COLLOQUIUM MATHEMATICUM

VOL. 147 2017 NO. 2

A MULTIVARIATE REMEZ-TYPE INEQUALITY WITH φ-CONCAVE WEIGHTS

BY

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Abstract. Let $\varphi : [0,\infty) \to [0,\infty)$ be an increasing twice continuously differentiable function with a positive power index $\beta(\varphi) := \inf_{t>0}(\varphi(t)/\varphi'(t))'$ and let $f : V \to [0,\infty)$ be concave on a convex body $V \subset \mathbb{R}^m$. In this paper we discuss the following Remez-type inequality for multivariate polynomials P of degree n on measurable sets $E \subseteq V$ equipped with a φ-concave measure $\mu(E) := \int_E \varphi(f(x))\,dx$:

$$\|P\|_{C(V)} \le T_n\left(\frac{2}{1-(1-\mu(E)/\mu(V))^{\beta(\varphi)/(1+m\beta(\varphi))}} - 1\right)\|P\|_{C(E)},$$

where T_n is the Chebyshev polynomial of degree n. In addition, we describe the classes of all extremal measures μ, bodies V, sets E, and polynomials P for this inequality.

1. Introduction. Let $|E|_m$ be the Lebesgue measure of a measurable set E from the m-dimensional Euclidean space $\mathbb{R}^m$, and let $\mathcal{P}_{n,m}$ be the set of all polynomials of m variables with real coefficients of degree at most n. Next, let $C(E)$ be the normed space of all bounded continuous functions f on a bounded set $E \subset \mathbb{R}^m$ with $\|f\|_{C(E)} := \sup_{x\in E}|f(x)|$. We also use the generic notation $\mathbb{R} := \mathbb{R}^1$.

The intensive study of polynomial inequalities on measurable sets has been initiated by the following result of Remez [28] (see also the books [13, Lemma III.7.3], [25, Theorem 5.1.6.1], [6, Theorem 5.1.1], [22, Theorem 2.7.1]):

THEOREM 1.1.

(a) *For a measurable set $E \subseteq [a,b]$ with $|E|_1 > 0$ and a polynomial $P \in \mathcal{P}_{n,1}$,*

$$\|P\|_{C([a,b])} \le T_n\big(2(b-a)/|E|_1 - 1\big)\|P\|_{C(E)}. \tag{1.1}$$

2010 *Mathematics Subject Classification*: Primary 41A17; Secondary 26D05.

Key words and phrases: multivariate polynomials, Remez-type inequality, k-concave measure, φ-concave weight.

Received 5 February 2016; revised 30 June 2016.

Published online 9 January 2017.

DOI: 10.4064/cm6884-7-2016

(b) *In the case of closed sets* $E \subset [a,b]$ *with* $0 < |E|_1 < b-a$ *and polynomials* $P \not\equiv 0$*, equality holds in* (1.1) *if and only if either*

(i) $E = [b-\lambda, b]$ *and* $P(x) = AT_n(2(b-x)/\lambda - 1)$*, or*

(ii) $E = [a, a+\lambda]$ *and* $P(x) = AT_n(2(x-a)/\lambda - 1)$*,*

where $A \in \mathbb{R} \setminus \{0\}$ *and* $\lambda \in (0, b-a)$*.*

Here and in the whole paper,

$$T_n(z) = \left((z+\sqrt{z^2-1})^n + (z-\sqrt{z^2-1})^n\right)/2$$

is the Chebyshev polynomial of degree n.

Numerous versions and generalizations of Remez's inequality (1.1) and their applications in analysis have been discussed since the 1970s (see for example [9, 10, 6, 21, 15, 16, 20, 7, 1, 11, 30, 27, 8] and references therein). In particular, the following multivariate version of Theorem 1.1 holds:

THEOREM 1.2.

(a) *For a convex body* $V \subset \mathbb{R}^m$*, a measurable set* $E \subseteq V$*,* $|E|_m > 0$*, and a polynomial* $P \in \mathcal{P}_{n,m}$*,*

$$\|P\|_{C(V)} \le T_n\left(\frac{2}{1-(1-|E|_m/|V|_m)^{1/m}} - 1\right)\|P\|_{C(E)}. \tag{1.2}$$

(b) *Equality holds in* (1.2) *if and only if* V *is a bounded convex cone in* $\mathbb{R}^m$*.*

Inequality (1.2) was proved by Brudnyi and the author [9, Theorem 2 and Remark 2], while the description (b) of all extremal convex bodies V in (1.2) was obtained in [10, Theorem 2]. As a corollary of Theorem 1.2, we proved a version of inequality (1.2) for sets equipped with a weighted measure and described the classes of all extremal bodies V, sets E, and polynomials P (see [15, Theorem 2.1]). Note that Theorem 1.1 is a special case of Theorem 1.2 for $m = 1$.

However, it is possible to find more precise estimates for special weights. In this paper we discuss a Remez-type inequality for multivariate polynomials on measurable sets in $\mathbb{R}^m$ equipped with φ-concave weighted measures.

Throughout the paper, we assume that $\varphi : [0,\infty) \to [0,\infty)$ is a continuous strictly increasing function. In addition, we assume that V is a closed convex body in $\mathbb{R}^m$ with a boundary ∂V.

DEFINITION 1.3. We say that $W : V \to [0,\infty)$ is a *φ-concave weight* if there exists a continuous concave function $f : V \to [0,\infty)$ such that $W(x) = W_{\varphi(f)}(x) = \varphi(f(x))$. The φ-concave (weighted) measure $\mu_{\varphi(f)}$ on V is defined on all Lebesgue measurable sets $E \subseteq V$ by $\mu_{\varphi(f)}(E) := \int_E \varphi(f(x))\,dx$. In the case of $\varphi(t) = t^{1/k}$, $k \in (0,\infty]$, the corresponding weight W is called

(k)-*concave* and the corresponding measure $\mu = \mu_{f^{1/k}}$ is called (k_m)-*concave*, where $k_m := 1/(1/k + m)$.

In a more general situation, (k_m)-concave measures on $\mathbb{R}^m$ with $\mu(\mathbb{R}^m) < \infty$ have been studied in a number of publications (see [3, 4, 5, 12, 17] and references therein). In particular, Borell [5] proved that a measure μ on $\mathbb{R}^m$ with $\mu(\mathbb{R}^m) < \infty$ and with convex compact support containing an open set in $\mathbb{R}^m$ satisfies the concavity condition

$$(1.3) \qquad \mu(\alpha A + (1-\alpha)B) \geq \left(\alpha\mu^{k_m}(A) + (1-\alpha)\mu^{k_m}(B)\right)^{1/k_m}$$

for any compact subsets $A \subset \mathbb{R}^m$, $B \subset \mathbb{R}^m$ and all $\alpha \in [0,1]$ if and only if μ is (k_m)-concave, $k \in (0, \infty]$.

Note that a (0)-concave measure is called *log-concave*, and it coincides with a φ-concave measure for the function $\varphi(t) = e^t$ with the extended domain $(-\infty, \infty)$ [4, 12]. In addition, the Lebesgue measure on $\mathbb{R}^m$ is a $(1/m)$-concave measure ($k = \infty$), and inequality (1.3) for $k = \infty$ coincides with the Brunn–Minkowski inequality

$$(1.4) \qquad |\alpha A + (1-\alpha)B|_m^{1/m} \geq \alpha |A|_m^{1/m} + (1-\alpha)|B|_m^{1/m}$$

in dimension m [18, (4.82)]. In this paper we assume that φ is a strictly increasing function, so the Lebesgue measure given by $\varphi(x) = 1$ is not a φ-concave measure. That is why we will treat the Lebesgue measure on V as the limit of (k_m)-concave measures as $k \to \infty$.

Fradelizi [12, Corollary 5] proved the following Remez-type inequality for (k_m)-concave measures, $k \in (0, \infty]$:

THEOREM 1.4. *For a convex body $V \subset \mathbb{R}^m$, a measurable set $E \subseteq V$, $|E|_m > 0$, a polynomial $P \in \mathcal{P}_{n,m}$, and a (k_m)-concave measure $\mu_{f^{1/k}}$ on V, $k \in (0, \infty]$,*

$$(1.5) \quad \|P\|_{C(V)} \leq T_n\left(\frac{2}{1 - (1 - \mu_{f^{1/k}}(E)/\mu_{f^{1/k}}(V))^{1/(1/k+m)}} - 1\right)\|P\|_{C(E)}.$$

In particular, inequality (1.2) is a special case of (1.5) for $k = \infty$. Inequality (1.5) is an easy corollary of a general distribution inequality with (k_m)-concave measures for multivariate polynomials on V (see [12, Corollary 4] for $k \in (0, \infty]$ and [26, p. 223] for log-concave measures). The proof of the latter inequality is based on the sophisticated geometric technique developed independently by Bobkov and Nazarov [4, Theorem 1.1] and Fradelizi [12, Theorem 1] for (k_m)-concave measures, and by Nazarov, Sodin, and Volberg [26, pp. 215, 220] for log-concave measures (see also [3, 4, 12] for further references).

In this paper we generalize inequality (1.5) to φ-concave measures and describe all extremal bodies V, sets E, polynomials P, and weights $\varphi(f)$ in the corresponding equality. The main results are presented in Section 2. In the proof of the Remez-type inequality we do not use the sophisticated geometric technique applied in [12] to prove (1.5). Instead, our proofs are based on simple extremal problems for concave functions discussed in Section 3 and on a geometric lemma from Section 4. Moreover, our approach allows a description of extremal elements in the Remez-type inequality. The proofs of main results are given in Section 4. In Section 5 we discuss several examples that illustrate the Remez-type inequality.

NOTATION. Throughout the paper, $\mathcal{C}(V)$ is the set of all continuous concave functions $f(x) \geq 0$ on a convex body $V \subset \mathbb{R}^m$, and $\mathcal{L}([a,b])$ is the set of all linear functions of the form $f(x) = \delta(x-a)$, $\delta > 0$, on $[a,b]$. In addition, $|E|_k$ denotes the k-dimensional Lebesgue measure of a measurable set $E \subset \mathbb{R}^m$. Finally, $x \cdot y := \sum_{i=1}^m x_i y_i$ is the dot product of vectors $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ and $y = (y_1, \dots, y_m) \in \mathbb{R}^m$, and $\rho(x, A)$ denotes the distance from $x \in \mathbb{R}^m$ to a set $A \subset \mathbb{R}^m$.

2. Remez-type inequality. Here, we discuss the Remez-type inequality for multivariate polynomials on measurable sets equipped with φ-concave weights. We recall that $\varphi : [0,\infty) \to [0,\infty)$ is a continuous strictly increasing function. We first need the following definitions (see also [17]).

DEFINITION 2.1. For a twice continuously differentiable function φ on $(0,\infty)$, the *power index* of φ is defined by the formula

$$\beta(\varphi) := \inf_{t>0} (\varphi(t)/\varphi'(t))'. \tag{2.1}$$

This definition was introduced by the author in [17, Sect. 4]. Note that $\beta(\varphi) = k$ for $\varphi(t) = t^{1/k}$, $k > 0$. Other examples of finding $\beta(\varphi)$ are discussed in Section 5.

DEFINITION 2.2. The convex hull of a vertex $x_0 \in \mathbb{R}^m$ and a closed $(m-1)$-dimensional convex body B_{m-1} (the base) is called the *bounded convex cone* (BCC) if the base is in an $(m-1)$-dimensional hyperplane of $\mathbb{R}^m$ that does not contain x_0. The set of all BCCs in $\mathbb{R}^m$ with a vertex x_0 is denoted by $\mathcal{K}(x_0)$.

In the following theorem a closed layer between two parallel hyperplanes in $\mathbb{R}^m$ is denoted by $\mathcal{L}$.

THEOREM 2.3. *Let φ be twice continuously differentiable on $(0,\infty)$ with $\beta(\varphi) > 0$. Then:*

(a) *For a convex body $V \subset \mathbb{R}^m$, a measurable set $E \subseteq V$ with $|E|_m > 0$, a polynomial $P \in \mathcal{P}_{n,m}$, and a function $f \in \mathcal{C}(V)$,*

$$\|P\|_{C(V)} \leq T_n\left(\frac{2}{1-(1-\mu_{\varphi(f)}(E)/\mu_{\varphi(f)}(V))^{\beta(\varphi)/(1+m\beta(\varphi))}} - 1\right)\|P\|_{C(E)}. \tag{2.2}$$

(b) *In the case of closed sets* $E \subset V$ *with* $0 < |E|_m < |V|_m$ *and polynomials* $P \not\equiv 0$*, equality holds in* (2.2) *if and only if* $V \in \mathcal{K}(x_0)$*,* $x_0 \in \mathbb{R}^m$*;* $\varphi(t) = Ct^{1/k}$ *for some* $k > 0$ *and* $C > 0$*;* $f(x) \geq 0$ *is a concave continuous function on* V *such that* $f(x + x_0)$ *is a* 1*-homogeneous function on* $V - \{x_0\}$*;* $E = V \cap \mathcal{L}$ *is the layer in* V *with an open subset if one of the boundary hyperplanes of* $\mathcal{L}$ *coincides with* H_{m-1}*; and* $P(x) = AT_n(2\rho(x, H_{m-1})/d - 1)$ *for* $x \in V$ *and some* $A \in \mathbb{R} \setminus \{0\}$*. Here, the base of* V *is located in a hyperplane* $H_{m-1} \subset \mathbb{R}^m$*,* $x_0 \notin H_{m-1}$*, and* d *is the width of the layer* E*.*

Remark 2.4. Inequality (2.2) immediately follows from Theorem 1.4 and properties of functions φ with $\beta(\varphi) > 0$ obtained by the author [17, Theorem 4.2]. In addition, the extremal elements from Theorem 2.3(b) in a more general setting are presented in [17, Theorem 2.1]. So the main purpose of the present paper is twofold. First, we give new and simpler proofs of Theorems 1.4 and 2.3(a) based on extremal properties of concave functions and on a geometric lemma. Second, we use some ingredients of this proof to show the uniqueness of the extremal elements from Theorem 2.3(b). Note that this fact cannot be proved on the basis of the existing proof of Theorem 1.4.

Remark 2.5. Theorem 1.4 is a special case of Theorem 2.3(a) for $\varphi(t) = t^{1/k}$, $k \in (0, \infty)$, while letting $k \to \infty$, we arrive at Theorem 1.2(a) from (2.2). It is also possible to derive Theorem 1.2(b) from Theorem 2.3(b) by setting $\varphi(t) = t^{1/k}$, $k > 0$, and letting $k \to \infty$. Note also that for $m = 1$, $\mathcal{K}(x_0)$ is the set of all intervals $[x_0, b]$ or $[a, x_0]$ and the function f from Theorem 2.3(b) is $\delta(x - x_0)$ or $\delta(x_0 - x)$, $\delta > 0$. In addition, note that some properties and examples of concave 1-homogeneous functions f are discussed in Propositions 5.7, 5.8 and Examples 5.9–5.11.

If φ does not coincide with $Ct^{1/k}$ for some $k, C > 0$, then inequality (2.2) is strict by Theorem 2.3(b). However, the following result shows that under certain conditions on φ inequality (2.2) cannot be improved for all $f \in \mathcal{C}(V)$.

Theorem 2.6. *Let* φ *be twice continuously differentiable on* $(0, \infty)$ *with* $\beta(\varphi) > 0$*. Suppose* φ *does not coincide with* $Ct^{1/k}$ *for any* $k, C > 0$*, and satisfies the condition*

$$\min\Big\{\lim_{y \to 0^+} \varphi(cy)/\varphi(y), \lim_{y \to \infty} \varphi(cy)/\varphi(y)\Big\} = c^{1/\beta(\varphi)} \tag{2.3}$$

for any $c \in (0, 1]$*, provided that the limits in* (2.3) *exist. Let a cone* V *belong to* $\mathcal{K}(x_0)$ *with the base in the hyperplane* $H_{m-1} := \{x \in \mathbb{R}^m : \alpha \cdot (x - x_0) = h\}$*, where* $\alpha \in \mathbb{R}^m \setminus \{0\}$*,* $x_0 \in \mathbb{R}^m \setminus H_{m-1}$*, and* $h = \rho(x_0, H_{m-1})$*. Next, let the layer* $E \subset V$*,* $0 < |E|_m < |V|_m$*, and the polynomial* $P(x) = AT_n(2\rho(x, H_{m-1})/d - 1)$*,* $x \in V$*,* $A \in \mathbb{R} \setminus \{0\}$*, be as in Theorem* 2.3(b)*.*

Then for the linear functions $f_\delta(x) := (\delta\alpha)\cdot(x-x_0)$ *on* V, $\delta>0$, *we have*

$$\|P\|_{C(V)} = \lim T_n\left(\frac{2}{1-(1-\mu_{\varphi(f_\delta)}(E)/\mu_{\varphi(f_\delta)}(V))^{\beta(\varphi)/(1+m\beta(\varphi))}}-1\right)\|P\|_{C(E)} \tag{2.4}$$

as $\delta\to 0^+$ *if*

$$\lim_{y\to 0^+}\varphi(cy)/\varphi(y) \le \lim_{y\to\infty}\varphi(cy)/\varphi(y) \quad \text{for } c = 1-d/h,$$

and as $\delta\to\infty$ *otherwise.*

Remark 2.7. Some functions φ with $\beta(\varphi)>0$, satisfying condition (2.3), are given in Examples 5.1–5.4.

3. Properties of univariate concave functions. The proof of the Remez-type inequality is based on properties of univariate concave functions. We recall that $\varphi:[0,\infty)\to[0,\infty)$ is a continuous strictly increasing function. The first property is a special case of a more general result obtained in [17, Theorem 4.2]. For the convenience of the reader we present a short and simple proof of the next proposition.

Proposition 3.1. *Let* $\gamma\ge 0$ *and let* φ *be a twice continuously differentiable function on* $(0,\infty)$. *If* $\beta(\varphi)>0$, *then the weight* $\varphi(f(x))(x-a)^\gamma$ *is* (k)*-concave on* $[a,b]$ *for every* $f\in\mathcal{C}([a,b])$, *where* $k=\beta(\varphi)/(1+\gamma\beta(\varphi))$.

Proof. We first show that $H(t,v) := [\varphi(t)v^\gamma]^k$ is a concave function of two variables on $[0,\infty)\times[0,\infty)$. It is well known that $H(t,v)$ is concave on $[0,\infty)\times[0,\infty)$ if and only if

$$\frac{\partial^2 H(t,v)}{\partial t^2}\le 0, \quad \det\Delta(t,v)\ge 0 \quad \text{for } t\in(0,\infty),\ v\in(0,\infty), \tag{3.1}$$

where $\Delta(t,v)$ is the Hessian (see [29, Theorem 4.5] and [14, Theorem 10.4.6]). By a straightforward calculation, we obtain

$$\frac{\partial^2 H(t,v)}{\partial t^2} = kv^{\gamma k}\varphi^{k-2}(t)\varphi'^2(t)[k-(\varphi(t)/\varphi'(t))'], \tag{3.2}$$

$$\frac{\partial^2 H(t,v)}{\partial v^2} = k\gamma(k\gamma-1)v^{\gamma k-2}\varphi''(t), \tag{3.3}$$

$$\frac{\partial^2 H(t,v)}{\partial t\partial v} = k^2\gamma v^{\gamma k-1}\varphi^{k-1}(t)\varphi'(t),$$

$$\det\Delta(t,v) = \frac{\partial^2 H(t,v)}{\partial t^2}\,\frac{\partial^2 H(t,v)}{\partial v^2} - \left(\frac{\partial^2 H(t,v)}{\partial t\partial v}\right)^2$$
$$= k^2\gamma v^{2\gamma k-2}\varphi^{2k-2}(t)\varphi'^2(t)[(\varphi(t)/\varphi'(t))' - k(1+\gamma(\varphi(t)/\varphi'(t))')].$$

Then it follows from (2.1) and (3.2) that $\partial^2 H(t,v)/\partial t^2 \le 0$. Next, using (3.3), we see that the inequality $\det \Delta(t,v) \ge 0$ follows from the relations

$$k = \frac{\beta(\varphi)}{1+\gamma\beta(\varphi)} = \inf_{t\in[0,\infty)} \frac{(\varphi(t)/\varphi'(t))'}{1+\gamma(\varphi(t)/\varphi'(t))'} \le \frac{(\varphi(t)/\varphi'(t))'}{1+\gamma(\varphi(t)/\varphi'(t))'}.$$

Hence $H(t,v)$ is concave on $[0,\infty)\times[0,\infty)$, and in addition $H(f(x), x-a)$ is concave on $[a,b]$ for any $a \ge 0$ and $f \in \mathcal{C}([a,b])$ (see [24, Sect. 16.B.7]). ■

Several extremal problems for concave functions are discussed below.

Proposition 3.2.

(a) *For a fixed* $c \in (a,b)$,

$$(3.4)\qquad A(a,b,c,\varphi) := \sup_{f\in\mathcal{C}([a,b])} \frac{\int_c^b \varphi(f(x))\,dx}{\int_a^b \varphi(f(x))\,dx} = \sup_{f\in\mathcal{L}([a,b])} \frac{\int_c^b \varphi(f(x))\,dx}{\int_a^b \varphi(f(x))\,dx} = \sup_{\delta>0} \frac{\int_{\delta(c-a)}^{\delta(b-a)} \varphi(t)\,dt}{\int_0^{\delta(b-a)} \varphi(t)\,dt}.$$

(b) *If there exists an extremal function* $f \in \mathcal{C}([a,b])$ *in* (3.4), *then* $f \in \mathcal{L}([a,b])$.

Proof. The inequality

$$(3.5)\qquad \sup_{f\in\mathcal{C}([a,b])} \frac{\int_c^b \varphi(f(x))\,dx}{\int_a^b \varphi(f(x))\,dx} \ge \sup_{f\in\mathcal{L}([a,b])} \frac{\int_c^b \varphi(f(x))\,dx}{\int_a^b \varphi(f(x))\,dx}$$

is trivial. Next, let $f \in \mathcal{C}([a,b]) \setminus \mathcal{L}([a,b])$ and set $\delta_c := f(c)/(c-a)$. Note that since $f(y) \ge 0$, $f(y)/(y-a)$ is nonincreasing on $[a,b]$. Therefore, the linear function $L(x) := \delta_c(x-a) \in \mathcal{L}([a,b])$ satisfies the relations

$$(3.6)\qquad L(x) \le f(x),\ x\in[a,c];\quad L(c) = f(c);\quad L(x) \ge f(x),\ x \in [c,b].$$

Then the inequality

$$(3.7)\qquad \frac{\int_c^b \varphi(f(x))\,dx}{\int_a^b \varphi(f(x))\,dx} < \frac{\int_c^b \varphi(L(x))\,dx}{\int_a^b \varphi(L(x))\,dx}$$

follows from (3.6) and from the fact that $f(x) \ne L(x)$ on a set of positive measure from $(a,b]$. Thus (3.7) implies the inequality

$$\sup_{f\in\mathcal{C}([a,b])} \frac{\int_c^b \varphi(f(x))\,dx}{\int_a^b \varphi(f(x))\,dx} \le \sup_{f\in\mathcal{L}([a,b])} \frac{\int_c^b \varphi(f(x))\,dx}{\int_a^b \varphi(f(x))\,dx},$$

which combined with (3.5) proves (a).

In addition, (3.5) shows that if there exists $f \in \mathcal{C}([a,b])$ such that

$$\frac{\int_c^b \varphi(f(x))\,dx}{\int_a^b \varphi(f(x))\,dx} = \sup_{f\in\mathcal{L}([a,b])} \frac{\int_c^b \varphi(f(x))\,dx}{\int_a^b \varphi(f(x))\,dx},$$

then $f \in \mathcal{L}([a,b])$. Thus (b) is established. ■

The following is a corollary of Proposition 3.2 for $\varphi(t) = t^{1/k}$, $k > 0$.

PROPOSITION 3.3.

(a) *For a fixed* $c \in (a,b)$ *and any* $k > 0$,

$$\sup_{f\in\mathcal{C}([a,b])} \frac{\int_c^b f^{1/k}(x)\,dx}{\int_a^b f^{1/k}(x)\,dx} = 1 - \left(\frac{c-a}{b-a}\right)^{1/k+1}. \tag{3.8}$$

(b) *The set of all extremal functions* $f \in \mathcal{C}([a,b])$ *in* (3.8) *is* $\mathcal{L}([a,b])$.

Since finding $A(a,b,c,\varphi)$ in (3.4) is a challenging task for many functions φ, we present below a "power" estimate of a more general expression $A(a,b,c,\gamma,\varphi)$ for functions φ with positive power indices.

PROPOSITION 3.4. *Let* φ *be twice continuously differentiable on* $(0,\infty)$ *with* $\beta(\varphi) > 0$ *and let* $c \in (a,b)$ *be a fixed number. Then for any* $\gamma \geq 0$*:*

(a) *We have*

$$A(a,b,c,\gamma,\varphi) := \sup_{f\in\mathcal{C}([a,b])} \frac{\int_c^b \varphi(f(x))(x-a)^\gamma\,dx}{\int_a^b \varphi(f(x))x-a)^\gamma\,dx} \leq 1 - \left(\frac{c-a}{b-a}\right)^{1/\beta(\varphi)+\gamma+1}. \tag{3.9}$$

(b) *The equality*

$$\frac{\int_c^b \varphi(f(x))(x-a)^\gamma\,dx}{\int_a^b \varphi(f(x))x-a)^\gamma\,dx} = 1 - \left(\frac{c-a}{b-a}\right)^{1/\beta(\varphi)+\gamma+1} \tag{3.10}$$

holds for some $f \in \mathcal{C}([a,b])$ *if and only if* $\varphi(t) = Ct^{1/k}$ *for some* $k, C > 0$ *and* $f \in \mathcal{L}([a,b])$.

Proof. (a) Using Proposition 3.1 for $\gamma \geq 0$, we deduce that for $k = \beta(\varphi)/(1+\gamma\beta(\varphi))$ and for any $f \in \mathcal{C}([a,b])$, the function $\varphi(f(x))(x-a)^\gamma$ is (k)-concave, that is, $\varphi(f(x))(x-a)^\gamma = F^{1/k}(x)$, where $F \in \mathcal{C}([a,b])$. Then by Proposition 3.3(a),

$$A(a,b,c,\gamma,\varphi) \leq \sup_{F\in\mathcal{C}([a,b])} \frac{\int_c^b F^{1/k}(x)\,dx}{\int_a^b F^{1/k}(x)\,dx} = 1 - \left(\frac{c-a}{b-a}\right)^{1/\beta(\varphi)+\gamma+1}. \tag{3.11}$$

Hence (3.9) follows.

(b) If (3.10) holds, then by Proposition 3.1, the function

$$F(x) = \big(\varphi(f(x))(x-a)^{\gamma}\big)^{\beta(\varphi)/(1+\beta(\varphi)\gamma)}$$

is concave on $[a,b]$ and

$$\frac{\int_c^b F^{1/\beta(\varphi)+\gamma}(x)\,dx}{\int_a^b F^{1/\beta(\varphi)+\gamma}(x)\,dx} = 1-\left(\frac{c-a}{b-a}\right)^{1/\beta(\varphi)+\gamma+1}. \tag{3.12}$$

Next by Proposition 3.3(b), equality (3.12) holds if and only if $F(x) = \delta_1(x-a)$, $\delta_1 > 0$. Hence there exists $f \in \mathcal{C}[a,b]$ such that $\varphi(f(x)) = \delta_2(x-a)^{1/\beta(\varphi)}$, $\delta_2 > 0$. Since $\varphi^{\beta(\varphi)}$ is concave by Proposition 3.1 for $\gamma = 0$, the inverse $(\varphi^{\beta(\varphi)})^{-1}$ is convex. So the function $f(x) = (\varphi^{\beta(\varphi)})^{-1}(\delta_2(x-a))$ is convex and concave on $[a,b]$, which is possible if and only if $f(x) = \delta(x-a)$, $\delta > 0$. Thus $\varphi(t) = Ct^k$ for some $k, C > 0$. ∎

Remark 3.5. If

$$\alpha(a,b,c,\varphi) := \min\left\{\lim_{y\to 0^+}\frac{\varphi((c-a)y)}{\varphi((b-a)y)}, \lim_{y\to\infty}\frac{\varphi((c-a)y)}{\varphi((b-a)y)}\right\} = \left(\frac{c-a}{b-a}\right)^{1/\beta(\varphi)}, \tag{3.13}$$

provided that the limits in (3.13) exist, then

$$A(a,b,c,\gamma,\varphi) = 1-\left(\frac{c-a}{b-a}\right)^{1/\beta(\varphi)+\gamma+1}. \tag{3.14}$$

Indeed, by L'Hospital's Rule,

$$A(a,b,c,\gamma,\varphi) \ge 1-\left(\frac{c-a}{b-a}\right)^{\gamma+1}\alpha(a,b,c,\gamma,\varphi). \tag{3.15}$$

Thus (3.14) follows from (3.9), (3.13), and (3.15). Note that conditions (3.13) and (2.3) are equivalent. Examples of functions φ with $\beta(\varphi) > 0$ satisfying condition (3.13) are given in Examples 5.1–5.4.

Finally, we discuss properties of increasing rearrangements of φ-concave functions on $[a,b]$.

Definition 3.6. The decreasing rearrangement $F^* : [0, b-a] \to [0,\infty)$ of a measurable function $F : [a,b] \to [0,\infty)$ is defined as the inverse $F^*(t) := \inf\{y \in [a,b] : D(F,y) < t\}$ of its distribution function $D(F,y) := |\{x \in [a,b] : F(x) > y\}|_1$. The increasing rearrangement of F is defined by $F_*(t) := F^*(b-a-t)$.

It is well known [19, Sect. 10.12] that for a measurable function $F(x) \ge 0$ and a measurable set $E \subseteq [a,b]$ with $|E|_1 > 0$, (3.16) $$\int_E F(x)\,dx \le \int_{b-a-|E|_1}^{b-a} F_*(t)\,dt,$$

and in addition

(3.17) $$\int_a^b F(x)\,dx = \int_0^{b-a} F_*(t)\,dt.$$

We also need the following property.

PROPOSITION 3.7. *If W is a continuous φ-concave weight on $[a,b]$, then W_* is a φ-concave weight on $[0,b-a]$ as well.*

Proof. Let us first show that W_* is concave on $[0,b-a]$ if $W\ge 0$ is concave on $[a,b]$. This result was proved in [4, Lemma 5.3] by using the Brunn–Minkowski inequality (1.4) for $m=1$. We present a different proof based on a simple geometric observation. Let $W\ge 0$ be a concave function on $[a,b]$. It is sufficient to show that $D(W,y)$ is concave on $[0,M]$, where $M:=\max_{x\in[a,b]}W(x)$, since this fact implies that W^* and W_* are concave on $[a,b]$. We first note that the set $\{x\in[a,b]: W(x)>y\}$ is an open interval, and in addition there exist numbers c and d such that $a\le c\le d\le b$ and W is strictly increasing on $[a,c]$, strictly decreasing on $[d,b]$, and $W(c)=W(d)=M$. Then the functions

$$g(y):=\begin{cases} b, & 0\le y<W(b),\\ W^{-1}(y), & W(b)\le y\le M,\end{cases}$$
$$h(y):=\begin{cases} a, & 0\le y<W(a),\\ W^{-1}(y), & W(a)\le y\le M,\end{cases}$$

satisfy the following properties: $h(y)\le g(y)$, g is concave on $[0,M]$, h is convex on $[0,M]$, and $D(W,y)=g(y)-h(y)$. Therefore, $D(W,y)$ is concave on $[0,M]$. Since $(\varphi(W))_*(t)=\varphi(W_*(t))$, the proposition is established. ∎

4. Proof of Theorems 2.3 and 2.6. Let $V\subset\mathbb{R}^m$ be a convex body and let x_0 be a fixed point from V. We denote by $\Sigma_V(x_0)$ the set of all closed rays l with the initial point x_0 such that $|V\cap l|_1>0$. Let us equip $\Sigma_V(x_0)$ with the $(m-1)$-dimensional uniformly distributed spherical measure S. In the case of $m=1$, $\Sigma_V(x_0)$ consists of one or two rays of unit measure.

To prove the theorems, we also need a geometric lemma. Let $E\subseteq V$ be a measurable set with $|E|_m>0$. Then for a.e. $l\in\Sigma_V(x_0)$, $E\cap l$ is a linearly measurable set. Let us set

$$C(E,V,x_0):=\operatorname*{ess\,inf}_{l\in\Sigma_V(x_0)}|V\cap l|_1/|E\cap l|_1.$$

LEMMA 4.1. *Let φ be a twice continuously differentiable function on $(0,\infty)$ with $\beta(\varphi)>0$. Then:*

(a) *For every nonnegative continuous concave function* $f : V \to [0, \infty)$,

$$C(E, V, x_0) \le \big(1 - (1 - \mu_{\varphi(f)}(E)/\mu_{\varphi(f)}(V))^{\beta(\varphi)/(1+m\beta(\varphi))}\big)^{-1}. \tag{4.1}$$

(b) *Equality holds in* (4.1) *if* $\varphi(t) = Ct^{1/k}$ *for some* $k > 0$ *and* $C > 0$, $f(x + x_0) \ge 0$ *is a concave continuous* 1*-homogeneous function on* $V - \{x_0\}$, *and* $E = E_{k,\tau}(x_0)$, *where* $\tau \in (0, 1]$. *Here,* $E_{k,\tau}(x_0)$ *is the closed subset of* V *such that for all* $l \in \Sigma_V(x_0)$, $E_{k,\tau}(x_0) \cap l$ *is a closed interval of length* $(1 - (1 - \tau)^{1/(1/k+m)})|V \cap l|_1$ *and one of its ends coincides with* $\partial V \cap l$. *In addition,* $\mu_{f^{1/k}}(E)/\mu_{f^{1/k}}(V) = \tau$.

(c) *If equality holds in* (4.1) *for a closed subset* E *of* V, *then for a.e.* $l \in \Sigma_V(x_0)$,

$$|V \cap l|_1/|E \cap l|_1 = C(E, V, x_0). \tag{4.2}$$

If in addition, for a.e. $l \in \Sigma_V(x_0)$, *the set* $E \cap l$ *is a closed interval and one of its ends coincides with* $\partial V \cap l$, *then* φ *and* f *satisfy the conditions of statement* (b) *and* $E_{k,\tau} \subseteq E$. *Moreover,* $E \cap l = E_{k,\tau} \cap l$ *for a.e.* $l \in \Sigma_V(x_0)$ *and* $\tau = \mu_{\varphi(f)}(E)/\mu_{\varphi(f)}(V)$.

Proof. (a) Introducing the spherical coordinates in $\mathbb{R}^m$, $m \ge 1$, with the origin at x_0, we see that for $x \in V$, $f(x) = h(r, l)$, where l is the ray passing through x_0 and x, and $r = |x - x_0|$ is a coordinate on l. Note that if f is concave on V, then $h(r, l)$ is a concave function of $r \in (0, |V \cap l|_1]$ for each fixed ray $l \in \Sigma_V(x_0)$. Moreover, by Proposition 3.1, $\varphi(h(r, l))r^{m-1}$ is a (k)-concave weight of variable $r \in (0, |V \cap l|_1]$ for a fixed $l \in \Sigma_V(x_0)$, where $k = \beta(\varphi)/(1 + (m - 1)\beta(\varphi))$. Therefore by Proposition 3.7, the increasing rearrangement $(\varphi(h(\cdot, l))(\cdot)^{m-1})_*(t)$ is (k)-concave on $(0, |V \cap l|_1]$. Then by (3.16) and (3.17) we have

$$\begin{aligned}
\frac{\mu_{\varphi(f)}(E)}{\mu_{\varphi(f)}(V)} &= \frac{\int_{l\in\Sigma_V(x_0)}(\int_{E\cap l} \varphi(h(r, l))r^{m-1}\, dr)\, dS}{\int_{l\in\Sigma_V(x_0)}(\int_{V\cap l} \varphi(h(r, l))r^{m-1}\, dr)\, dS} \\
&\le \operatorname*{ess\,sup}_{l\in\Sigma_V(x_0)} \frac{\int_{E\cap l} \varphi(h(r, l))r^{m-1}\, dr}{\int_{V\cap l} \varphi(h(r, l))r^{m-1}\, dr} \\
&\le \operatorname*{ess\,sup}_{l\in\Sigma_V(x_0)} \frac{\int_{|V\cap l|_1-|E\cap l|_1}^{|V\cap l|_1} \big(\varphi(h((\cdot), l))(\cdot)^{m-1}\big)_*(t)\, dt}{\int_0^{|V\cap l|_1} \big(\varphi(h((\cdot), l))(\cdot)^{m-1}\big)_*(t)\, dt} \\
&\le \operatorname*{ess\,sup}_{l\in\Sigma_V(x_0)} \sup_{F\in\mathcal{C}([0,|V\cap l|_1])} \frac{\int_{|V\cap l|_1-|E\cap l|_1}^{|V\cap l|_1} F^{1/k}(t)\, dt}{\int_0^{|V\cap l|_1} F^{1/k}(t)\, dt}.
\end{aligned} \tag{4.3}$$

Applying Proposition 3.3(a) to the right-hand side of (4.3), we obtain

$$\text{(4.4)}\qquad \frac{\mu_{\varphi(f)}(E)}{\mu_{\varphi(f)}(V)} \le \operatorname*{ess\,sup}_{l\in\Sigma_V(x_0)}\left(1-\left(1-\frac{|E\cap l|_1}{|V\cap l|_1}\right)^{1/\beta(\varphi)+m}\right) = 1-(1-1/C(E,V,x_0))^{1/\beta(\varphi)+m}.$$

Thus (4.1) follows from (4.4).

(b) Let φ, f, and $E = E_{k,\tau}(x_0)$ satisfy the conditions of statement (b). Then $\beta(\varphi) = k$ and $C(E_{k,\tau}(x_0), V, x_0) = (1-(1-\tau)^{1/(1/k+m)})^{-1}$. Taking into account that f is continuous and 1-homogeneous with respect to x_0, we see that $f(x) = h(r,l) = \theta(l)r$ for some continuous function $\theta(l)$ on $\Sigma_V(x_0)$. Then by the construction of $E_{k,\tau}(x_0)$,

$$\frac{\mu_{\varphi(f)}(E_{k,\tau}(x_0))}{\mu_{\varphi(f)}(V)} = \frac{\int_{E_{k,\tau}(x_0)} f^{1/k}(x)\,dx}{\int_V f^{1/k}(x)\,dx} = \frac{\int_{l\in\Sigma_V(x_0)} \theta^{1/k}(l)\big(\int_{|V\cap l|_1-|E_{k,\tau}(x_0)\cap l|_1}^{|V\cap l|_1} r^{1/k+m-1}\,dr\big)\,dS}{\int_{l\in\Sigma_V(x_0)} \theta^{1/k}(l)\big(\int_0^{|V\cap l|_1} r^{1/k+m-1}\,dr\big)\,dS} = \tau.$$

Therefore, equality holds in (4.1).

(c) If equality holds in (4.1), then equalities hold in (4.3) and (4.4). Therefore for a.e. $l\in\Sigma_V(x_0)$,

$$\frac{\mu_{\varphi(f)}(E)}{\mu_{\varphi(f)}(V)} = \frac{\int_{l\in\Sigma_V(x_0)}\big(\int_{E\cap l}\varphi(h(r,l))r^{m-1}\,dr\big)\,dS}{\int_{l\in\Sigma_V(x_0)}\big(\int_{V\cap l}\varphi(h(r,l))r^{m-1}\,dr\big)\,dS} = \operatorname*{ess\,sup}_{l\in\Sigma_V(x_0)} \frac{\int_{E\cap l}\varphi(h(r,l))r^{m-1}\,dr}{\int_{V\cap l}\varphi(h(r,l))r^{m-1}\,dr}.$$

This is possible only if for a.e. $l\in\Sigma_V(x_0)$,

$$\text{(4.5)}\quad 1-(1-1/C(E,V,x_0))^{1/\beta(\varphi)+m} = \frac{\mu_{\varphi(f)}(E)}{\mu_{\varphi(f)}(V)} = \frac{\int_{E\cap l}\varphi(h(r,l))r^{m-1}\,dr}{\int_{V\cap l}\varphi(h(r,l))r^{m-1}\,dr}.$$

Next similarly to (4.3) and (4.4), we obtain

$$\text{(4.6)}\qquad \frac{\int_{E\cap l}\varphi(h(r,l))r^{m-1}\,dr}{\int_{V\cap l}\varphi(h(r,l))r^{m-1}\,dr} \le \sup_{F\in\mathcal{C}([0,|V\cap l|_1])} \frac{\int_{|V\cap l|_1-|E\cap l|_1}^{|V\cap l|_1} F^{1/\beta(\varphi)+m-1}(t)\,dt}{\int_0^{|V\cap l|_1} F^{1/\beta(\varphi)+m-1}(t)\,dt} = 1-\left(1-\frac{|E\cap l|_1}{|V\cap l|_1}\right)^{1/\beta(\varphi)+m}.$$

Comparing (4.5) and (4.6), for a.e. $l\in\Sigma_V(x_0)$ we obtain

$$\text{(4.7)}\qquad \frac{\int_{E\cap l}\varphi(h(r,l))r^{m-1}\,dr}{\int_{V\cap l}\varphi(h(r,l))r^{m-1}\,dr} = 1-\left(1-\frac{|E\cap l|_1}{|V\cap l|_1}\right)^{1/\beta(\varphi)+m}.$$

Further, it follows from (4.5) and (4.7) that for a.e. $l \in \Sigma_V(x_0)$,

$$|E \cap l|_1/|V \cap l|_1 = 1/C(E, V, x_0). \tag{4.8}$$

Thus (4.2) holds. If E satisfies the conditions of (c), then by Proposition 3.4(b), equality (4.7) is possible only if $\varphi(t) = Ct^{1/k}$ and $\beta(\varphi) = k$ for some $k > 0$ and $C > 0$, and $h(r, l) = C(l)r$ for $r \in [0, |V \cap l|_1]$ and a.e. $l \in \Sigma_V(x_0)$.

In addition, note that f is a nonnegative continuous and concave function on V by the definition of the measure $\mu_{\varphi(f)}$. Then by continuity of f, $h(r, l) = C(l)r$ for all $l \in \Sigma_V(x_0)$, that is, $f(x + x_0)$ is a 1-homogeneous function on $V - x_0$. Thus φ and f satisfy (b).

Next, setting $\tau = \mu_{\varphi(f)}(E)/\mu_{\varphi(f)}(V)$, we can rewrite (4.8) as

$$|E \cap l|_1/|V \cap l|_1 = 1 - (1 - \tau)^{1/(1/k+m)}, \tag{4.9}$$

which holds for a.e. $l \in \Sigma_V(x_0)$. Taking account of the conditions on E, we see from (4.9) that $E \cap l = E_{k,\tau} \cap l$ for a.e. $l \in \Sigma_V(x_0)$. Since E is closed, we conclude that $E_{k,\tau} \subseteq E$. This completes the proof of Lemma 4.1. ∎

Proof of Theorem 2.3. (a) Let E be a measurable subset of V with $|E|_m > 0$. Next, let $P \in \mathcal{P}_{n,m}$ and $x_0 \in V$ be such that $\|P\|_{C(V)} = |P(x_0)|$. Since the restriction of P to any $l \in \Sigma_V(x_0)$ belongs to $\mathcal{P}_{n,1}$, we deduce from Theorem 1.1(a) that for all $l \in \Sigma_V(x_0)$ such that $E \cap l$ is a linearly measurable set with $|E \cap l|_1 > 0$,

$$\begin{aligned} \|P\|_{C(V)} = |P(x_0)| &\le T_n(2|V \cap l|_1/|E \cap l|_1 - 1)\|P\|_{C(E \cap l)} \\ &\le T_n(2|V \cap l|_1/|E \cap l|_1 - 1)\|P\|_{C(E)}. \end{aligned} \tag{4.10}$$

Since $E \cap l$ is linearly measurable for a.e. $l \in \Sigma_V(x_0)$, from (4.10) and Lemma 4.1(a) we obtain

$$\begin{aligned} \|P\|_{C(V)} = |P(x_0)| &\le T_n(2C(E, V, x_0) - 1)\|P\|_{C(E)} \\ &\le T_n\left(\frac{2}{1 - (1 - \mu_{\varphi(f)}(E)/\mu_{\varphi(f)}(V))^{\beta(\varphi)/(1+m\beta(\varphi))}} - 1\right)\|P\|_{C(E)}. \end{aligned} \tag{4.11}$$

Thus (2.2) is established.

(b) This consists of four steps.

STEP 1. We first prove the sufficiency of the conditions of the theorem. Let φ, f, P, V, and E be as in Theorem 2.3(b). Then by Lemma 4.1(b), the layer E coincides with $E_{k,\tau}$, where $\tau = \mu_{\varphi(f)}(E)/\mu_{\varphi(f)}(V)$. Therefore, for all $l \in \Sigma_V(x_0)$, $E \cap l$ is a closed interval of length $|E \cap l|_1 = (1 - (1 - \tau)^{1/(1/k+m)})|V \cap l|_1$, and one of its ends coincides with $\partial V \cap l$.

Next, the restriction of P to V is a polynomial from $\mathcal{P}_{n,m}$. Moreover, the restriction of P to any $l \in \Sigma_V(x_0)$ coincides with the corresponding Chebyshev polynomial in terms of a coordinate on $V \cap l$. To complete the proof of the sufficiency, we observe that by the sufficiency part of Theorem 1.1(b) for all $l \in \Sigma_V(x_0)$,

$$(4.12)\quad \begin{aligned} |P(x_0)| &= \|P\|_{C(V\cap l)} = T_n(2|V\cap l|_1/|E\cap l|_1 - 1)\|P\|_{C(E\cap l)} \\ &= T_n\big(2(1-(1-\tau)^{1/(1/k+m)})^{-1} - 1\big)\|P\|_{C(E)}. \end{aligned}$$

The necessity of Theorem 2.3(b) is proved in Steps 2–4. In these steps, let V be a convex body in $\mathbb{R}^m$ and let a set $E \subseteq V$, functions $\varphi, f \in \mathcal{C}(V)$, and a polynomial $P \in \mathcal{P}_{n,m}$ be such that equality holds in (2.2). In addition, let $x_0 \in V$ be such that $\|P\|_{C(V)} = |P(x_0)|$. Without loss of generality we can assume that $x_0 = 0$. Equality in (2.2) shows that

$$(4.13)\quad \begin{aligned} \|P\|_{C(V)} &= |P(0)| \\ &= T_n\left(\frac{2}{1-(1-\mu_{\varphi(f)}(E)/\mu_{\varphi(f)}(V))^{\beta(\varphi)/(1+m\beta(\varphi))}} - 1\right)\|P\|_{C(E)}. \end{aligned}$$

Step 2. We first show that φ and f satisfy the conditions of Theorem 2.3(b). Indeed, we note that (4.13) implies equalities in (4.11). Next, using the monotone behavior of $T_n(2u-1)$ for $u \geq 1$, we see that equalities in (4.11) yield equality in (4.1). Then by Lemma 4.1(c), equality (4.2) holds for a.e. $l \in \Sigma_V(0)$. Therefore, for a.e. $l \in \Sigma_V(0)$,

$$(4.14)\quad |P(x_0)| = \|P\|_{C(V\cap l)} = T_n(2|V\cap l|_1/|E\cap l|_1 - 1)\|P\|_{C(E)}.$$

Further, (4.14) shows that for a.e. $l \in \Sigma_V(0)$,

$$(4.15)\quad \|P\|_{C(E\cap l)} = \|P\|_{C(E)}.$$

Indeed, if $\|P\|_{C(E\cap l)} < \|P\|_{C(E)}$ for rays l from a subset of $\Sigma_V(0)$ of positive spherical measure, then (4.14) contradicts Theorem 1.1(a). Therefore, for a.e. $l \in \Sigma_V(x_0)$,

$$(4.16)\quad |P(x_0)| = \|P\|_{C(V\cap l)} = T_n(2|V\cap l|_1/|E\cap l|_1 - 1)\|P\|_{C(E\cap l)},$$

which by the necessity part of Theorem 1.1(b) is possible only if $E \cap l$ is a closed interval and one of its ends coincides with $\partial V \cap l$. Thus E satisfies the conditions of Lemma 4.1(c). Then by 4.1(c), $\varphi(t) = Ct^{1/k}$ for some $k > 0$ and $C > 0$, and $f(x)$ is a nonnegative concave continuous 1-homogeneous function on V and $E_{k,\tau}(0) \subseteq E$, where $\tau = \mu_{\varphi(f)}(E)/\mu_{\varphi(f)}(V)$. Moreover, $E \cap l = E_{k,\tau}(0) \cap l$ for a.e. $l \in \Sigma_V(0)$.

Step 3. Here, we show that $E = E_{k,\tau}(0)$. We first note that by Theorem 1.1(b), (4.16) is valid only if $\|P\|_{C(E\cap l)} = |P(\partial V \cap l)|$ for a.e. $l \in \Sigma_V(0)$. Therefore, (4.15) holds for all $l \in \Sigma_V(0)$. Further, we show that

$$(4.17)\quad |V\cap l|_1/|E\cap l|_1 = C(E,V,0)$$

for all $l \in \Sigma_V(0)$. Indeed, since $E_{k,\tau}(0) \subseteq E$, for all $l \in \Sigma_V(0)$ we have

$$(4.18)\quad |V\cap l|_1/|E\cap l|_1 \leq |V\cap l|_1/|E_{k,\tau}(0)\cap l|_1 = C(E,V,0).$$

Let us assume that

$$|V \cap l_0|_1/|E \cap l_0|_1 < C(E,V,0) \tag{4.19}$$

for some $l_0 \in \Sigma_V(0)$. Then we deduce from equality in (4.11), relation (4.15) for all $l \in \Sigma_V(0)$, and (4.19) that

$$\begin{aligned}\|P\|_{C(V\cap l_0)} &= T_n(2C(E,V,0)-1)\|P\|_{C(E)} = T_n(2C(E,V,0)-1)\|P\|_{C(E\cap l_0)} \\ &> T_n(2|V\cap l_0|_1/|E\cap l_0|_1 - 1)\|P\|_{C(E\cap l_0)}.\end{aligned}$$

This contradicts Theorem 1.1(a). Therefore, (4.18) implies (4.17) for all l in $\Sigma_V(0)$.

Using again equality in (4.11), relation (4.15), and (4.17) for all $l \in \Sigma_V(0)$, we obtain

$$\|P\|_{C(V\cap l)} = T_n(2|V\cap l|_1/|E\cap l|_1 - 1)\|P\|_{C(E\cap l)} \tag{4.20}$$

for all $l \in \Sigma_V(0)$. By Theorem 1.1(b), this is possible only if $E = E_{k,\tau}(0)$.

STEP 4. It remains to prove that $V \in \mathcal{K}_m(0)$. By Theorem 1.1(b), equality (4.20) is possible only if

$$P(x) = AT_n(2\eta - 1 - 2\eta M_V(x)), \qquad x \in V,\ A \in \mathbb{R}^1 \setminus \{0\}, \tag{4.21}$$

where $\eta := C(E_{k,\tau}, V, 0) = (1-(1-\tau)^{1/(1/k+m)})^{-1} \geq 1$, and for $l_x \in \Sigma_V(0)$ passing through x,

$$M_V(x) := |x|/|V \cap l_x|_1 = \inf\{\lambda \in [0,\infty) : x \in \lambda V\} \tag{4.22}$$

is the Minkowski functional on V, which is a 1-homogeneous nonnegative function on V.

Next, we show that the function (4.21) belongs to $\mathcal{P}_{n,m}$ if and only if $M_V(x) = \alpha \cdot x$, $x \in V$, for some fixed $\alpha \in \mathbb{R}^m \setminus \{0\}$. Indeed, if $P \in \mathcal{P}_{n,m}$, then for any $\varepsilon > 0$,

$$P(\varepsilon x) = A \sum_{i=0}^{n} a_i \varepsilon^i (M_V(x))^i = \sum_{i=0}^{n} \varepsilon^i Q_i(x), \tag{4.23}$$

where $a_i \in \mathbb{R}$ depends only on η, and $Q_i \in \mathcal{P}_{i,m}$ is the i-homogeneous component of P, $0 \leq i \leq n$. In particular, (4.23) shows that $M_V(x) = (Aa_1)^{-1}Q_1(x)$, where $a_1 = -2\eta T_n'(2\eta - 1) < 0$. Thus $M_V(x) = \alpha \cdot x$ for some $\alpha \in \mathbb{R}^m \setminus \{0\}$ and every $x \in V$.

It remains to note that $V \in \mathcal{K}_m(0)$ if $M_V(x) = \alpha \cdot x$ on V (see also [10, Lemma 3]). Indeed, setting $A = \{x^* \in V : M_V(x^*) = \alpha \cdot x^* = 1\}$, we obtain $A \subseteq \partial V$ by (4.22), and $A = U_{m-1} \cap V$ is a convex subset of the hyperplane $U_{m-1} := \{x \in \mathbb{R}^m : \alpha \cdot x = 1\}$. Since $V = \{\lambda x^* \in \mathbb{R}^m : x^* \in A,\ \lambda \in [0,1]\}$, the set V is the convex hull of $A \cup \{0\}$, that is, $V \in \mathcal{K}_m(0)$. This completes the proof of Theorem 2.3. ∎

REMARK 4.2. In particular for $\varphi(t) = t^{1/k}$, the proof of Theorem 2.3(a) is a new proof of Theorem 1.4.

Proof of Theorem 2.6. Without loss of generality we can assume that x_0 coincides with the origin and $H_{m-1} = \{x \in \mathbb{R}^m : x_1 = h\}$, that is, H_{m-1} is perpendicular to the x_1-axis.

Then for the function $f_\delta(x) := \delta x_1$, where $\delta > 0$, we obtain

$$\begin{aligned}(4.24)\qquad \frac{\mu_{\varphi(f_\delta)}(E)}{\mu_{\varphi(f_\delta)}(V)} &= \frac{\int_{h-d}^{h} \varphi(\delta x_1) x_1^{m-1}\, dx_1}{\int_0^h \varphi(\delta x_1) x_1^{m-1}\, dx_1} \\ &= 1 - \frac{\int_0^{\delta(h-d)} \varphi(t) t^{m-1}\, dt}{\int_0^{\delta h} \varphi(t) t^{m-1}\, dt} = 1 - \frac{B_1(\delta)}{B_2(\delta)}.\end{aligned}$$

Since φ is increasing,

$$(4.25)\qquad \lim_{\delta\to 0^+} B_1(\delta) = \lim_{\delta\to 0^+} B_2(\delta) = 0, \qquad \lim_{\delta\to\infty} B_1(\delta) = \lim_{\delta\to\infty} B_2(\delta) = \infty,$$

where $B_1(\delta)$ and $B_2(\delta)$ are defined in (4.24). Taking account of (4.25), we see by L'Hospital's Rule that

$$(4.26)\qquad \lim \frac{B_1(\delta)}{B_2(\delta)} = \left(1 - \frac{d}{h}\right)^m \lim \frac{\varphi(\delta(h-d))}{\varphi(\delta h)}$$

as $\delta \to 0^+$ or $\delta \to \infty$, provided that the limit on the right-hand side of (4.26) exists. It is easy to evaluate the limit on the left-hand side by using condition (2.3). Indeed, if

$$(4.27)\qquad \lim_{y\to\infty} \varphi((1-d/h)y)/\varphi(y) = (1-d/h)^{1/\beta(\varphi)},$$

then (4.26) and (4.27) imply

$$(4.28)\qquad \lim_{\delta\to\infty} B_1(\delta)/B_2(\delta) = (1-d/h)^{1/\beta(\varphi)+m}.$$

If

$$(4.29)\qquad \lim_{y\to 0^+} \varphi((1-d/h)y)/\varphi(y) = (1-d/h)^{1/\beta(\varphi)},$$

then (4.26) and (4.28) imply

$$(4.30)\qquad \lim_{\delta\to 0^+} B_1(\delta)/B_2(\delta) = (1-d/h)^{1/\beta(\varphi)+m}.$$

Further, note that if (2.3) is satisfied, then either (4.27) or (4.29) holds. Combining (4.24) with (4.28) or (4.30), we obtain

$$\frac{h}{d} = \lim\left(\left(1 - \frac{\mu_{\varphi(f_\delta)}(E)}{\mu_{\varphi(f_\delta)}(V)}\right)^{\beta(\varphi)/(1+m\beta(\varphi))}\right)^{-1},$$

as $\delta \to 0^+$ if (4.29) holds, and as $\delta \to \infty$ if (4.27) holds.

It remains to note (see Step 1 of the proof of Theorem 2.3(b)) that

$$\begin{aligned}\|P\|_{C(V)} &= T_n(2h/d-1)\|P\|_{C(E)} \\ &= \lim T_n\left(\frac{2}{1-(1-\mu_{\varphi(f_\delta)}(E)/\mu_{\varphi(f_\delta)}(V))^{\beta(\varphi)/(1+m\beta(\varphi))}} - 1\right)\|P\|_{C(E)}\end{aligned}$$

as $\delta \to 0^+$ or $\delta \to \infty$. This establishes (2.4). ∎

5. Examples. In this section we discuss some examples of functions φ and properties and examples of extremal functions f from Theorem 2.3. In Examples 5.1–5.4 we present some strictly increasing and twice continuously differentiable functions φ with $\beta(\varphi) > 0$. In addition, these functions satisfy condition (2.3).

EXAMPLE 5.1. $\varphi(t) = t^\lambda \log^\gamma(1+t)$, $\lambda \geq 0$, $\lambda+\gamma \geq 0$, $\lambda + |\gamma| > 0$. Then φ is strictly increasing on $(0,\infty)$ and

$$\beta(\varphi) = 1/(\lambda + \max\{\gamma, 0\}) \tag{5.1}$$

(see [17, Example 5.4]). In addition,

$$\lim_{y\to 0^+} \varphi(cy)/\varphi(y) = c^{\lambda+\gamma}, \quad \lim_{y\to\infty} \varphi(cy)/\varphi(y) = c^{\lambda}, \qquad c \in (0,1]. \tag{5.2}$$

Then condition (2.3) follows from (5.1) and (5.2).

EXAMPLE 5.2. $\varphi(t) = t^\lambda(1+t)^\gamma$, $\lambda \geq 0$, $\lambda+\gamma \geq 0$, $\lambda + |\gamma| > 0$. Then φ is strictly increasing on $(0,\infty)$ and (5.1) holds (see [17, Example 5.5]). In addition,

$$\lim_{y\to 0^+} \varphi(cy)/\varphi(y) = c^{\lambda}, \quad \lim_{y\to\infty} \varphi(cy)/\varphi(y) = c^{\lambda+\gamma}, \qquad c \in (0,1]. \tag{5.3}$$

Then condition (2.3) follows from (5.2) and (5.3).

The functions φ in the following two examples satisfy condition (2.3) as well.

EXAMPLE 5.3. $\varphi(t) = t - \log(1+t)$, $\beta(\varphi) = 1/2$.

EXAMPLE 5.4. $\varphi(t) = t + \log(1+t)$, $\beta(\varphi) = 1$.

Next, we present two examples of functions φ with $\beta(\varphi) \leq 0$.

EXAMPLE 5.5. $\varphi(t) = t^\lambda \exp(\tau t^\alpha)$, $\lambda \geq 0$, $\alpha\tau > 0$. Then φ is strictly increasing on $(0,\infty)$ and

$$\beta(\varphi) \begin{cases} = 0, & \tau > 0,\ \alpha \in (0,1] \text{ or } \tau, \alpha < 0, \\ < 0, & \tau > 0,\ \alpha \in (1,\infty) \end{cases}$$

(see [17, Example 5.6]).

EXAMPLE 5.6. $\varphi(t) = t^\lambda \exp(\tau \log^{2m+1} t)$, $\lambda \geq 0$, $\tau > 0$, where m is a nonzero integer. Then φ is strictly increasing on $(0,\infty)$ and

$$(\varphi(t)/\varphi'(t))' = \frac{\lambda + \tau(2m+1)\log^{2m} t - 2\tau m(2m+1)\log^{2m-1} t}{(\lambda + \tau(2m+1)\log^{2m} t)^2}.$$

Thus $\beta(\varphi) \leq 0$.

Next, we discuss some properties and examples of concave continuous 1-homogeneous functions f from Theorem 2.3(b).

Proposition 5.7. *Let S be a convex set in $\mathbb{R}^m$, $0 \in S$, and let f be a continuous function on S with $f(0) = 0$. Then any two of the following statements imply the third one:*

(i) *f is concave on S;*
(ii) *f is 1-homogeneous on S, that is, $f(\lambda x) = \lambda f(x)$ for any $x \in S$ and $\lambda > 0$ such that $\lambda x \in S$;*
(iii) *f is superadditive on S, that is, $f(x + y) \geq f(x) + f(y)$ for any $x \in S$ and $y \in S$ such that $x + y \in S$.*

Proof. We prove the proposition by standard arguments from the theory of convex functions.

(i)&(ii)⇒(iii). For any $x \in S$ and $y \in S$ we deduce from (i) that $f((x + y)/2) \geq (1/2)(f(x) + f(y))$. Then assuming $x + y \in S$, we arrive at (iii) from (ii).

(ii)&(iii)⇒(i). Since $0 \in S$, for any $x \in S$, $y \in S$, and $\lambda \in [0, 1]$ we have $\lambda x \in S$, $(1 - \lambda)y \in S$, and $\lambda x + (1 - \lambda)y \in S$. So by (iii) and (ii),

$$f(\lambda x + (1 - \lambda)y) \geq f(\lambda x) + f((1 - \lambda)y) = \lambda f(x) + (1 - \lambda)f(y).$$

Then f is concave on S.

(i)&(iii)⇒(ii). We first note that $\lambda x \in S$ for any $x \in S$ and $\lambda \in [0, 1]$ since $0 \in S$. Let n be a natural number and $x \in S$. Then using (iii) and (i), we obtain $f(x) = nf(x/n)$ or equivalently $f(ny) = nf(y)$ if $ny \in S$. Hence for any natural m such that $mx/n \in S$, we have $f(mx/n) = mf(x/n) = (m/n)f(x)$. Finally, approximating $\lambda > 0$ by $m/n \in (0, \lambda]$, we arrive at (ii) since f is continuous on S. ∎

Proposition 5.8. *Let S be a closed convex set in $\mathbb{R}^2$, $0 \in \partial S$, and let a function $f(x, y)$ of two variables be continuous on S and twice differentiable and 1-homogeneous on $S \setminus \{0\}$. Then f is concave on S if and only if for each fixed y, $f(x, y)$ is a concave function of x on the interval $\{x : (x, y) \in S\}$.*

Proof. Differentiating both sides of Euler's formula

$$x\frac{\partial f(x, y)}{\partial x} + y\frac{\partial f(x, y)}{\partial y} = f(x, y)$$

with respect to each variable, we see that the Hessian determinant $\det \Delta(x, y)$ of f equals zero on $S \setminus \{0\}$. Therefore by the criterion for concavity of a function in two variables (cf. (3.1)), f is concave on $S \setminus \{0\}$ if and only if $\partial^2 f(x, y)/\partial x^2 \leq 0$ for $(x, y) \in S \setminus \{0\}$. Since f is continuous on S, the proposition is established. ∎

Example 5.9. Let $S = \{x \in \mathbb{R}^m : x_i \geq 0,\ 1 \leq i \leq m\}$. Then the Cobb–Douglas function

$$f(x) = \prod_{i=1}^{m} x_i^{\alpha_i}, \qquad \alpha_i \geq 0,\, 1 \leq i \leq m, \quad \sum_{i=1}^{m} \alpha_i = 1,$$

is superadditive on S by a generalized Hölder's inequality [19, Sect. 2.7.11]. Hence f is concave on S by Proposition 5.7.

Example 5.10. Each elementary symmetric function

$$f(x) = (\sigma_k(x))^{1/k}, \quad 1 \leq k \leq m,$$

is superadditive on S due to Bohnenblust (see [23] or [2, Sect. 1.34]). Hence f is concave on S by Proposition 5.7.

Note that Examples 5.9 and 5.10 are discussed in [17, Example 2.5] as well.

Example 5.11. Let $S = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq y < \infty\}$. Then the function of two variables

$$f(x, y) = \sqrt{y^2 - x^2}$$

is concave on $[0, y]$ for each fixed $y \geq 0$. Hence f is concave on S by Proposition 5.8.

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