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THE OPTIMAL CONSTANTS IN KHINTCHINE'S INEQUALITY FOR THE CASE $2 < p < 3$

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Abstract. A main step in Haagerup's proof for the optimal constants in Khintchine's inequality is to show integral inequalities of the type $\int (g^s - f^s)\, d\mu \geq 0$. In 2000, F. L. Nazarov and A. N. Podkorytov made Haagerup's proof much more clear for the case $0 < p < 2$ by using a lemma on distribution functions. In this article we treat the case $2 < p < 3$ with their technique.

1. Introduction. We will always denote by $\varepsilon_1, \varepsilon_2, \ldots$ a sequence of i.i.d. Bernoulli variables on a probability space $(\Omega, \mathbb{P})$ with $\mathbb{P}(\{\varepsilon_1 = -1\}) = \mathbb{P}(\{\varepsilon_1 = 1\})$, and we denote by $\mathbb{E}$ the expectation. The Khintchine inequality is the following:

Theorem 1.1. *For every* $p \in \mathbb{R}_{>0}$ *there exist constants* $A, B > 0$ *such that for every* $n \in \mathbb{N}$ *and every sequence* $(a_k) \in \mathbb{R}^n$ *we have*

$$A\|(a_k)\|_2 \leq \Big(\mathbb{E}\Big|\sum_{k=1}^{n} a_k \varepsilon_k\Big|^p\Big)^{1/p} \leq B\|(a_k)\|_2. \tag{1.1}$$

The optimal constants for which (1.1) *holds are*

$$A_p = \min\bigg\{1, 2^{1/2-1/p}, 2^{1/2}\bigg(\frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi}}\bigg)^{1/p}\bigg\}$$

and

$$B_p = \max\bigg\{1, 2^{1/2}\bigg(\frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi}}\bigg)^{1/p}\bigg\}.$$

The values $A_p = 1$ for $2 \leq p$ and $B_p = 1$ for $p \leq 2$ follow directly from Hölder's inequality. Haagerup [H] published the first complete proof of Theorem 1.1. A relatively elementary proof for the case $p \geq 3$ can be found in [FHJSZ] where mainly convexity arguments were used. A new approach

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for the case $0 < p < 2$ was given in [NP], and we extend those ideas to the case $2 < p < 3$.

1.1. Organization of the paper. In Section 2 we will recall the basic ideas to prove Khintchine's inequality which lead to inequalities of the type $\int (g^s - f^s)\, d\mu \geq 0$. In Section 3 we will describe the method of Nazarov and Podkorytov to solve such kind of integral inequalities. Their lemma provides a positive answer to this inequality by showing two conditions. Sections 4 and 5 will be dedicated to the proof of the first and the second of these.

1.2. Preliminaries. Throughout this article, we only make use of standard notation and functions which should be well-known to the reader. At several points, we make use of the series expansion of the function $f(t) = -\ln(\cos(t))$ for $t \in (-\pi/2, \pi/2)$. We remark that $f'(t) = \tan(t)$ and the series representation of f is (cf. [GR, 1.518])

$$-\ln(\cos(t)) = \sum_{k=1}^{\infty} \frac{2^{2k-1}(2^{2k}-1)|B_{2k}|}{k(2k)!} t^{2k} = \frac{t^2}{2} + \frac{t^4}{12} + \frac{t^6}{45} + \cdots \tag{1.2}$$

where B_m denotes the mth Bernoulli number. We remark that the coefficients of this series expansion are non-negative, and hence we can derive from (1.2) upper estimates for the cosine:

$$\begin{aligned} \cos(t) &\leq \exp(-t^2/2 - t^4/12 - t^6/45) \\ &\leq \exp(-t^2/2 - t^4/12) \leq \exp(-t^2/2). \end{aligned} \tag{1.3}$$

We will make use of the exponential, sine and cosine integrals: For $x < 0$ we define

$$\mathrm{Ei}(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t}\, dt = \gamma + \ln(-x) + \sum_{k=1}^{\infty} \frac{x^k}{k \cdot k!}, \tag{1.4}$$

and for positive x we define

$$\mathrm{si}(x) = -\int_{x}^{\infty} \frac{\sin(t)}{t}\, dt = -\frac{\pi}{2} - \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k-1}}{(2k-1)(2k-1)!}, \tag{1.5}$$

$$\mathrm{ci}(x) = -\int_{x}^{\infty} \frac{\cos(t)}{t}\, dt = \gamma + \ln(x) + \sum_{k=1}^{\infty} (-1)^k \frac{x^{2k}}{2k(2k)!} \tag{1.6}$$

where $\gamma = 0.577215\ldots$ is Euler's constant (see [GR, 8.211, 8.214, 8.230, 8.232]). We assume that one can always compute sufficiently precise numerical values of $\mathrm{Ei}(x)$, $\mathrm{si}(x)$ and $\mathrm{ci}(x)$ for fixed x via the series representation.

2. Haagerup's approach. The first observation, due to Stechkin, is that

$$\liminf_{n\to\infty} \mathbb{E}\left| \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \varepsilon_k \right|^p \geq 2^{p/2} \frac{\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}}$$

(see [S] and also [H]). Hence, we only need to establish

$$B_p \le 2^{1/2}\biggl(\frac{\Gamma\bigl(\frac{p+1}{2}\bigr)}{\sqrt{\pi}}\biggr)^{1/p}.$$

We introduce for $s > 0$ the following auxiliary function:

$$F_p(s) = c_p \int_0^\infty \biggl(\frac{1}{2}t^2 - 1 + |\cos(t/\sqrt{s})|^s\biggr)\frac{dt}{t^{p+1}}$$

where c_p is a constant which is of no greater interest here. We have the following lemma:

Lemma 2.1.

(1) *Let* $\sum_{k=1}^n a_k^2 = 1$. *Then*

$$\mathbb{E}\Bigl|\sum_{k=1}^n a_k\varepsilon_k\Bigr|^p \le \sum_{k=1}^n a_k^2 F_p(a_k^{-2}).$$

(2) *We have*

$$\lim_{s\to\infty} F_p(s) = c_p \int_0^\infty \biggl(\frac{1}{2}t^2 - 1 + \exp(-t^2/2)\biggr)\frac{dt}{t^{p+1}} = 2^{p/2}\frac{\Gamma\bigl(\frac{p+1}{2}\bigr)}{\sqrt{\pi}}.$$

If we were able to show

$$F_p(s) \le 2^{p/2}\frac{\Gamma\bigl(\frac{p+1}{2}\bigr)}{\sqrt{\pi}}, \tag{2.1}$$

we would deduce with the help of the preceding lemma that for all sequences of scalars $a_1, \ldots, a_n$ with l_2-norm equal to 1,

$$\mathbb{E}\Bigl|\sum_{k=1}^n a_k\varepsilon_k\Bigr|^p \le \sum_{k=1}^n a_k^2 F_p(a_k^{-2}) \le 2^{p/2}\frac{\Gamma\bigl(\frac{p+1}{2}\bigr)}{\sqrt{\pi}}\Bigl(\sum_{k=1}^n a_k^2\Bigr) = 2^{p/2}\frac{\Gamma\bigl(\frac{p+1}{2}\bigr)}{\sqrt{\pi}},$$

and this establishes the Khintchine inequality by taking $\sqrt[p]{\cdot}$ and by homogeneity. Unfortunately, (2.1) is not true for s sufficiently close to 1 and p sufficiently close to 2. However, there is a short and elementary proof in [H, Lemma 4.6] for the case when $\sum_{k=1}^n a_k^2 = 1$ and one of the coefficients satisfies $|a_k| \ge 2^{-1/4}$. So, we only need to consider the case that all the coefficients are less than or equal to $2^{1/4}$, which means that we only need to show (2.1) for $s \ge \sqrt{2}$ in order to show Khintchine's inequality. Because of Lemma 2.1(2) and the definition of F_p, inequality (2.1) is equivalent to

$$\int_0^\infty (\exp(-t^2/2)^s - |\cos(t)|^s)\frac{dt}{t^{p+1}} \ge 0. \tag{2.2}$$

3. Nazarov and Podkorytov's lemma on distribution functions.

We introduce the definition of distribution functions suitable for our needs:

DEFINITION 3.1. Let $f : X \to \mathbb{R}_{>0}$ be a measurable function on a measure space (X, μ). Then we denote by $F_*(y) := \mu(\{x \in X : f(x) < y\})$, $y \in \mathbb{R}_{>0}$, the *distribution function* of f.

Since we know that the integral of a function is determined by its distribution function, it might be more useful to study the distribution functions rather than the functions themselves in order to treat integral inequalities like (2.2). In our case we can hope to simplify the situation because (2.2) has an oscillating integrand, whereas the distribution functions are increasing. In [NP] we find a lemma on distribution functions which was successfully used in that article to compute the optimal constants for the case $0 < p < 2$. Later, it was also used in [K] for the optimal constants in the Khintchine inequality with Steinhaus variables. It states the following:

LEMMA 3.2. *Let* $Y > 0$ *and* $f, g : X \to [0, Y]$ *be measurable functions on a measure space* (X, μ). *Assume that the distribution functions* F_*, G_* *of* f, g *are finite on* $(0, Y)$. *Let* $S = \{s > 0 : g^s - f^s \in L_1(X, \mu)\}$. *If there exist* $y_0 \in (0, Y)$ *and* $s_0 \in S$ *such that:*

(1) $F_* - G_* \leq 0$ *on* $(0, y_0)$ *and* $F_* - G_* \geq 0$ *on* (y_0, Y),
(2) $\int_X (g^{s_0} - f^{s_0})\, d\mu \geq 0$,

then $\int_X (g^s - f^s)\, d\mu \geq 0$ *for all* $s \in S$ *with* $s \geq s_0$.

In our situation we will consider the measure space $X = (0, \infty)$ with the measure $d\mu_p = \frac{dt}{t^{p+1}}$ and the functions f, g on X defined by $f(t) := |\cos(t)|$ and $g(t) = \exp(-t^2/2)$. An easy calculation shows that the distribution functions of f, g are

$$F_*(x) = \frac{1}{p} \sum_{k=0}^{\infty} \left(\frac{1}{(k\pi + \arccos(x))^p} - \frac{1}{((k+1)\pi - \arccos(x))^p} \right) \tag{3.1}$$

and $G_*(x) = \frac{1}{p}(-2\ln(x))^{-p/2}$ for $x \in (0, 1)$. We take $Y = 1$ and one can verify that in the notation of Lemma 3.2 we have $S = \mathbb{R}_{>0}$ and F_*, G_* are finite functions and C^1 on $(0, 1)$.

4. Proof of the first condition of Lemma 3.2.

Specifically, we will prove the following lemma, from which the first condition follows:

LEMMA 4.1. *There exist constants* $0 < \rho < \sigma < 1$ *such that:*

(1) $F_*(\sigma) - G_*(\sigma) \geq 0$.
(2) $F_* - G_*$ *is increasing on* $(\rho, 1)$.
(3) $F_* - G_*$ *is less than* 0 *on* $(0, \rho)$.

We will show this lemma with $\rho = 1/15$ and $\sigma = 0.97$.

4.1. Proof of Lemma 4.1(1). We minorize F_* for $x \in (0,1)$:

$$\begin{aligned} pF_*(x) &= \frac{1}{\arccos(x)^p} - \sum_{k\in\mathbb{N}}\left(\frac{1}{(k\pi-\arccos(x))^p} - \frac{1}{(k\pi+\arccos(x))^p}\right) \\ &\geq \frac{1}{\arccos(x)^p} - \sum_{k\in\mathbb{N}}\left(\frac{1}{(k\pi-k\arccos(x))^p} - \frac{1}{(k\pi+k\arccos(x))^p}\right) \\ &= \frac{1}{\arccos(x)^p} - \left(\sum_{k\in\mathbb{N}}\frac{1}{k^p}\right)\left(\frac{1}{(\pi-\arccos(x))^p} - \frac{1}{(\pi+\arccos(x))^p}\right) \\ &\geq \frac{1}{\arccos(x)^p} - \left(\sum_{k\in\mathbb{N}}\frac{1}{k^2}\right)\left(\frac{1}{(\pi-\arccos(x))^p} - \frac{1}{(\pi+\arccos(x))^p}\right). \end{aligned} \tag{4.1}$$

The series in the last line of (4.1) equals $\pi^2/6$. By the mean value theorem applied to the function $x \mapsto -\frac{1}{x^p}$, there is a $\zeta \in (\pi-\arccos(x), \pi+\arccos(x))$ such that

$$\begin{aligned} \frac{1}{2\arccos(x)}\left(\frac{1}{(\pi-\arccos(x))^p} - \frac{1}{(\pi+\arccos(x))^p}\right) &= p\frac{1}{\zeta^{p+1}} \\ &\leq 3\frac{1}{(\pi-\arccos(x))^{p+1}} \leq 3\frac{1}{(\pi-\arccos(x))^3}. \end{aligned}$$

Hence, (4.1) is greater than or equal to

$$\frac{1}{\arccos(x)^p} - \pi^2\frac{\arccos(x)}{(\pi-\arccos(x))^3},$$

and therefore

$$\begin{aligned} &p(F_*(x) - G_*(x)) \\ &\qquad \geq \frac{1}{\arccos(x)^p} - \frac{1}{\sqrt{2\cdot\ln(1/x)}^p} - \pi^2\frac{\arccos(x)}{(\pi-\arccos(x))^3}. \end{aligned} \tag{4.2}$$

We fix $x = 0.97$ and $p = 2$, and we see that the right-hand side is greater than 0. Since

$$\frac{1}{\arccos(0.97)} > \frac{1}{\sqrt{2\cdot\ln(1/0.97)}} \geq 1,$$

the right-hand side of (4.2) increases for $x = 0.97$ if we increase p.

4.2. Proof of Lemma 4.1(2). We recall $2 < p < 3$ and $0 < x < 1/15$. We can represent the summands of F_* in the following manner:

$$\frac{1}{(k\pi+\arccos(x))^p} - \frac{1}{((k+1)\pi-\arccos(x))^p} = \frac{1/(1-\varepsilon_k)^p - 1/(1+\varepsilon_k)^p}{[(k+1/2)\pi]^p}$$

where $\varepsilon_k = \frac{\pi/2-\arccos(x)}{(k+1/2)\pi} \leq 0.05$. Using linear Taylor approximation and estimating the remainder, one gets

$$(1+\varepsilon_k)^p - (1-\varepsilon_k)^p \leq 2p(1+\varepsilon_k^2)\varepsilon_k \leq 2.005p\varepsilon_k.$$

We remark that $\pi/2 - \arccos(x) \le 0.07$ and $\sin(\pi/2 - \arccos(x)) = x$. By monotonicity of $\sin(t)/t$ on the interval $(0, \pi/2)$ we deduce for $0 \le t \le 0.07$ that $t \le \frac{0.07}{\sin(0.07)} \sin(t)$. With all these results in mind we calculate

$$\begin{aligned}
\text{(4.3)}\qquad & \frac{1}{(1-\varepsilon_k)^p} - \frac{1}{(1+\varepsilon_k)^p} \\
&\le \frac{(1+\varepsilon_k)^p - (1-\varepsilon_k)^p}{(1-\varepsilon_k^2)^p} \le \frac{(1+\varepsilon_k)^p - (1-\varepsilon_k)^p}{(1-0.05^2)^3} \le 2.025 p\varepsilon_k \\
&\le 2.025p\,\frac{0.07}{\sin(0.07)}\,\frac{\sin(\pi/2-\arccos(x))}{(k+1/2)\pi} \le \frac{2.03}{(k+1/2)\pi} px.
\end{aligned}$$

Applying (4.3) in (3.1), we get

$$F_*(x) \le 2.03\Bigg(\sum_{k=0}^{\infty} \frac{1}{[(k+1/2)\pi]^{p+1}}\Bigg) \cdot x =: d_p \cdot x.$$

The coefficient d_p is convex in p, which yields $d_p \le d_2 + (d_3 - d_2)(p-2)$. Via the representation

$$d_p = 2.03\bigg(\frac{2}{\pi}\bigg)^{p+1}\bigg(1 - \frac{1}{2^{p+1}}\bigg)\bigg(\sum_{k=1}^{\infty} \frac{1}{k^{p+1}}\bigg)$$

involving Riemann's ζ-function, we can compute with sufficient precision $d_2 \le 0.551$ and $d_3 \le 0.3384$ and we conclude $F_*(x) \le (0.98 - 0.21p) \cdot x$. The last step will be to show $(0.98 - 0.21p) \cdot x \le G_*(x)$, which is equivalent to $p(0.98 - 0.21p) \le x^{-1}/(2\ln(1/x))^{p/2}$ or

$$p(0.98 - 0.21p) \le \frac{\exp(t)}{(2t)^{p/2}} \tag{4.4}$$

for $t \ge 2.7$. Maximizing the left-hand side of (4.4) in $2 \le p \le 3$ we get $p(0.98 - 0.21p) \le 1.15$. The right-hand side of (4.4) is increasing on $[3/2, \infty)$ (which can be verified by differentiation), so we have $\exp(t)/(2t)^{p/2} \ge \exp(2.7)/(2 \cdot 2.7)^{3/2} \ge 1.18 \ge 1.15$. ■

4.3. Proof of Lemma 4.1(3). We prove that $F_* - G_*$ is increasing on $(1/15, 1)$, which is equivalent to $F_*'/G_*' \ge 1$, providing $G_*' > 0$. We compute

$$\begin{aligned}
F_*'(x) &= \sum_{k=0}^{\infty}\bigg(\frac{1}{(k\pi + \arccos(x))^{p+1}} + \frac{1}{((k+1)\pi - \arccos(x))^{p+1}}\bigg)\frac{1}{\sqrt{1-x^2}} \\
&\ge \bigg(\frac{1}{\arccos(x)^{p+1}} + \frac{1}{(\pi - \arccos(x))^{p+1}}\bigg)\frac{1}{\sqrt{1-x^2}}, \\
G_*'(x) &= \frac{1}{x(-2\ln(x))^{p/2+1}}.
\end{aligned}$$

For simplification, we set $t = \arccos(x)$. This yields

$$\frac{F'_*(x)}{G'_*(x)} \geq \left(\frac{1}{t^{p+1}} + \frac{1}{(\pi - t)^{p+1}}\right)[-2\ln(\cos(t))]^{p/2+1}\cot(t) \tag{4.5}$$
$$= \left(\left(\frac{-2\ln(\cos(t))}{t^2}\right)^{(p+1)/2} + \left(\frac{-2\ln(\cos(t))}{(\pi - t)^2}\right)^{(p+1)/2}\right)[-2\ln(\cos(t))]^{1/2}\cot(t).$$

Hence, it suffices to show that the right-hand side is greater than 1 for $0 < t < 1.50409$. We split the proof into subsections.

4.3.1. *Reduction to the case $p = 2$.* We show that the case $p = 2$ gives the least value in (4.5) for every t, and for this it is appropriate to examine the expression $A^{p+1} + B^{p+1}$. We have the following lemma, easy to check:

Lemma 4.2. *Let $p \geq 2$, $A \geq 1$ and $B > 0$ be real numbers. If $\frac{A^3}{B^3} \cdot \ln(A) \geq -\ln(B)$, then $A^{p+1} + B^{p+1} \geq A^3 + B^3$.*

Note that the case $B \geq 1$ is trivial.

We set $A := \sqrt{-2\ln(\cos(t))/t^2}$ and $B := \sqrt{-2\ln(\cos(t))/(\pi - t)^2}$. Then $A \geq 1$ follows from the fact that $-\ln(\cos(t)) \geq t^2/2$ on $[0, \pi/2]$ (see (1.2)). We have $A^3/B^3 = (\pi - t)^3/t^3$, and hence we want to show the inequality

$$(\pi - t)^3 \ln\left(\frac{-2\ln(\cos(t))}{t^2}\right) \geq -t^3 \ln\left(\frac{-2\ln(\cos(t))}{(\pi - t)^2}\right). \tag{4.6}$$

Bringing the terms which belong to t^3 on the right-hand side and applying the logarithm rules, we can restate (4.6) as follows:

$$(\pi^3 - 3\pi^2 t + 3\pi t^2)\ln\left(\frac{-2\ln(\cos(t))}{t^2}\right) \geq 2t^3 \ln\left(\frac{\pi - t}{t}\right).$$

Use $-2\ln(\cos(t))/t^2 \geq 1 + \left(t^2/6 + \frac{2}{45}t^4\right)$ for $0 \leq t < \pi/2$ (see 1.2) and the well-known estimate $\ln(1 + x) \geq x - x^2/2$ for $x \geq 0$ to conclude that

$$\ln\left(\frac{-2\ln(\cos(t))}{t^2}\right) \geq \frac{t^2}{6} + t^4\left(\frac{2}{45} - \frac{1}{2}\left[\frac{1}{6} + \frac{2}{45}t^2\right]^2\right). \tag{4.7}$$

Since t is bounded from above by $\pi/2$, we can check that the coefficient of t^4 is positive, which yields that the left-hand side of (4.7) is bounded from below by $t^2/6$. Now, it is sufficient to prove that $(\pi^3 - 3\pi^2 t + 3\pi t^2) \cdot t^2/6 \geq 2t^3 \ln\left(\frac{\pi - t}{t}\right)$, or equivalently

$$\pi^3 - 3\pi^2 t + 3\pi t^2 \geq 12t\ln\left(\frac{\pi - t}{t}\right). \tag{4.8}$$

To do so, we first remark that $t\ln\left(\frac{\pi - t}{t}\right)$ is concave for $0 < t < \pi$; this can be proved by looking at the derivative, which is decreasing. Denote by T_{t_0}

the tangent function to $12t\ln\bigl(\frac{\pi-t}{t}\bigr)$ at the point t_0. By concavity, we have

$$12t\ln\biggl(\frac{\pi-t}{t}\biggr)\le T_{t_0}(t) \tag{4.9}$$

for every t_0. Hence, (4.8) leads to the inequality $\pi^3-3\pi^2t+3\pi t^2\ge T_{t_0}(t)$, which obviously contains only second degree polynomials. This last inequality can be verified to hold for $t_0=1$.

Thus, we have shown that it suffices to prove (4.5) for $p=2$:

$$\biggl(\frac{1}{t^3}+\frac{1}{(\pi-t)^3}\biggr)[-2\ln(\cos(t))]^2\cot(t)\ge 1. \tag{4.10}$$

We will treat the cases $0\le t\le 1$ and $t>1$ separately and by different methods.

4.3.2. *Case 1:* $0\le t\le 1$ *and* $p=2$. We will minorize each of the three factors in (4.10) by polynomials in t and in $1/t$.

Lemma 4.3. *For* $0<t\le 1$ *the following three inequalities hold:*

(1) $\displaystyle\frac{1}{t^3}+\frac{1}{(\pi-t)^3}\ge\frac{1}{t^3}\biggl(1+\frac{1}{\pi^3}t^3+\frac{3}{\pi^4}t^4+\frac{6}{\pi^5}t^5\biggr)$.

(2) $[-2\ln(\cos(t))]^2\ge t^4\bigl(1+\frac{1}{3}t^2+\frac{7}{60}t^4\bigr)$.

(3) $\cot(t)\ge\dfrac{1}{t}-\dfrac{t}{3}-\dfrac{1}{40}t^3$.

Proof. (1) We can transform the left-hand side to

$$\frac{1}{t^3}\biggl(1+\frac{t^3}{\pi^3}\biggl(\frac{1}{1-t/\pi}\biggr)^3\biggr).$$

Expanding $1/(1-t/\pi)$ into a geometric series, we minorize as follows:

$$\biggl(\frac{1}{1-t/\pi}\biggr)^3=\biggl(\sum_{k=0}^{\infty}\biggl(\frac{t}{\pi}\biggr)^k\biggr)^3\ge\biggl(1+\frac{1}{\pi}t+\frac{1}{\pi^2}t^2\biggr)^3\ge 1+\frac{3}{\pi}t+\frac{6}{\pi^2}t^2,$$

which gives the desired result.

(2) We use $-2\ln(\cos(t))/t^2\ge 1+t^2/6+\frac{2}{45}t^4$ (see (1.2)), and we evaluate $\bigl(1+t^2/6+\frac{2}{45}t^4\bigr)^2$ up to the power 4.

(3) The series expansion of the cotangent (see [GR]) is

$$\cot(t)=\frac{1}{t}-\frac{1}{3}t-\underbrace{t^3\sum_{k=2}^{\infty}\frac{2^{2k}|B_{2k}|}{(2k)!}t^{2k-4}}_{=:R(t)}.$$

Since R is increasing on $[0,1]$, we have $R(t)\le R(1)=1/1-1/3-\cot(1)\le 1/40$. ∎

Proposition 4.4. *Let* $0 \le t \le 1$. *Then*

$$\left(1+\frac{1}{\pi^3}t^3+\frac{3}{\pi^4}t^4+\frac{6}{\pi^5}t^5\right)\left(1-\frac{1}{3}t^2-\frac{1}{40}t^4\right) \ge 1-\frac{1}{3}t^2+\frac{1}{40}t^3. \tag{4.11}$$

Proof. We expand partially the left-hand side of (4.11) so that we can represent it as the sum of

$$p_1(t) := 1-\frac{1}{3}t^2+\frac{1}{\pi^3}t^3+\frac{3}{\pi^4}t^4$$

and

$$p_2(t) := t^5\left(\frac{6}{\pi^5}-\left[\frac{1}{3\pi^3}+\frac{1}{\pi^4}t+\left(\frac{2}{\pi^5}+\frac{1}{40\pi^3}\right)t^2+\frac{3}{40\pi^4}t^3+\frac{3}{20\pi^5}t^4\right]\right). \tag{4.12}$$

We can maximize the expression in square brackets by simply taking $t = 1$, and thus we can bound (4.12) from below by $-0.009t^5 \ge -0.009 \cdot 1 \cdot t^4$. Now, we have

$$p_1(t)+p_2(t) \ge p_1(t)-0.009t^4 \ge 1-\frac{1}{3}t^2+\frac{1}{\pi^3}t^3-0.004t^4,$$

and we show analogously that $-0.004t^4 \ge -0.004t^3$, from which we obtain the desired inequality (4.11). ■

Corollary 4.5. *Let* $0 \le t \le 1$. *Then*

$$\left(1-\frac{1}{3}t^2+\frac{1}{40}t^3\right)\left(1+\frac{1}{3}t^2+\frac{7}{60}t^4\right) \ge 1. \tag{4.13}$$

Proof. The expansion of the left-hand side of (4.13) is exactly $1+\frac{4\frac{1}{2}}{180}t^3+\frac{1}{180}t^4+\frac{1\frac{1}{2}}{180}t^5-\frac{7}{180}t^6+\frac{7}{2400}t^7$. This expression is obviously greater than or equal to 1 for $0 \le t \le 1$. ■

Combining Lemma 4.3 with Proposition 4.4 and Corollary 4.5 yields (4.10) for $0 \le t \le 1$.

4.3.3. *Case 2:* $1 \le t \le 1.50409$ *and* $p = 2$. The desired inequality can be reformulated as

$$g(t) := \frac{1}{t^3}+\frac{1}{(\pi-t)^3} \ge \frac{\tan(t)}{[-2\ln(\cos(t))]^2} =: f(t).$$

Obviously, g is a convex function on $(0, \pi/2)$, and one can show that f is convex there, too. Denote by T_{t_0} the tangent function to g at the point t_0. We can check that $g \ge f$ on an interval $[a, b]$ by the following method: Find an appropriate t_0 and show $T_{t_0}(a) \ge f(a)$ and as well for b. Thus, by convexity we have $g(t) \ge T_{t_0}(t) \ge f(t)$ for all $a \le t \le b$. Good values are $t_0 = 1.1$ for the interval $[1, 1.25]$, and $t_0 = 1.45$ for $[1.24, 1.505]$.

We now give a sketch of the proof that f is convex on $(0, \pi/2)$. We show that $f''(t) \ge 0$. The second derivative of $f''(t)$ multiplied by the positive value $2\ln(\cos(t))^4\cos(t)^3/\sin(t)$ is $\ln(\cos(t))^2 + 3\ln(\cos(t)) + 3\sin(t)^2$. We set $s := -\ln(\cos(t))$, and we conclude that $f''(t) \geq 0$ for all $0 < t < \pi/2$ is equivalent to $s^2 - 3s + 3 - 3\exp(-2s) \geq 0$ for all $0 < s < \infty$. We multiply both sides by $\exp(2s)$ and use $\exp(2s) \geq 1 + 2s + 2s^2$. Then

$$(s^2 - 3s + 3)(1 + 2s + 2s^2) - 3 = s(2s(s-1)^2 + 3 - s),$$

which is indeed positive for positive s. ∎

5. Proof of the second condition of Lemma 3.2. We need to show that

$$H(p) := \int_0^\infty \frac{\exp(-t^2/\sqrt{2}) - |\cos(t)|^{\sqrt{2}}}{t^{p+1}}\, dt \geq 0$$

for all $2 < p < 3$. We do so by showing that $H(2) \geq 0$ and $H'(p) \geq 0$ for $2 \leq p \leq 3$.

5.1. Proof of $H'(p) \geq 0$. We have, by differentiation under the integral,

$$H'(p) = \int_0^\infty -\ln(t) \frac{\exp(-t^2/\sqrt{2}) - |\cos(t)|^{\sqrt{2}}}{t^{p+1}}\, dt,$$

and we examine the resulting integral on different intervals. Let us start with $[0,1]$. Since $-\ln(t) \geq 0$ and $\exp(-t^2/\sqrt{2}) \geq \exp(-t^2/\sqrt{2} - t^4/(\sqrt{2}\cdot 6))$ $\geq |\cos(t)|^{\sqrt{2}}$, we obtain

$$\begin{aligned} (5.1)\qquad & -\ln(t)\frac{\exp(-t^2/\sqrt{2}) - |\cos(t)|^{\sqrt{2}}}{t^{p+1}} \\ & \qquad \geq -\ln(t)\frac{\exp(-t^2/\sqrt{2})\big(1 - \exp(-t^4/(\sqrt{2}\cdot 6))\big)}{t^3}. \end{aligned}$$

We estimate each of the three (positive) factors on the right-hand side of (5.1) from below by Taylor polynomials. We use $-\ln(t) \geq 1 - t$, and additionally $\exp(-a) \geq 1 - a$ and $1 - \exp(-b) \geq b - b^2/2$ for $a, b \geq 0$, which yield sufficiently precise estimates for the second and third factors. The result of these estimations for the integrand is a not too extensive polynomial, for which we can compute the exact value, which is larger than 0.0153.

The next interval is $[1, \pi/2]$. The integrand is negative since $-\ln(t) \leq 0$ and $\exp(-t^2/\sqrt{2}) \geq |\cos(t)|^{\sqrt{2}}$. Consequently, we want to find a precise estimate from above of

$$\frac{\ln(t)\big(\exp(-t^2/\sqrt{2}) - |\cos(t)|^{\sqrt{2}}\big)}{t^{p+1}} \leq \frac{\ln(t)\big(\exp(-t^2/\sqrt{2}) - |\cos(t)|^{\sqrt{2}}\big)}{t^3}$$

for $1 \leq t \leq \pi/2$. We make use of the fact that $\ln(t) \leq t - 1$. Since $f(t) := \exp(-t^2/\sqrt{2})$ is convex on this interval, we can estimate this term from above by the secant function through $(1, f(1))$ and $(\pi/2, f(\pi/2))$. We estimate $|\cos(t)|^{\sqrt{2}}$ from below by $\cos(1.2)^{\sqrt{2}}$ on the interval $[1,1.2]$, by $\cos(1.4)^{\sqrt{2}}$ on $[1.2,1.4]$, and by 0 elsewhere. These approximations give

$$\int_1^{\pi/2} -\ln(t)\frac{\exp(-t^2/\sqrt{2})-|\cos(t)|^{\sqrt{2}}}{t^{p+1}}\,dt \geq -0.0147.$$

It should be natural that the integral on the interval $[\pi/2,\infty)$ is positive, because $\exp(-t^2/\sqrt{2})$ decays very fast whereas $|\cos(t)|^{\sqrt{2}}$ oscillates between 0 and 1. By simple one-variable calculus methods one can show the following lemma:

Lemma 5.1. *Let* $t\in[\pi/2,\infty)$, $2\leq p\leq 3$, *set* $\lambda_p := 1.75\cdot(2/\pi)^p$ *and* $\Lambda_p = \frac{1}{e(p-1)}$. *Then*

(1) $\displaystyle\frac{\ln(t)}{t^{p+1}} \geq \lambda_p\frac{1}{t^4}$.

(2) $\displaystyle\frac{\ln(t)}{t^{p+1}} \leq \Lambda_p\frac{1}{t^2}$.

The motivation behind this lemma is that we are now able to give estimates which we can compute explicitly (see [GR, 2.642]):

$$\int_{\pi/2}^{\infty} \ln(t)\frac{|\cos(t)|^{\sqrt{2}}}{t^{p+1}} \geq \lambda\int_{\pi/2}^{\infty}\frac{\cos(t)^2}{t^4}\,dt = \lambda_p\frac{2+\pi^2-2\,\mathrm{si}(\pi)\pi}{3\pi} \geq 0.04\left(\frac{2}{\pi}\right)^p$$

where si was defined in (1.5). On the other hand, we have

$$\int_{\pi/2}^{\infty} \ln(t)\frac{\exp(-t^2/\sqrt{2})}{t^{p+1}}\,dt \leq \Lambda_p\int_{\pi/2}^{\infty}\frac{\exp(-t^2/\sqrt{2})}{t^2}\,dt \leq 0.008\frac{1}{p-1}.$$

The second integral was computable since we can transform it via partial integration to an integral with integrand of the form $\exp(-s^2)$. Hence, we need to show that $0.04(2/\pi)^p \geq 0.008/(p-1)$. This is feasible because $(p-1)(2/\pi)^p$ is increasing on $[2,3]$, and so one only needs to check this for $p=2$. We conclude that the integral on the interval $[\pi/2,\infty)$ is positive.

5.2. Proof of $H(2)\geq 0$.

We want to show that

$$\int_0^{\infty}\frac{\exp(-t^2/\sqrt{2})-|\cos(t)|^{\sqrt{2}}}{t^3}\,dt \geq 0,$$

and we start with estimates on the interval $[0,\pi/4]$. We have $|\cos(t)|^{\sqrt{2}} \leq \exp(-t^2/\sqrt{2}-t^4/(\sqrt{2}\cdot 6)-\sqrt{2}\,t^6/45)$ by (1.3) and $1-\exp(-b)\leq b-b^2/2$

for $a, b \geq 0$, which yields

$$\begin{aligned}
(5.2)\qquad & \exp(-t^2/\sqrt{2}) - |\cos(t)|^{\sqrt{2}} \\
&\geq \exp(-t^2/\sqrt{2})\biggl(1 - \exp\biggl(-\biggl[\frac{t^4}{\sqrt{2}\cdot 6} + \frac{\sqrt{2}t^6}{45}\biggr]\biggr)\biggr) \\
&\geq \exp(-t^2/\sqrt{2})\biggl(\biggl[\frac{1}{\sqrt{2}\cdot 6}t^4 + \frac{\sqrt{2}}{45}t^6\biggr] - \frac{1}{2}\biggl[\frac{1}{\sqrt{2}\cdot 6}t^4 + \frac{\sqrt{2}}{45}t^6\biggr]^2\biggr) \\
&\geq \exp(-t^2/\sqrt{2})\biggl(\biggl[\frac{1}{\sqrt{2}\cdot 6}t^4 + \frac{\sqrt{2}}{45}t^6\biggr] - \frac{1}{2}\Bigl(\frac{\pi}{4}\Bigr)^2\biggl[\frac{\sqrt{2}}{12} + \frac{\sqrt{2}}{45}\Bigl(\frac{\pi}{4}\Bigr)^2\biggr]^2 t^6\biggr).
\end{aligned}$$

Dividing the right-hand side by t^3, we get an expression of the form $\exp(-t^2/\sqrt{2})(at + bt^3)$. Applying the change of variable $s \mapsto \sqrt[4]{2}\sqrt{s}$ we end up with an expression of the form $\exp(-t)(a' + b't)$, for which we can easily calculate the integral and which finally gives the estimate

$$\int_0^{\pi/4} \frac{\exp(-t^2/\sqrt{2}) - |\cos(t)|^{\sqrt{2}}}{t^3}\,dt \geq 0.03129.$$

Next, we want to determine the exact integral of $\exp(-t^2/\sqrt{2})/t^3$. For this, let $a > 0$. Applying $s \mapsto \sqrt[4]{2}\sqrt{s}$ and [GR, 2.325], we get

$$(5.3)\qquad \int_a^\infty \frac{e^{-t^2/\sqrt{2}}}{t^3}\,dt = \frac{1}{2\sqrt{2}}\int_{a^2/\sqrt{2}}^\infty \frac{e^{-s}}{s^2}\,ds = \frac{e^{-a^2/\sqrt{2}}}{2a^2} + \frac{\mathrm{Ei}(-a^2/\sqrt{2})}{2\sqrt{2}}$$

where Ei is defined in (1.4). Substituting $a = \pi/4$ we find that (5.3) is greater than 0.29586.

The last part is to estimate $\int_{\pi/4}^\infty |\cos(t)|^{\sqrt{2}}/t^3$; we treat the intervals $[\pi/4, 3\pi/4]$ and $[3\pi/4, \infty)$ separately. Let us start with $[\pi/4, 3\pi/4]$. We will make use of the fact that

$$\int \frac{\cos(t)}{t^3}\,dt = \frac{1}{2}\biggl(-\frac{\cos(t)}{t^2} + \frac{\sin(t)}{t} - \mathrm{ci}(t)\biggr) + K$$

and

$$(5.4)\qquad \int \frac{\cos(t)^2}{t^3}\,dt = \frac{1}{4}\biggl(\frac{\cos(2t)}{t^2} + 2\frac{\sin(2t)}{t} - 4\,\mathrm{ci}(2t) - \frac{1}{t^2}\biggr)$$

(see for example [GR, 2.642]), where ci was defined in (1.6). We have $|\cos(t)| \in [0, \sqrt{2}/2]$ for $t \in [\pi/4, 3\pi/4]$, and our strategy will be to bound $x^{\sqrt{2}}$ from above with sufficient precision by a polynomial of the form $a + bx + cx^2$, since we know how to integrate $1/t^3$, $\cos(t)/t^3$ and $\cos(t)^2/t^3$.

LEMMA 5.2. *Let x be real and non-negative. Then:*

(1) $0 \leq (\sqrt{2} - 1)x^2 + (2 - \sqrt{2} - 0.126)x - x^{\sqrt{2}}$ *for* $x \in [0, 0.25]$.
(2) $0 \leq (\sqrt{2} - 1)x^2 + 0.6356x - 0.04399 - x^{\sqrt{2}}$ *for* $x \in [0.25, \sqrt{2}/2]$.

Proof. (1) The function $f(x) = (\sqrt{2}-1)x^2 + (2-\sqrt{2})x - x^{\sqrt{2}}$ is concave on $[0, 0.25]$ and $f(0) = 0 < f(0.25)$, which yields $f(x) \geq 0$ for all $x \in [0, 0.25]$. Hence, we can subtract the secant function through the points $(0, f(0))$ and $(0.25, f(0.25))$ from f, and this gives the non-negative function $x \mapsto (\sqrt{2}-1)x^2 + (2-\sqrt{2}-0.126)x - x^{\sqrt{2}}$.

(2) Set $g(x) := (\sqrt{2}-1)x^2 + 0.6356x - 0.04399 - x^{\sqrt{2}}$ for $x \in [0.25, \sqrt{2}/2]$. In order to show $g \geq 0$, observe that $g(0.25) \geq 0$ and $g(\sqrt{2}/2) \geq 0$. Note that $g'(x) = 2(\sqrt{2}-1)x + 0.6356 - \sqrt{2}\,x^{\sqrt{2}-1}$ is convex, $g'(0.25) > 0$ and $g'(\sqrt{2}/2) < 0$. Hence, there is an $x_0 \in (0.25, \sqrt{2}/2)$ such that $g' \geq 0$ on $[0.25, x_0]$ and $g' \leq 0$ on $[x_0, \sqrt{2}/2]$ and the claim follows. ∎

Using this lemma to bound $|\cos(t)|^{\sqrt{2}}$ from above by a polynomial of the form $a\cos(t)^2 + b|\cos(t)| + c$ we get the following estimate:

$$\int_{\pi/4}^{3\pi/4} \frac{|\cos(t)|^{\sqrt{2}}}{t^3}\,dt \leq 0.259.$$

Now, we want to bound $\int_{3\pi/4}^{\infty}(|\cos(t)|^{\sqrt{2}}/t^3)\,dt$ from above. For this, we consider the measure space $X = (3\pi/4, \infty)$ endowed with the measure $d\mu := \frac{dt}{t^3}$. Since X has finite measure, we can apply the following well-known Hölder-type inequality:

$$\|\cos(t)\|_{L_{\sqrt{2}}(\mu)} \leq \mu(X)^{1/\sqrt{2}-1/2}\|\cos(t)\|_{L_2(\mu)}.$$

Using the primitive of $\cos(t)^2/t^3$ (see 5.4), we can compute the right-hand side. From this, we get $\int_{3\pi/4}^{\infty}|\cos(t)|^{\sqrt{2}}/t^3 \leq 0.067$. All in all, we have

$$\int_0^{\infty} \frac{\exp(-t^2/\sqrt{2}) - |\cos(t)|^{\sqrt{2}}}{t^3}\,dt \geq 0.031 + 0.295 - 0.259 - 0.067 = 0,$$

which is the last inequality to verify.

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REFERENCES

[FHJSZ] T. Figiel, P. Hitczenko, W. B. Johnson, G. Schechtman and J. Zinn, *Extremal properties of Rademacher functions with applications to the Khintchine and Rosenthal inequalities*, Trans. Amer. Math. Soc. 349 (1997), 997–1027.

[GR] I. S. Gradshtein and I. M. Ryzhik, *Table of Integrals, Series and Products*, 5th ed., Academic Press, 1994.

[H] U. Haagerup, *The best constants in the Khintchine inequality*, Studia Math. 70 (1982), 231–283.

[K] H. König, *On the best constants in the Khintchine inequality for Steinhaus variables*, Israel J. Math. 203 (2014), 23–57.

[NP] F. L. Nazarov and A. N. Podkorytov, *Ball, Haagerup, and distribution functions*, in: Complex Analysis, Operators and Related Topics, V. P. Havin and N. K. Nikolski (eds.), Birkhäuser, 2000, 247–267.

[S] S. B. Stechkin, *On best lacunary systems of functions*, Izv. Akad. Nauk SSSR Ser. Mat. 25 (1961), 357–366 (in Russian).

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