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A Calabi–Yau threefold with 192 automorphisms

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Abstract. We study a double octic Calabi–Yau threefold with a group of 192 automorphisms preserving the canonical form.

1. Introduction. A *Calabi–Yau threefold* is a smooth projective three-dimensional manifold with trivial canonical class and vanishing first Betti number. Calabi–Yau threefolds are among the most studied objects in modern mathematics, the motivation coming not only from mathematics but also from physics. In applications, particularly important are Calabi–Yau threefolds with a large group of automorphisms. Finite groups of automorphisms of Calabi–Yau threefolds defined over $\mathbb{Q}$ were used to split the Galois action on cohomology groups and consequently better understand their arithmetics. Many examples of new Calabi–Yau threefolds were constructed as resolutions of singularities of quotients of complex manifolds by an action of a finite group (the most famous example being the mirror of the Dwork pencil).

In this note we shall give an example of a Calabi–Yau threefold X and a subgroup $G \subseteq \mathrm{Aut}(X)$ of order 192. Let S_8 be the following configuration of eight planes:

$$S_8 := \{xyzt(x+y+z-t)(x+y-z+t)(x-y+z+t)(-x+y+z+t) = 0\} \tag{1}$$

(arrangement no. 238 of [7]) and let Y be the double covering of the projective space $\mathbb{P}^3$ branched along S_8.

Theorem 1.1. *The variety Y has a projective resolution of singularities X which is a Calabi–Yau threefold with Hodge numbers $h^{1,1} = 44$, $h^{1,2} = 0$, and the automorphism group* $\mathrm{Aut}(X)$ *of X contains a subgroup G isomorphic to the group* `SmallGroup`$(192, 955)$ ([1]).

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The group G has a trivial center and is not nilpotent, elements of G preserve the canonical form of X and can be obtained from projective transformations of the weighted projective space $\mathbb{P}(1,1,1,1,4)$. The group G contains 14 conjugacy classes, and the quotients of X by the corresponding cyclic groups admit Calabi–Yau resolution of singularities. We give a table containing representatives of the conjugacy classes.

The choice of this particular octic arrangement was motivated by our attempt to give a complete classification of double octics. Using extensive Magma computations we found the symmetry groups of all double octics with $h^{1,2} \leq 1$ listed in [7] and checked that Y has the biggest one. The Calabi–Yau threefold X is given in a very explicit and geometric way compared to other known examples of Calabi–Yau threefolds with a large group of automorphisms. The elements of the group G can be given with explicit equations, so one can describe the quotients, their resolutions and Hodge numbers. We believe that the careful study of other double octics and their groups of symmetries should produce interesting new examples of Calabi–Yau threefolds.

We do not know whether G is the complete group of automorphisms of X preserving the canonical form. We shall give an example of a birational map of a double octic Calabi–Yau threefold which is not induced by a linear transformation.

In Section 2 we shall describe the geometry of the octic arrangement S_8, in particular we shall give the singularities of S_8 (and hence of Y) and discuss a resolution. In Section 3 we compute the group of permutations of eight planes that preserve all incidences, identify this group in the `SmallGroups` database and find two generators of this group. Next we observe that these two generators can be realized by taking projective transformations of $\mathbb{P}^3$, extending them to the double cover Y and finally lifting to the chosen resolution of singularities of X.

There is a correspondence between the Calabi–Yau threefold X and van Geemen and Nygaard's [5] complete intersection of four quadrics in $\mathbb{P}^7$ given by the equations

$$\begin{aligned}
Y_0^2 &= X_0^2 + X_1^2 + X_2^2 + X_3^2,\\
Y_1^2 &= X_0^2 - X_1^2 + X_2^2 - X_3^2,\\
Y_2^2 &= X_0^2 + X_1^2 - X_2^2 - X_3^2,\\
Y_3^2 &= X_0^2 - X_1^2 - X_2^2 + X_3^2
\end{aligned}$$

(see [7, Sec. 6.1.7]). The large group of automorphisms of van Geemen and Nygaard's complete intersection was studied in a series of papers by E. Freitag and R. Salvati Manni (cf. [4]).

2. Geometry and resolution of singularities of Y. Observe that S_8 has three types of singularities: double lines, triple points and fourfold points. There are 12 fourfold points given by the intersections of the following quadruples of planes:

$$(1,2,5,6),(1,2,7,8),(1,3,5,7),(2,3,5,8),(2,3,6,7),(1,3,6,8),$$
$$(3,4,5,6),(2,4,5,7),(1,4,5,8),(1,4,6,7),(2,4,6,8),(3,4,7,8),$$

eight triple points

$$(1,2,3),(1,2,4),(1,3,4),(2,3,4),(5,6,7),(5,6,8),(5,7,8),(6,7,8),$$

and 28 double lines (corresponding to all pairs of planes). The singularities of the double cover Y correspond to the singularities of the branch locus S_8.

By [3, Thm. 3.5], Y admits a resolution of singularities that is a Calabi–Yau threefold with (topological) Euler characteristic 88. Since deformations of (any) Calabi–Yau resolution of a double octic correspond to equisingular deformations of the branch divisor, in the special case of arrangement of eight planes this is the same as deformations of a configuration of eight planes preserving all the incidences. So, the arrangement under study is rigid, i.e. admits no deformations (for details see for instance [7, Sec. 4.1]).

Using [3] we can resolve the singularities of Y by blowing up first all fourfold points and then all double lines. The main disadvantage of this method is that the resolution is not unique, it depends on the order of double lines. More precisely, at every threefold point three double lines intersect, and the resolution depends on which line is blown up first.

In order to circumvent this inconvenience, in [2] another resolution of singularities was given.

Proposition 2.1. *There exists a resolution of singularities $\sigma : X \to Y$ such that*

- *X is a Calabi–Yau threefold,*
- *for every automorphism $g : Y \to Y$ there exists an automorphism $f : X \to X$ such that $g\circ\sigma = \sigma\circ f$, i.e. the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \sigma\downarrow & & \downarrow\sigma \\ Y & \xrightarrow{g} & Y \end{array}$$

Proof. Consider the variety Y as a hypersurface in a weighted projective space $\mathbb{P}(1,1,1,1,4)$ given by the equation

$$u^2 = f_8(x,y,z,t),$$

where x, y, z, t are variables of weight 1, u is a variable of weight 4 and $f_8(x, y, z, t)$ is the equation of the branch octic. A point $(x, y, z, t, u) \in Y$ is a singular point of Y exactly when $u = 0$ and $(x, y, z, t) \in \mathbb{P}^3$ is a singular point of S_8.

More explicitly, for any fourfold point in S_8 we can make a projective change of coordinates such that the point becomes $(0 : 0 : 0 : 1) \in S_8$ and the equation of S_8 is $xyz(x+y+z)l_4$, where l_4 is a homogeneous polynomial of degree 4 such that $l_4(0, 0, 0, 1) \neq 0$. Substituting $t = 1$, we get an affine equation. Now, in affine charts the blow-up is given by

$$\begin{aligned}&(x, y, z, u) \mapsto (x, xy, xz, x^2u), \qquad (x, y, z, u) \mapsto (xy, y, yz, y^2u),\\&(x, y, z, u) \mapsto (xz, yz, z, z^2u).\end{aligned}$$

We shall consider only the first chart. Substituting it into the equation of Y we get

$$x^4(u^2 - yz(y + z + 1)l_4(x, xy, xz, 1)),$$

so the strict transform of Y is given by

$$u^2 - yz(y + z + 1)l_4(x, xy, xz, 1);$$

it contains double lines and triple points, but one fourfold point is resolved.

After blowing up all fourfold points, we are left with double lines and triple points. The affine equation of Y is now either

$$u^2 - xyl_6(x, y, z) \quad \text{with } l_6(0, 0, 0) \neq 0,$$

or

$$u^2 - xyzl_5(x, y, z) \quad \text{with } l_5(0, 0, 0) \neq 0.$$

We study only the second case, as the first one is much easier. We blow up the singular locus

$$\begin{aligned}C &:= \{xy = xz = yz = u = 0\}\\&= \{x = y = u = 0\} \cup \{x = z = u = 0\} \cup \{y = z = u = 0\}.\end{aligned}$$

The blow-up of the open set $l_5 \neq 0$ is the closure of the graph of the map

$$\mathbb{C}^4 \setminus C \ni (x, y, z, u) \mapsto ((x, y, z, u), (xy : xz : yz : u)) \in \mathbb{C}^4 \times \mathbb{P}^3,$$

and so it is given by the following condition:

$$\left\{((x, y, z, u), (p : q : r : s)) \in \mathbb{C}^4 \times \mathbb{P}^3 : \operatorname{rank}\begin{pmatrix} xy & xz & yz & u \\ p & q & r & s \end{pmatrix} = 1\right\}.$$

This time we have four affine charts on the blow-up given by $p = 1$, $q = 1$, $r = 1$ and $s = 1$. In the chart $s = 1$ we get

$$xy = pu, \qquad xz = qu, \qquad yz = ru,$$

and consequently on the strict transform of Y we get $x^2yz = pqu^2 = pqxyzl_5$, and so $x = pql_5$. In a similar way we get $y = prl_5$ and $z = qrl_5$. Using these relations we obtain $u^4 = (xyz)^2 l_5^2 = (xy)(xz)(yz)l_5^2 = (pu)(qu)(ru)l_5^2 = pqru^3 l_5^2$, and therefore $u = pqrl_5^2$. Finally, on the strict transform of Y we have

$$x = pql_5, \quad y = prl_5, \quad z = qrl_5, \quad u = pqrl_5^2,$$

and so the blow-up of Y is smooth in the affine open set $s = 1$, $l_5 \neq 0$.

In the affine chart $p = 1$ in a similar way we get

$$rl_5 = ys^2, \quad ql_5 = xs^2, \quad zl_5 = xys^2, \quad u = xys,$$

and the strict transform is smooth when $l_5 \neq 0$. In the remaining charts $q = 1$ and $r = 1$ we get the same conclusion.

Observe that the inverse image of the triple point $x = y = z = 0$ is the union of three lines $\{x = y = z = u = 0, p = q = 0\} \cup \{x = y = z = u = 0, p = r = 0\} \cup \{x = y = z = u = 0, q = r = 0\}$. As the resolution agrees away from a finite union of lines with the one described in [3], the canonical divisor of X is zero away from a finite union of lines, and hence is zero. Moreover the two resolutions are birational, so they have first Betti numbers equal.

The second statement of the proposition follows from the universal property of blowing up [6, Prop. II.7.14]. ∎

3. Symmetries of the octic arrangement. We shall study the group of those automorphisms of the Calabi–Yau 3-fold X that induce permutations of planes in the branch divisor S_8.

LEMMA 3.1. *Let $G \subset \Sigma_8$ be the group of permutations of the branch planes that preserve the incidences. Then G is the* 955*th group of order* 192 (*Magma* `SmallGroup`(192, 955)). *The group G is generated by the permutations* (23)(5687), (1835)(2746).

Proof. At the beginning of the previous section we listed all quadruples of planes intersecting at fourfold points. A simple Magma program determines the permutations that preserve the listed quadruples. Finally, we use the function `FewGenerators` to find a small set of generators. ∎

LEMMA 3.2. *Every permutation $\sigma \in G$ is induced by a projective transformation of $\mathbb{P}(1,1,1,1,4)$ preserving Y and the* (*rational*) *canonical form ω_Y.*

Proof. There is a 3-form on Y with poles located in the singular locus of the branch divisor S_8; in the affine chart $t = 1$ it is given by

$$\omega_Y = \frac{1}{u}\, dx\wedge dy\wedge dz = \frac{2}{\frac{\partial f_8}{\partial x}}\, du\wedge dy\wedge dz = \frac{2}{\frac{\partial f_8}{\partial y}}\, dx\wedge du\wedge dz = \frac{2}{\frac{\partial f_8}{\partial z}}\, dx\wedge dy\wedge du.$$

We find explicitly an appropriate projective transformation for the two generators. For the first permutation we search a transformation of the shape

$$(x, y, z, t, u) \mapsto (\alpha x, \beta z, \gamma y, \delta t, \epsilon u).$$

Applying this transformation to the fifth, sixth, seventh and eighth arrangement plane we conclude

$$\alpha = -\beta = \gamma = -\delta.$$

Normalizing by $\alpha = -1$ we get the linear map $(x, y, z, t) \mapsto (-x, z, -y, t)$. Pulling back (in the affine chart $t = 1$) the form $dx \wedge dy \wedge dz$ we get $(-dx) \wedge dz \wedge (-dy) = -dx \wedge dy \wedge dz$, and therefore $\epsilon = -1$. Finally, the projective transformation of $\mathbb{P}(1, 1, 1, 1, 4)$ is given by

$$(x, y, z, t, u) \mapsto (-x, z, -y, t, -u).$$

Computations for the second generator are analogous but more complicated and yield the transformation

$$(x, y, z, t, u) \mapsto (-x+y+z+t, x-y+z+t, x+y+z-t, x+y-z+t, 16u). \ \blacksquare$$

Theorem 1 follows from Proposition 2 and Lemmas 3 and 4.

Remark 3.3. We do not know if G is the group of all automorphisms of X preserving the canonical form. The following example demonstrates that in general there can exist a non-linear birational transformation of a double octic.

Example 3.4. Consider a double octic given by the following arrangement of eight planes:

$$x(x + z)(x + t)(2x + z - 3t)y(y + t)(3y + z)(4y + z - t) = 0.$$

Then the map

$$\begin{pmatrix} u \\ x \\ y \\ z \\ t \end{pmatrix} \mapsto \begin{pmatrix} (2x - z + 3t)^2(t - z)^2(3t - z)^2(-2y - z + t)^2 u \\ (3t - z)y(2x - z + 3t) \\ (t - z)x(-2y - z + t) \\ z(2x - z + 3t)(-2y - z + t) \\ t(2x - z + 3t)(-2y - z + t) \end{pmatrix}$$

defines a birational transformation of X.

4. Cyclic subgroups. The group G is too big and complicated to give its complete description; in Table 1 we list only representatives of all fourteen conjugacy classes. For each conjugacy class we give the permutation of the arrangement planes and a projective transformation that lifts to an automorphism of the double octic considered.

Table 1

No	Order	Size	Representative
$(x, y, z, t, u) \mapsto \dots$			
1	1	1	1
(x, y, z, t, u)			
2	2	3	$(1,3)(2,4)(5,7)(6,8)$
(z, t, x, y, u)			
3	2	4	$(1,8)(2,7)(3,6)(4,5)$
$(-x+y+z+t, x-y+z+t, x+y-z+t, x+y+z-t, -16u)$			
4	2	6	$(1,3)(2,4)(5,8)(6,7)$
$(z, t, -x, -y, u)$			
5	2	6	$(1,3)(2,4)$
$(z, -t, x, -y, u)$			
6	2	12	$(1,7)(2,6)(3,5)(4,8)$
$(x-y+z+t, -x-y+z-t, x+y+z-t, x-y-z-t, 16u)$			
7	2	12	$(3,4)(7,8)$
$(x, y, -t, -z, -u)$			
8	3	32	$(1,4,2)(5,7,8)$
(t, x, z, y, u)			
9	4	12	$(1,4,3,2)(5,8,7,6)$
$(t, -x, y, -z, -u)$			
10	4	12	$(1,8,4,5)(2,7,3,6)$
$(-x+y+z+t, -x+y-z-t, -x-y+z-t, x+y+z-t, -16u)$			
11	4	12	$(1,8,3,6)(2,5,4,7)$
$(-x+y+z+t, -x-y-z+t, x+y-z+t, -x+y-z-t, 16u)$			
12	4	24	$(1,8,3,5)(2,7,4,6)$
$(-x+y+z+t, -x+y-z-t, -x-y-z+t, x+y-z+t, 16u)$			
13	4	24	$(2,3)(5,6,8,7)$
$(-x, z, -y, t, -u)$			
14	6	32	$(1,7,4,8,2,5)(3,6)$
$(x-y+z+t, x+y+z-t, x+y-z+t, -x+y+z+t, -16u)$			

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