \overline{\mathcal{N}}) = 0$. Hence the pair of multiplicative tensors $(\Pi, \overline{\mathcal{N}})$ makes $G \times_\sigma \mathbb{R}$ into a symplectic-Nijenhuis groupoid. ∎

Proof of Proposition 5.3. We first apply Lemma 5.4 to get the symplectic-Nijenhuis groupoid $(G \times_\sigma \mathbb{R}, \Pi, \overline{\mathcal{N}})$. We then use the source map s_σ given in (3) and set

$$P = (s_\sigma)_* \Pi = \pi_0 + \frac{\partial}{\partial t} \wedge E_0,$$

where $\pi_0 = (s_\sigma)_* \pi$ and E_0 are the push-forward by s_σ of the Poisson structure π on G and its Reeb vector field R, respectively. Theorem 5.2 in [36] ensures the existence of a Nijenhuis operator N on $T(M \times R)$ such that $(M \times \mathbb{R}, P, N)$ is a Poisson–Nijenhuis manifold. From the Schouten bracket $[P, P] = 0$, it follows that $(\pi_0, [E_0])$ defines a coPoisson structure on M. Furthermore, $\mathcal{L}_{\frac{\partial}{\partial \tau}} N = 0$ since

$$\mathcal{L}_{\frac{\partial}{\partial \tau}} \mathcal{N} = 0 \quad \text{and} \quad \mathcal{L}_{\frac{\partial}{\partial \tau}} P = 0.$$

Thus N induces a Nijenhuis operator on $TM \times \mathbb{R}$, also denoted by N. The compatibility of P with N implies that $(\pi_0, [E_0], N)$ is a coPoisson–Nijenhuis structure on M. ∎

Proposition 5.5. *Every integrable coPoisson–Nijenhuis manifold $(M, \pi, [E])$ is the unit space of a unique source-connected and source simply-connected cosymplectic-Nijenhuis groupoid $(G, \omega, \eta, \widetilde{\mathcal{N}})$*

Proof. Assume that $(\pi, [E], \mathcal{N})$ is an integrable coPoisson–Nijenhuis structure on M, that is, $(J^1 M, \{\ ,\ \}_{(\pi, E)}, \varrho_{(\pi,E)})$ is integrable. Let (G, ω, η) be its associated source-connected and source simply-connected cosymplectic groupoid. By Lemma 4.5, $G \times \mathbb{R}$ is endowed with a symplectic Lie groupoid structure. Denote by $\mathbf{R}$ the Reeb vector field on G. Then the associated multiplicative non-degenerate Poisson tensor on $G \times \mathbb{R}$ can be written as:

$$\Pi = \pi + \frac{\partial}{\partial \tau} \wedge \mathbf{R}.$$

Let $d = d_{(\pi_0, E_0)}$ the de Rham differential of $J^1 M$ and let $d_{\mathcal{N}} = \iota_{\mathcal{N}} \circ d - d \circ \iota_{\mathcal{N}}$. Then, $[d,\ d_N] = d_{\mathcal{N}} \circ d + d \circ d_{\mathcal{N}} = 0$. We apply the universal lifting theorem in [19] and conclude that $d_{\mathcal{N}}$ comes from a multiplicative tensor $\Pi_{\mathcal{N}} \in \Gamma(\wedge^2(TG \times \mathbb{R}))$ such that $[\Pi, \Pi_{\mathcal{N}}] = 0$. Let $\widetilde{\mathcal{N}} = \Pi_{\mathcal{N}} \circ \Pi^{-1} : TG \times \mathbb{R} \to TG \times \mathbb{R}$. This is a multiplicative tensor because both Π and $\Pi_{\mathcal{N}}$ are multiplicative. It follows that $(G, \omega, \eta, \widetilde{\mathcal{N}})$ is a cosymplectic-Nijenhuis groupoid. ∎

Combining Propositions 5.3 and 5.5, we get the following:

Theorem 5.6. *There is a one-to-one correspondence between cosymplectic-Nijenhuis groupoids and integrable coPoisson–Nijenhuis manifolds.*

5.1. Examples

Example 1. Let (S, ω) be a symplectic manifold and let $G = S \times S$ be the pair groupoid with its symplectic form $\omega \oplus (-\omega)$. Take $\mathcal{G} = G \times S^1$, where the circle S^1 is endowed with its standard exact 1-form $\eta = d\theta$. Define $\overline{\omega} = pr_1^*(\omega \oplus (-\omega))$ and $\overline{\eta} = pr_2^* d\eta$, where

$pr_1 : \mathcal{G} = G \times S^1 \to G$, $pr_2 : \mathcal{G} \to S^1$ are the natural projections. Then $(\mathcal{G}, \overline{\omega}, \overline{\eta})$ is a cosymplectic groupoid over S.

EXAMPLE 2. Let M be equipped with the zero Poisson structure $\pi = 0$ and the zero vector field. Then its integrating cosymplectic groupoid is given by the data $(G = T^*M \times \mathbb{R},\ pr_1^*(d\theta),\ d\tau)$, where $pr_1 : G \to T^*M$ and θ is the Liouville 1-form on T^*M.

References

[1] P. Antunes, C. Laurent-Gengoux, J.-M. Nunes da Costa, *Hierarchies and compatibility on Courant algebroids*, Pacific J. Math. 26 (2013), 1–3.

[2] E. Asadi, J. A. Sanders, *Integrable systems in symplectic geometry*, Glasg. Math. J. 51 (2009), 5–23.

[3] G. Bazzoni, M. Fernández, V. Munoz, *Non-formal co-symplectic manifolds*, Trans. Amer. Math. Soc. 367 (2015), 4459–4481.

[4] G. Bazzoni, O. Goertsches, *K-cosymplectic manifolds*, Ann. Global Anal. Geom. 47 (2015), 239–270.

[5] D. E. Blair, *The theory of quasi-Sasakian structures*, J. Differential Geometry 1 (1967), 331–345.

[6] B. Cappelletti-Montano, A. De Nicola, I. Yudin, *A survey on cosymplectic geometry*, Rev. Math. Phys. 25 (2013), 1343002, 55 pp.

[7] D. Chinea, M. de León, J.-C. Marrero, *The constraint algorithm for time-dependent Lagrangians*, J. Math. Phys. 35 (1994), 3410–3447.

[8] D. Chinea, M. de León, J. C. Marrero, *Locally conformal cosymplectic manifolds and time-dependent Hamiltonian systems*, Comment. Math. Univ. Carolin. 32 (1991), 383–387.

[9] M. Crainic and C. Zhu, *Integrability of Jacobi and Poisson structures*, Ann. Inst. Fourier (Grenoble) 57 (2007), 1181–1216.

[10] P. A. Damianou, F. Petalidou, *Poisson brackets with prescribed Casimirs*, Canadian J. Math. 64 (2012), 991–1018.

[11] P. A. Damianou, R. Loja Fernandes, *From the Toda lattice to the Volterra lattice and back,* Rep. Math. Phys. 50 (2002), 361–378.

[12] M. de León, J.-C. Marrero, *Constrained time-dependent Lagrangian systems and Lagrangian submanifolds*, J. Math. Phys. 34 (1993), 622–644.

[13] A. De Nicola, J.-C. Marrero, E. Padrón, *Reduction of Poisson–Nijenhuis Lie algebroids to symplectic-Nijenhuis Lie algebroids with a nondegenerate Nijenhuis tensor*, J. Phys. A 44 (2011), 425206, 35 pp.

[14] S.-A. Djiba, A. Wade, *Bicrossed product induced by Poisson vector fields and their integrability*, Int. J. Geom. Methods Mod. Phys. 13 (2016), no. 3, 1650022, 17 pp.

[15] S.-A. Djiba, A. Wade, *On cosymplectic groupoids*, C. R. Math. Acad. Sci. Paris 353 (2015), 859–863.

[16] J.-P. Dufour, N.-G. Zung, *Poisson Structures and Their Normal Forms*, Progress in Mathematics 242, Birkhäuser, 2005.

[17] J. Grabowski, P. Urbański, *Lie algebroids and Poisson–Nijenhuis structures. Quantizations, deformations and coherent states*, Rep. Math. Phys. 40 (1997), 195–208.

[18] V. Guillemin, E. Miranda, A. Pires, *Codimension one symplectic foliations and regular Poisson structures*, Bull. Braz. Math. Soc. (N.S.) 42 (2011), 607–623.

[19] D. Iglesias-Ponte, C. Laurent-Gengoux, P. Xu, *Universal lifting theorem and quasi-Poisson groupoids*, J. Eur. Math. Soc. 14 (2012), 681–731.

[20] J. Janyška, M. Modugno, *Generalized geometrical structures of odd dimensional manifolds*, J. Math. Pures Appl. 91 (2009), 211–232.

[21] Y. Kosmann-Schwarzbach, V. Rubtsov, *Compatible structures on Lie algebroids and Monge-Ampre operators*, Acta Appl. Math. 109 (2010), 101–135.

[22] Y. Kosmann-Schwarzbach, F. Magri, *Poisson–Nijenhuis structures*, Ann. Inst. Henri Poincaré Phys. Théor. 53 (1990), 35–81.

[23] H. Li, *Topology of co-symplectic/co-Kähler manifolds*, Asian J. Math. 12 (2008), 527–543.

[24] P. Libermann, *Sur les automorphismes infinitésimaux des structures symplectiques et des structures de contact*, in: Colloque Géom. Diff. Globale, Centre Belge Rech. Math., Louvain, 1959, 37–59.

[25] A. Lichnerowicz, *Les variétés de Poisson et leurs algèbres de Lie associées*, J. Diff. Geom. 12 (1977), 253–300.

[26] A. Lichnerowicz, *Variétés canoniques et transformations canoniques*, C. R. Acad. Sci. Paris Sér. A-B 280 (1975), A37–A40.

[27] A. Lichnerowicz, *Sur la géométrie des transformations canoniques*, C. R. Acad. Sci. Paris Ser. A-B 289 (1979), A279–A283.

[28] J.-H. Lu, *Poisson homogeneous spaces and Lie algebroids associated to Poisson actions*, Duke Math. J. 87 (1997), 261–304.

[29] K. Mackenzie, *Groupoids and Lie Algebroids in Differential Geometry*, Cambridge University Press, Cambridge, 1987.

[30] F. Magri, C. Morosi, *A geometrical characterization of integrable Hamiltonian systems through the theory of Poisson–Nijenhuis manifolds*, Quaderno S/19, Univ. di Milano, 1984.

[31] C.-M. Marle, *Lie group actions on Poisson and canonical manifolds*, in: Dynamical Systems and Microphysics (Udine, 1981), Academic Press, New York, 1982, 61–73.

[32] J. C. Marrero, E. Padrón, *New examples of compact cosymplectic solvmanifolds*, Arch. Math. (Brno) 34 (1998), 337–345.

[33] A. Meucci, *Toda equations, bi-Hamiltonian systems, and compatible Lie algebroids*, Math. Phys. Anal. Geom. 4 (2001), 131–146.

[34] T. Mokri, *Matched pairs of Lie algebroids*, Glasgow Math. J. 39 (1997), 167–181.

[35] J. M. Nunes da Costa, *Some remarks on the Poisson–Nijenhuis and the Jacobi structures*, in: Proc. Summer School on Differential Geometry (Coimbra, 1999), 109–117.

[36] M. Stienon, P. Xu, *Poisson quasi-Nijenhuis manifolds*, Comm. Math. Phys. 270 (2007), 709–725.

[37] A. D. Weinstein, *The modular automorphism group of a Poisson manifold*, J. Geom. Phys. 23 (1997), 379–394.