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Banach-lattice isomorphisms of $C_0(K,X)$ spaces which determine the locally compact spaces K

by

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Abstract. For a locally compact Hausdorff space K and a real Banach-lattice X let $C_0(K,X)$ denote the Banach lattice of all X-valued continuous functions vanishing at infinity, endowed with the supremum norm.

We refine some Banach space results due to Cambern to the setting of Banach lattices to prove that if there is a Banach-lattice isomorphism T from $C_0(K,X)$ onto $C_0(S,X)$ satisfying

$$\|T\|\,\|T^{-1}\| < \lambda^+(X),$$

then K and S are homeomorphic, where

$$\lambda^+(X) = \inf\{\max\{\|x-y\|, \|x+y\|\} : \|x\| = \|y\| = 1 \text{ and } x, y \geq 0\}.$$

This result is optimal for the classical l_p spaces, $1 \leq p < \infty$.

1. Introduction. In this paper we use the standard terminology and notation of Banach lattice and Banach space theory, as may be found, e.g., in [1, 2, 7, 15]. When K is a compact Hausdorff space, the Banach lattice $C_0(K,X)$ will be denoted by $C(K,X)$. If $X = \mathbb{R}$ these spaces will also be denoted by $C_0(K)$ and $C(K)$ respectively. Throughout this paper all Banach lattices are assumed to be real.

In 1947 Kaplansky [14] proved that as a lattice alone, $C(K)$ determines the topology of K. More precisely, if there is a lattice isomorphism $T\colon C(K) \to C(S)$, that is, $T(f \vee g) = Tf \vee Tg$ for all $f, g \in C(K)$, then K and S are homeomorphic.

It is well-known that there exists no natural extension of Kaplansky's theorem to $C(K,X)$ spaces without additional hypotheses on the operator

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T even in the case where T is a Banach-lattice isomorphism and X is l_1^2, the two-dimensional abstract L_1-space. Indeed, l_1^2 is isometric to l_∞^2, the real space $\mathbb{R}^2$ endowed with the supremum norm. On the other hand, there are non-homeomorphic compact Hausdorff spaces K_1 and K_2 such that the topological sums $K_1 \oplus K_1$ and $K_2 \oplus K_2$ are homeomorphic [10]. Then we have the following chain of isometric Banach-lattice isomorphisms:

$$\begin{aligned} C(K_1, l_1^2) &\cong C(K_1, l_\infty^2) \cong C(K_1 \oplus K_1) \cong C(K_2 \oplus K_2) \\ &\cong C(K_2, l_\infty^2) \cong C(K_2, l_1^2). \end{aligned}$$

So, the motivation of this paper is the following natural question.

Problem 1.1. *When does a Banach-lattice isomorphism T of $C_0(K, X)$ spaces determine the locally compact spaces K?*

We begin by noticing that very recently [5] a vector-valued generalization of the classical Banach–Stone theorem was proved which gives a partial answer to the above question. To state it, recall the following parameter introduced by Schäffer [9, 19] for Banach spaces X:

$$\lambda(X) = \inf\{\max\{\|x - y\|, \|x + y\|\} : \|x\| = \|y\| = 1\}. \tag{1.1}$$

Theorem 1.2. *Let K and S be locally compact Hausdorff spaces and X a real Banach space. If there is a Banach-space isomorphism T from $C_0(K, X)$ onto $C_0(S, X)$ satisfying*

$$\|T\|\,\|T^{-1}\| < \lambda(X),$$

then K and S are homeomorphic.

In view of Theorem 1.2 and Problem 1.1 we are led to the following question. If T is actually a Banach-lattice isomorphism from $C_0(K, X)$ onto $C_0(S, X)$, how much the distortion $\|T\|\,\|T^{-1}\|$ in Theorem 1.1 can be increased without affecting the validity of this result?

In order to answer this question it is convenient to introduce another parameter defined for every Banach lattice X:

$$\lambda^+(X) = \inf\{\max\{\|x - y\|, \|x + y\|\} : \|x\| = \|y\| = 1 \text{ and } x, y \geq 0\}.$$

First, notice that $\lambda(X) \leq \lambda^+(X)$ for every Banach lattice X. Moreover, while

$$\lambda(l_p) = \min\{2^{1/p}, 2^{1-1/p}\}, \quad 1 \leq p < \infty$$

(see [9, Theorem 3.1]), it is easy to check that $\lambda^+(l_p) = 2^{1/p}$ for $1 \leq p < \infty$. Hence, for any $1 \leq p < 2$,

$$\lambda(l_p) < \lambda^+(l_p).$$

Thus, the main result of the present work is the following improvement of Theorem 1.2 in the case where T is a Banach lattice isomorphism.

Theorem 1.3. *Let K and S be locally compact Hausdorff spaces and X a Banach lattice. If there is a Banach-lattice isomorphism T from $C_0(K,X)$ onto $C_0(S,X)$ satisfying*

$$\|T\|\,\|T^{-1}\| < \lambda^+(X),$$

then K and S are homeomorphic.

We stress that in the case where $X = \ell_p$ for $1 \le p < \infty$, $\lambda^+(X)$ is the best possible bound in Theorem 1.3. Indeed, let $K = \{1,2\}$ and $S = \{1\}$ be two discrete compact Hausdorff spaces. Define $T : C(K,X) \to C(S,X)$ by

$$T((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = (x_1, y_1, x_2, y_2, \ldots). \tag{1.2}$$

It is easy to see that T is a Banach-lattice isomorphism from $C_0(K,X)$ onto $C_0(S,X)$ satisfying $\|T\|\,\|T^{-1}\| = 2^{1/p}$. But of course K and S are not homeomorphic.

Theorem 1.3 has the following consequence when the target space X is an abstract L_p-space for some $1 \le p < \infty$, that is, $\|x+p\|^p = \|x\|^p + \|y\|^p$ for all $x, y \in X$ such that $x \wedge y = 0$.

Corollary 1.4. *Let K and S be locally compact Hausdorff spaces and X be an abstract L_p-space. If there is a Banach-lattice isomorphism T from $C_0(K,X)$ onto $C_0(S,X)$ satisfying*

$$\|T\|\,\|T^{-1}\| < 2^{1/p},$$

then K and S are homeomorphic.

Proof. It is easy to seen that $\lambda^+(X) = 2^{1/p}$. Therefore, by Theorem 1.3 we are done. ∎

In order to prove Theorem 1.3 we need to refine Banach-space techniques employed in [4] to the Banach-lattice setting. In that paper Cambern obtained the first vector-valued extension of the Banach–Stone theorem for $C_0(K,X)$ spaces, where X is a finite-dimensional Hilbert space and the isomorphisms T between these spaces satisfy $\|T\|\,\|T^{-1}\| < \sqrt{2}$.

The main task of this work is to show how to improve the techniques developed by Cambern in the case where the $C_0(K,X)$ spaces are Banach lattices and T is a Banach-lattice isomorphism satisfying $\|T\|\,\|T^{-1}\| < \lambda^+(X)$. It is worth mentioning that Banach lattices X with $\lambda^+(X) > 1$ enjoy geometrical properties which are often used throughout all the refinement of the results of [4] obtained here.

2. On representing measures of positive operators on $C_0(K,X)$ spaces. In this section we collect some auxiliary results that we use in the proof of our theorems. Let X be a Banach lattice. The *positive cone* of X is the norm-closed set $C_X = \{x \in X : x \ge 0\}$. We use the notations B_X^+ and S_X^+ for the sets $\{x \in C_X : \|x\| \le 1\}$ and $\{x \in C_X : \|x\| = 1\}$, respectively.

An operator L between Banach lattices X and Y is said to be *positive* if $L(C_X) \subset C_Y$.

If X is a Banach space then we denote by $\mathrm{rcabv}(K, X)$ the Banach space of all regular countably additive X-valued vector measures defined on Σ_K, the Borel σ-algebra of K, endowed with the total variation norm. Moreover, for all Banach space Y, we denote by $\mathcal{B}(X, Y)$ the Banach space of all continuous linear operators from X to Y. When $X = Y$, we write $\mathcal{B}(X)$ instead of $\mathcal{B}(X, X)$.

A well-known theorem of Dinculeanu [6] states that if $L\colon C_0(K, X) \to Y$ is a continuous operator, there exists a unique measure $m\colon \Sigma_K \to \mathcal{B}(X, Y^{**})$ satisfying

(1) $L(f) = \int_K f\, dm$ for all $f \in C_0(K, X)$;
(2) $L^*(y^*) = m_{y^*} \in \mathrm{rcabv}(K, X^*)$ for each $y^* \in Y^*$ where $m_{y^*}\colon \Sigma_K \to X^*$ is defined as $m_{y^*}(A)(x) = \langle m(A)(x), y^* \rangle$;
(3) $\|L\| = \widetilde{m}(K)$, where $\widetilde{m}(\cdot)$ denotes total semivariation.

The integral above is in the sense of [6]. The measure m is called the *representing measure* of L.

Recall that a measure $m\colon \mathcal{A} \to \mathcal{B}(X, Y)$ is said to be *positive* whenever $m(A)\colon X \to Y$ is a positive operator for all $A \in \mathcal{A}$. Riesz's representation theorem [18, Theorem 6.19] establishes a correspondence between positive functionals on $C_0(K)$ and the set of positive Radon measures defined on Σ_K. Singer's representation theorem [20] gives an analogous correspondence between functionals on $C_0(K, X)$ and the Banach space $\mathrm{rcabv}(K, X^*)$. See also [11] for a simple proof of Theorem 2.1 when K is a compact space.

THEOREM 2.1. *There is an isometric isomorphism between* $C_0(K, X)^*$ *and the Banach space* $\mathrm{rcabv}(K, X^*)$*, where the functional* $U \in C_0(K, X)^*$ *and the corresponding measure* $\mu \in \mathrm{rcabv}(K, X^*)$ *are related by the formula*

$$Uf = \int_K f\, d\mu, \qquad \forall f \in C_0(K, X),$$

with the integral being the immediate integral of Dinculeanu [6].

By using Riesz's representation theorem and following the proof of Theorem 2.1 given in [16] we obtain the proof of the lemma below, which will be useful later on.

LEMMA 2.2. *Let* K *be a locally compact Hausdorff spaces, and let* X *and* Y *be Banach lattices. Suppose that* $L\colon C_0(K, X) \to Y$ *is a positive operator. Then the representing measure of* L *is positive.*

3. A set-valued map induced by $C_0(K, X)$ Banach-lattice isomorphisms. Henceforward K and S will be two locally compact spaces and X

a Banach lattice such that there exists a Banach-lattice isomorphism T from $C_0(K,X)$ onto $C_0(S,X)$ satisfying the hypothesis of Theorem 1.3.

In this section we introduce a set-valued map which will play an important role in obtaining the homeomorphism between K and S.

First of all notice that the hypothesis $\|T\|\,\|T^{-1}\| < \lambda^+(X)$ implies that $\lambda^+(X) > 1$. On the other hand, it is easy to check that $\lambda^+(c_0) = 1$. Thus, X contains no copy of c_0.

Let $T_s\colon C_0(K,X) \to X$ for $s \in S$ fixed be defined by

$$T_s(f) = Tf(s).$$

According to Lemma 2.2, T_s is a positive measure, and since X contains no copy of c_0, by [3, Theorem 4.4] and [17, Theorem 13], T_s assumes its values in $\mathcal{B}(X)$. Then, for fixed $k \in K$, $e \in S_X^+$ and $0 < r < 1$ we set

$$\Omega_r(k,e,T,S) = \{s \in S : \|T_s(\{k\})(e)\| \ge r/\|T^{-1}\|\}.$$

Analogously, pick $k \in K$ and define $T_k^{-1}\colon C_0(S,X) \to X$ by

$$T_k^{-1}(g) = T^{-1}g(k).$$

For all $s \in S$, $e \in S_Y^+$ and $0 < r < 1$ we also set

$$\Omega_r(s,e,T^{-1},K) = \big\{k \in K : \|T_k^{-1}(\{s\})(e)\| \ge r/\|T\|\big\}.$$

In the two next sections we will provide some properties of the sets $\Omega_r(k,e,T,S)$.

4. $\Omega_r(k,e,T,S)$ is non-empty for all $k \in K$, $e \in S_X^+$ and r. The purpose of this section is to show that the set-valued map introduced in the previous section is well-defined, that is, it always assumes non-empty values. We begin by fixing some notation.

Pick $k \in K$. It is easy to see that there are a fundamental system $\{U_i\}_{i\in I_k}$ of neighborhoods of k and a net $(f_i)_{i\in I_k}$ in $C_0(K)$ satisfying

(1) I_k is a partially ordered set;
(2) $0 \le f_i \le 1$, $f_i(k) = 1$ and $f_i(K \setminus U_i) = 0$ for all $i \in I_k$;
(3) $U_{i_2} \subset U_{i_1}$ and $f_{i_2} \le f_{i_1}$ when $i_1 \le i_2$;
(4) if $i_1,\dots,i_n \in I_k$ then there is an $i \in I_k$ such that $i_t \le i$ for each $t = 1,\dots,n$.

We will write $\{U_i, f_i\}_{i\in I_k} \leftrightarrow \{k\}$ to indicate that the above conditions are satisfied.

Next denote by δ_k the *Dirac measure* at k and observe that any vector measure $m\colon \Sigma_K \to \mathcal{B}(X,Y)$ can be written as $m = L\cdot\delta_k + n$, where $L \in \mathcal{B}(X,Y)$ and $n(\{k\}) = \mathbf{0}$. Recall that a vector measure $m\colon \Sigma_K \to \mathcal{B}(X,Y)$ is said to be *variationally regular* if given $\varepsilon > 0$ and $A \in \Sigma_K$ there is a compact set $C \subset A$ and an open set $U \supset A$ such that $\widetilde{m}(U \setminus C) < \varepsilon$.

Furthermore, recall that an X-valued function is called *totally mensurable* if it is the uniform limit of a sequence of X-valued simple functions.

The proof of the following lemma is straightforward.

Lemma 4.1. *Suppose that $m\colon \Sigma_K \to \mathcal{B}(X,Y)$ is a variationally regular measure that has bounded semivariation. Let $k \in K$ and $\{U_i\}_{i\in I}$ be a fundamental system of open neighborhoods of k. Let f_i be an X-valued, totally mensurable function satisfying $f_i(K\setminus U_i) = 0$, $\|f_i\| = 1$, and $f_i(k) = e$ for all $i \in I$, where $\|e\| = 1$. Then*

$$m(\{k\})(e) = \lim_i \int_K f_i\,dm.$$

Now for all $f \in C_0(K,X)$ and $\varepsilon > 0$ we set

$$\mathcal{K}(f,\varepsilon) := \{k \in K : \|f(k)\| \ge \varepsilon\}.$$

Proposition 4.2. *If $k \in K$, $e \in S_X^+$ and $0 < r < 1$ then $\Omega_r(k,e,T,S) \neq \emptyset$ and*

$$\Omega_r(k,e,T,S) = \bigcap_{i\in I_k} \mathcal{K}(T(f_i\cdot e),\, r/\|T^{-1}\|). \tag{4.1}$$

Proof. Firstly we will show

$$\bigcap_{i\in I_k} \mathcal{K}(T(f_i\cdot e), r/\|T^{-1}\|) \neq \emptyset. \tag{4.2}$$

We will prove that the collection $\{\mathcal{K}(T(f_i\cdot e), r/\|T^{-1}\|) : i \in I_k\}$ has the finite intersection property. Let $i_1,\dots,i_n \in I_k$ and define $h = \min_{1\le t\le n} f_{i_t}$. So, $f_{i_t}\cdot e \ge h\cdot e$ for all $t = 1,\dots,n$ and

$$\bigcap_{t=1}^{n} \mathcal{K}(T(f_{i_t}\cdot e),\, r/\|T^{-1}\|) \supset \mathcal{K}(T(h\cdot e),\, r/\|T^{-1}\|) \neq \emptyset,$$

because $\|h\cdot e\| = \|h\| = 1 > r$ and $\|T(h\cdot e)\| \ge 1/\|T^{-1}\| > r/\|T^{-1}\|$. Hence by compactness we deduce (4.2).

Now we will prove (4.1). Indeed, let $s \in \bigcap_{i\in I_k} \mathcal{K}(T(f_i\cdot e), r/\|T^{-1}\|)$. Then

$$\|T(f_i\cdot e)(s)\| \ge r/\|T^{-1}\|$$

for every $i \in I_k$. By [3, Theorem 5.1 and p. 156] the measure T_s is variationally regular. So, Lemma 4.1 implies that

$$T_s(\{k\})(e) = \lim_{i\in I_k} T(f_i\cdot e)(s) \tag{4.3}$$

in the norm topology. Thus

$$\|T_s(\{k\})(e)\| \ge r/\|T^{-1}\|,$$

that is, $s \in \Omega_r(k,e,T,S)$.

Conversely, if $s \in \Omega_r(k,e,T,S)$ then

$$\|T_s(\{k\})(e)\| \geq r/\|T^{-1}\|.$$

Since T_s is a positive measure, by [2, p. 182] there is $\psi \in S^+_{X^*}$ such that

$$\psi(T_s(\{k\})(e)) \geq r/\|T^{-1}\|.$$

Therefore by Lemma 4.1 and (4.3) we have

$$\begin{aligned}\psi(T_s(\{k\})(e)) &= \lim_{i\in I_k} \psi(T(f_i\cdot e)(s)) = \lim_{i\in I_k} T^*(\psi\cdot\delta_s)(f_i\cdot e)\\ &= \langle e, T^*(\psi\cdot\delta_s)(\{k\})\rangle.\end{aligned}$$

Write $T^*(\psi\cdot\delta_s) = \phi\cdot\delta_k + \nu$, where $\phi \in X^*$ and $\nu \in \mathrm{rcabv}(K,X^*)$ satisfies $\nu(\{k\}) = \mathbf{0}$. By Lemma 2.2, $T^*(\psi\cdot\delta_s)$ is a positive measure. Consequently, $\phi = T^*(\psi\cdot\delta_s)(\{k\}) \in X^*$ is positive and ν is a positive measure. Hence for any $i \in I_k$,

$$\begin{aligned}T^*(\psi\cdot\delta_s)(f_i\cdot e) &= (\phi\cdot\delta_k)(f_i\cdot e) + \nu(f_i\cdot e) = \langle e,\phi\rangle + \nu(f_i\cdot e)\\ &\geq \langle e,\phi\rangle = \langle e, T^*(\psi\cdot\delta_s)(\{k\})\rangle \geq r/\|T^{-1}\|.\end{aligned}$$

Consequently,

$$\|T(f_i\cdot e)(s)\| \geq \psi(T(f_i\cdot e)(s)) = T^*(\psi\cdot\delta_s)(f_i\cdot e) \geq r/\|T^{-1}\|$$

for all $i \in I_k$. Thus, $s \in \bigcap_{i\in I_k} \mathcal{K}(T(f_i\cdot e), r/\|T^{-1}\|)$, and the proposition is proved. ∎

5. $\Omega_r(k,e,T,S)$ is finite for all $k \in K$, $e \in S^+_X$ and r close to 1. In this section we show that $\Omega_r(k,e,T,S)$ is finite for all $k \in K$ and $e \in S^+_X$ provided that r satisfies

$$\|T\|\,\|T^{-1}\|/\lambda^+(X) < r < 1. \tag{5.1}$$

We start by recalling a property of the parameter $\lambda(X)$ obtained in [5, Proposition 5.1] for all Banach spaces X. We restate it for the parameter $\lambda^+(X)$ for all Banach lattices X.

LEMMA 5.1. *Let X be a Banach lattice. Let $m \in \mathbb{N}$ and $d > 0$. Suppose that $x_1,\ldots,x_{2^m} \in C_X$ satisfy $\|x_t\| \geq d$ for all $1 \leq t \leq 2^m$. Then there are $a_1,\ldots,a_{2^m} \in \mathbb{R}$ with $|a_t| \leq 1$ for $1 \leq t \leq 2^m$ such that*

$$\Big\|\sum_{t=1}^{2^m} a_t x_t\Big\| \geq d\lambda^+(X)^m.$$

Proof. The result is easily established for $m = 1$. So, assume that it holds for any $1 \leq t \leq m$ and let $x_1,\ldots,x_{2^{m+1}}$ be vectors in C_X such that $\|x_p\| \geq d$ for $1 \leq p \leq 2^{m+1}$. We can find scalars $\beta_1,\ldots,\beta_{2^m},\ldots,\beta_{2^{m+1}}$ with

$|\beta_p| \le 1$ for $1 \le p \le 2^{m+1}$ satisfying

$$M_1 := \Big\|\sum_{p=1}^{2^m} |\beta_p| x_p\Big\| \ge d\lambda^+(X)^m \quad\text{and}\quad M_2 := \Big\|\sum_{p=2^m+1}^{2^{m+1}} |\beta_p| x_p\Big\| \ge d\lambda^+(X)^m.$$

Let

$$x = \frac{1}{M_1}\sum_{p=1}^{2^m} |\beta_p| x_p \quad\text{and}\quad y = \frac{1}{M_2}\sum_{p=2^m+1}^{2^{m+1}} |\beta_p| x_p.$$

Notice that $x, y \in S_X^+$. By the case $m = 1$ there exist $b_1, b_2 \in \mathbb{R}$ with $|b_1|, |b_2| \le 1$ such that

$$\|b_1 x + b_2 y\| \ge d\lambda^+(X). \tag{5.2}$$

By setting $M_0 = \min\{M_1, M_2\}$ we have $M_0 \ge d\lambda^+(X)^m$, and (5.2) implies

$$\Big\|\sum_{p=1}^{2^m} \frac{b_1 M_0}{M_1}|\beta_p| x_p + \sum_{p=2^m+1}^{2^{m+1}} \frac{b_2 M_0}{M_2}|\beta_p| x_p\Big\| \ge d\lambda^+(X)^{m+1}.$$

The conclusion follows by taking

$$a_p = \begin{cases} \dfrac{b_1 M_0}{M_1}|\beta_p| & \text{if } 1 \le p \le 2^m,\\[2mm] \dfrac{b_2 M_0}{M_2}|\beta_p| & \text{if } 2^m + 1 \le p \le 2^{m+1}. \ \blacksquare \end{cases}$$

PROPOSITION 5.2. *If $k \in K$, $e \in S_X^+$ and r satisfies* (5.1) *then $\Omega_r(k, e, T, S)$ is finite.*

Proof. Assume that $\{U_i, f_i\}_{i\in I_k} \leftrightarrow \{k\}$ and let $s \in \Omega_r(k, e, T, S)$. Take a non-negative $g_s \in C_0(S)$ such that

$$g_s(s) = \|g_s\| \le 1/\|T^{-1}\|.$$

Consider $G_s := g_s \cdot e \in C_0(S, X)$. Notice that $G_s \ge 0$ and

$$\|G_s(s)\| = 1/\|T^{-1}\| > r/\|T^{-1}\|.$$

Therefore Lemma 5.1 and (4.1) imply that for each $i \in I_k$ there are $a_i, b_i \in \mathbb{R}$ with $|a_i|, |b_i| \le 1$ such that

$$\|a_i T(f_i \cdot e)(s) + b_i G_s(s)\| \ge r\lambda^+(X)/\|T^{-1}\|.$$

So,

$$\|a_i T(f_i \cdot e) + b_i G_s\| \ge \|a_i T(f_i \cdot e)(s) + b_i G_s(s)\| \ge r\lambda^+(X)/\|T^{-1}\|.$$

Hence for each $i \in I_k$,

$$\|a_i(f_i \cdot e) + b_i T^{-1}(G_s)\| > r\lambda^+(X)/(\|T\|\,\|T^{-1}\|) > 1.$$

Let $W_i = \{k' \in K : f_i(k') \ne 0\}$ for $i \in I_k$. Observe that $\{W_i\}_{i\in I_k}$ is a fundamental system of open neighborhoods of k because W_i is open and

$W_i \subset U_i$. Moreover, since $\|T^{-1}(G_s)\| \le 1$, if $i \in I_k$ then for some $k_i \in W_i$,

$$\|T^{-1}(G_s)(k_i)\| \ge \delta, \quad \text{where} \quad \delta := r\lambda^+(X)/(\|T\|\,\|T^{-1}\|) - 1 > 0.$$

From the continuity of $T^{-1}(G_s)$, we obtain

$$\|T^{-1}(G_s)(k)\| \ge \delta. \tag{5.3}$$

Now assume that $\Omega_r(k,e,T,S)$ is infinite. Let $s_1,\dots,s_{2^m} \in \Omega_r(k,e,T,S)$ and let $V_1,\dots,V_{2^m}$ be disjoint open subsets of S such that $s_p \in V_p$ for $p = 1,\dots,2^m$. For each $p = 1,\dots,2^m$ let $g_{s_p} \in C_0(S)$ be non-negative with

$$g_{s_p}(s_p) = \|g_{s_p}\| = 1/\|T^{-1}\| \quad \text{and} \quad g_{s_p}(S \setminus V_p) = 0.$$

If $G_p = g_{s_p} \cdot e$ then by (5.3) and Lemma 5.1 there are $a_1,\dots,a_{2^m} \in \mathbb{R}$ with $|a_p| \le 1$ for $1 \le p \le 2^m$ satisfying

$$\Big\|\sum_{p=1}^{2^m} a_p T^{-1}(G_p)(k)\Big\| \ge \delta\lambda^+(X)^m. \tag{5.4}$$

On the other hand,

$$\Big\|\sum_{p=1}^{2^m} a_p T^{-1}(G_p)(k)\Big\| \le \Big\|\sum_{p=1}^{2^m} a_p T^{-1}(G_p)\Big\| \le \Big\|T^{-1}\Big(\sum_{p=1}^{2^m} a_p G_p\Big)\Big\| \le 1.$$

This is impossible by (5.4). So, $\Omega_r(k,e,T,S)$ must be finite. ∎

6. Two auxiliary maps ρ and τ. In order to define two maps ρ and τ which will be used to establish a homeomorphism between K and S, we need to introduce some new sets. From now on we assume that r satisfies inequality (5.1). We set

$$\Omega_k = \bigcup_{e \in S_X^+} \Omega_r(k,e,T,S) \quad \text{and} \quad \Delta_s = \bigcup_{e \in S_X^+} \Omega_r(s,e,T^{-1},K),$$

for $k \in K$ and $s \in S$. We also set

$$\Omega = \bigcup_{k \in K} \Omega_k \quad \text{and} \quad \Delta = \bigcup_{s \in S} \Delta_s.$$

From Proposition 4.2 we know that $\Omega_k \ne \emptyset$ for all $k \in K$, and $\Delta_s \ne \emptyset$ for all $s \in S$.

Let $\rho\colon \Omega \to K$ be defined by $\rho(s) = k$ if and only if $s \in \Omega_k$. Analogously, let $\tau\colon \Delta \to S$ be defined by $\tau(k) = s$ if and only if $k \in \Delta_s$. The next proposition shows that ρ and τ are well-defined.

PROPOSITION 6.1. *Let $k_1, k_2 \in K$. If $k_1 \ne k_2$, then $\Omega_{k_1} \cap \Omega_{k_2} = \emptyset$. Moreover, if $s_1, s_2 \in S$ are distinct, then $\Delta_{s_1} \cap \Delta_{s_2} = \emptyset$.*

Proof. Suppose that there is $s \in \Omega_{k_1} \cap \Omega_{k_2}$. Then for some $e_1, e_2 \in S_X^+$ we have $s \in \Omega_r(k_1,e_1,T,S) \cap \Omega_r(k_2,e_2,T,S)$. By the definition of these sets we see that

$$\|T_s(\{k_1\})(e_1)\| \geq r/\|T^{-1}\| \quad \text{and} \quad \|T_s(\{k_2\})(e_2)\| \geq r/\|T^{-1}\|.$$

By Lemma 5.1 there exist $a_1, a_2 \in \mathbb{R}$ with $|a_1|, |a_2| \leq 1$ such that

$$\|a_1 T_s(\{k_1\})(e_1) + a_2 T_s(\{k_2\})(e_2)\| \geq r\lambda^+(X)/\|T^{-1}\|.$$

On the other hand, from definition of semivariation we have

$$\|T\| \geq \|T_s\| \geq \|a_1 T_s(\{k_1\})(e_2) + a_2 T_s(\{k_2\})(e_2)\| \geq r\lambda^+(X)/\|T^{-1}\|,$$

which contradicts (5.1). The "moreover" part follows in the same way. ∎

7. The maps $\rho\colon \Omega \to K$ and $\tau\colon \Delta \to S$ are injective. In this section we will prove that the maps ρ and τ defined above are injective maps (Proposition 7.5). We begin with the following observation.

Remark 7.1. Let $k \in K$ and $e \in S_X^+$. If $s \in S$ satisfies

$$\|T_s(\{k\})(e)\| = \Big\|\lim_i T(f_i \cdot e)(s)\Big\| \geq r/\|T^{-1}\|$$

for $\{U_i, f_i\}_{i \in I_k} \leftrightarrow \{k\}$ then s belongs to the finite set $\Omega_r(k, e, T, S)$. So,

$$\sup_{s' \in S} \Big\|\lim_i T(f_i \cdot e)(s')\Big\|$$

exists and it is attained at some point in $\Omega_r(k, e, T, S)$.

In order to justify injectivity of ρ and τ we prove the following proposition.

Proposition 7.2. *Let $s \in S$ and $\{U_i, f_i\}_{i \in I_k} \leftrightarrow \{k\}$ be such that*

$$\sup_{s' \in S} \Big\|\lim_i T(f_i \cdot e)(s')\Big\| = \Big\|\lim_i T(f_i \cdot e)(s)\Big\|,$$

and set

$$u = \lim_i T(f_i \cdot e)(s) \Big/ \Big\|\lim_i T(f_i \cdot e)(s)\Big\|.$$

If $\{V_j, g_j\}_{j \in J_s} \leftrightarrow \{s\}$ then for each $k' \in K \setminus \{k\}$ we have

$$\Big\|\lim_j T^{-1}(g_j \cdot u)(k')\Big\| \leq \|T^{-1}\|/2.$$

Proof. Assume that there is some $k' \neq k$ satisfying

$$\Big\|\lim_j T^{-1}(g_j \cdot u)(k')\Big\| > \|T^{-1}\|/2.$$

Let $c = \lim_j T^{-1}(g_j \cdot u)(k')$ and pick some $\psi \in S_{X^*}$ such that $\langle c, \psi \rangle = \|c\|$. By Theorem 2.1 we can write

$$(T^{-1})^*(\psi \cdot \delta_{k'}) = \phi \cdot \delta_s + \eta,$$

where $\phi \in X^*$ and $\eta \in \mathrm{rcabv}(S, X^*)$ satisfies $\eta(\{s\}) = \mathbf{0}$. Thus,

$$\|c\| = \langle c, \psi\rangle = \lim_j \langle T^{-1}(g_j \cdot u)(k'), \psi\rangle = \lim_j (T^{-1})^*(\psi \cdot \delta_{k'})(g_j \cdot u)$$
$$= \lim_j \int_S g_j \cdot u \, d(T^{-1})^*(\psi \cdot \delta_{k'}) = \langle u, \phi\rangle.$$

This implies that $\|\phi\| \ge \|c\| > \|T^{-1}\|/2$ since $\|u\| = 1$. So,

$$|\eta| = \|(T^{-1})^*(\psi \cdot \delta_{k'})\| - \|\phi\| \le \|T^{-1}\| - \|c\| < \|T^{-1}\|/2.$$

Let $v = \lim_i T(f_i \cdot e)(s)$. Then $u = v/\|v\|$ and we have

$$\lim_i \langle T(f_i \cdot e)(s), \phi\rangle = \langle \|v\|u, \phi\rangle = \|v\|\,\|c\|.$$

Since $\|T^{-1}\| - \|c\| < \|c\|$, we can choose $\varepsilon > 0$ such that

$$(\|v\| + \varepsilon)(\|T^{-1}\| - \|c\|) < (\|v\| - \varepsilon)\|c\|. \tag{7.1}$$

Let $\Omega_r(k, e, T, S) = \{s, s_1, \ldots, s_q\}$ and write

$$\eta = \sum_{t=1}^{q} \phi_t \cdot \delta_{s_t} + m$$

where $\phi_t \in X^*$, $1 \le t \le q$, and $m \in \mathrm{rcabv}(S, X^*)$ satisfies $m(\{s\}) = m(\{s_t\}) = \mathbf{0}$ for $1 \le t \le q$. By the choice of s, there is $i_1 \in I_k$ such that if $i \ge i_1$ then

$$|\langle T(f_i \cdot e)(s), \phi\rangle| > (\|v\| - \varepsilon)\|c\|, \tag{7.2}$$
$$|\langle T(f_i \cdot e)(s_t), \phi_t\rangle| < (\|v\| + \varepsilon)\|\phi_t\| \quad \text{for } 1 \le t \le q. \tag{7.3}$$

Note that $|m|(\Omega_r(k, e, T, S)) = 0$, so by the regularity of m there exists a compact subset H of $S \setminus \Omega_r(k, e, T, S)$ such that

$$|m|(S \setminus H) \le (\|v\| + \varepsilon - r/\|T^{-1}\|)|m|/\|T\|. \tag{7.4}$$

Since H is compact and $H \cap \Omega_r(k, e, T, S) = \emptyset$, inequality (4.1) implies that there are $i'_1, \ldots, i'_n \in I_k$ satisfying

$$H \cap \mathcal{K}(T(f_{i'_1} \cdot e), r/\|T^{-1}\|) \cap \cdots \cap \mathcal{K}(T(f_{i'_n} \cdot e), r/\|T^{-1}\|) = \emptyset.$$

Let $i_2 \in I_k$ be such that $i_2 \ge i'_p$ for $1 \le p \le n$. Then $f_{i_2} \le f_{i'_p}$ for any $1 \le p \le n$, and by positivity of T we have

$$\mathcal{K}(T(f_{i_2} \cdot e), r/\|T^{-1}\|) \subset \mathcal{K}(T(f_{i'_1} \cdot e), r/\|T^{-1}\|) \cap \cdots \cap \mathcal{K}(T(f_{i'_n} \cdot e), r/\|T^{-1}\|).$$

Therefore

$$H \cap \mathcal{K}(T(f_i \cdot e), r/\|T^{-1}\|) = \emptyset$$

for every $i \ge i_2$. That is,

$$\|T(f_i \cdot e)(s')\| < r/\|T^{-1}\| \quad \text{for each } s' \in H. \tag{7.5}$$

Next, we can choose $i_0 \in I_k$ such that $i_0 \ge i_1$, $i_0 \ge i_2$ and $k' \notin U_i$ for all $i \ge i_0$. Thus if $i \ge i_0$ then

$$\begin{aligned} 0 = \int_K f_i \cdot e\, d\,(\psi\cdot\delta_{k'}) &= \int_S T(f_i\cdot e)\, d(T^{-1})^*(\psi\cdot\delta_{k'}) \\ &= \int_S T(f_i\cdot e)\, d(\phi\cdot\delta_s) + \sum_{t=1}^{q}\int_S T(f_i\cdot e)\, d(\phi_t\cdot\delta_{s_t}) \\ &\quad + \int_{S\setminus H} T(f_i\cdot e)\, dm + \int_H T(f_i\cdot e)\, dm \\ &= \langle T(f_i\cdot e)(s),\phi\rangle + \sum_{t=1}^{q}\langle T(f_i\cdot e)(s_t),\phi_t\rangle \\ &\quad + \int_{S\setminus H} T(f_i\cdot e)\, dm + \int_H T(f_i\cdot e)\, dm. \end{aligned}$$

Now, for $i \ge i_0$ inequality (7.2) shows that the modulus of the first term on the right is greater than $(\|v\|-\varepsilon)\|c\|$. On the other hand, combining (7.3)–(7.5) we see that the modulus of the remaining terms is less than

$$\begin{aligned} (\|v\|+\varepsilon)\sum_{t=1}^{q}\|\phi_t\| + \bigg(\|v\|+\varepsilon-\frac{r}{\|T^{-1}\|}\bigg)|\,m| + \frac{r}{\|T^{-1}\|}|\,m| \\ = (\|v\|+\varepsilon)|\,\eta| \le (\|v\|+\varepsilon)(\|T^{-1}\|-\|c\|), \end{aligned}$$

which contradicts (7.1). ∎

Remark 7.3. Let $k\in K$, $s\in S$, $u\in S_X^+$ and $\{V_j,g_j\}_{j\in J_s}$ be as in the statement of Proposition 7.2. Then

$$\Big\|\lim_j T^{-1}(g_j\cdot u)(k)\Big\| \ge r/\|T\|.$$

Indeed, by Proposition 4.2 we have $\Omega_r(s,u,T^{-1},K)\neq\emptyset$. So, we have $k_0\in \Omega_r(s,u,T^{-1},K)$ for some $k_0\in K$, that is,

$$\|T_{k_0}^{-1}(\{s\})(u)\| = \Big\|\lim_j T^{-1}(g_j\cdot u)(k_0)\Big\| \ge r/\|T\|.$$

On the other hand, if $k_0\neq k$ then Proposition 7.2 would imply that

$$\Big\|\lim_j T^{-1}(g_j\cdot u)(k_0)\Big\| \le \|T^{-1}\|/2,$$

which is impossible since

$$\|T^{-1}\|/2 \le \|T^{-1}\|/\lambda^+(X) < r/\|T\|.$$

Proposition 7.4. *If $k\in K$ and $s\in S$ then $s\in\Omega_k$ if and only if $k\in\Delta_s$.*

Proof. Suppose that $s\in\Omega_k$ but $k\notin\Delta_s$. Then there exists $e\in S_X^+$ such that $s\in\Omega_r(k,e,T,S)$. Let $k_0\in\Omega_r(s,e,T^{-1},K)$ be such that

$$\sup_{k\in K}\Big\|\lim_j T^{-1}(g_j\cdot e)(k)\Big\|$$

is attained at k_0. Let $b \in S_X^+$ be defined as

$$b = \lim_j T^{-1}(g_j \cdot e)(k_0) \Big/ \Big\| \lim_j T^{-1}(g_j \cdot e)(k_0) \Big\|.$$

By Proposition 7.2 and Remark 7.3 (applied to T^{-1} instead of T) it follows that if $\{U_i', f_i'\}_{i \in I_{k_0}} \leftrightarrow \{k_0\}$ then

$$\Big\| \lim_i T(f_i' \cdot b)(s') \Big\| \le \|T\|/2 \quad \text{for } s' \in S \setminus \{s\}$$

and

$$\|T_s(\{k_0\})(b)\| = \Big\| \lim_i T(f_i' \cdot b)(s) \Big\| \ge r/\|T^{-1}\|. \tag{7.6}$$

Observe that $k_0 \ne k$. Now since $s \in \Omega_r(k, e, T, S)$ we have

$$\|T_s(\{k\})(e)\| = \Big\| \lim_i T(f_i \cdot e)(s) \Big\| \ge r/\|T^{-1}\|. \tag{7.7}$$

In view of (7.6) and (7.7) and Lemma 5.1 there are $a_1, a_2 \in \mathbb{R}$ with $|a_1|, |a_2| \le 1$ such that

$$\|a_1 T_s(\{k_0\})(b) + a_2 T_s(\{k\})(e)\| \ge r\lambda^+(X)/\|T^{-1}\|.$$

By using the definition of semivariation we have

$$\|T\| \ge \|T_s\| \ge \|a_1 T_s(\{k_0\})(b) + a_2 T_s(\{k\})(e)\| \ge r\lambda^+(X)/\|T^{-1}\|,$$

which is absurd by (5.1). So, $s \in \Omega_k$ implies that $k \in \Delta_s$. Similarly, $k \in \Delta_s$ implies $s \in \Omega_k$. ∎

Now we are in a position to prove the main result of this section.

PROPOSITION 7.5. *The maps $\rho\colon \Omega \to K$ and $\tau\colon \Delta \to S$ are injective. Moreover, $\Omega = S$ and $\Delta = K$.*

Proof. Let $k \in K$ be such that $\rho(s_1) = \rho(s_2) = k$ for some $s_1, s_2 \in \Omega$. By definition of ρ we have $s_1, s_2 \in \Omega_k$. Suppose that $s_1 \ne s_2$. By Proposition 7.4 we infer that $k \in \Delta_{s_1} \cap \Delta_{s_2}$, but this contradicts Proposition 6.1. The same argument shows that τ is injective.

Now, let $s \in S$. Then there is $k \in K$ such that $\tau(k) = s$. The definition of τ implies that $k \in \Delta_s$. By Proposition 7.4 we have $s \in \Omega_k$. So, $s \in \Omega$ and we conclude that $\Omega = S$. Analogously we see that $\Delta = K$. ∎

8. The proof of the main theorem. Finally, we are ready to prove Theorem 1.3. By Propositions 4.2 and 7.4 we infer that $\rho\colon S \to K$ and $\tau\colon K \to S$ are bijective. Notice also that if $s \in S$ and $\rho(s) = k$ then $s \in \Omega_k$, and Proposition 7.4 implies $k \in \Delta_s$, that is, $\tau(k) = s$. So, $\rho = \tau^{-1}$ and therefore the next proposition completes the proof of Theorem 1.3.

PROPOSITION 8.1. *$\rho\colon S \to K$ is a homeomorphism.*

Proof. We will prove that ρ is continuous. Let $s \in S$. Take a net $(s_\gamma)_{\gamma\in\Gamma}$ converging to s and assume that $\rho(s_\gamma) = k_\gamma$ does not converge to $k = \rho(s)$. Then there is a compact neighborhood V of k such that for all $\gamma_0 \in \Gamma$, there exists $\gamma \geq \gamma_0$ with $k_\gamma \notin V$. Let $h > 0$ be given. Since $\rho(s) = k$, for some $b \in S_X^+$ we have

$$\|T_s(\{k\})(b)\| = \left\|\lim_i T(f_i \cdot e)(s)\right\| \geq r/\|T^{-1}\| > (r-h)/\|T^{-1}\|,$$

where $\{U_i, f_i\}_{i\in I_k} \leftrightarrow \{k\}$. Thus there is $i_0 \in I_k$ such that for $i \geq i_0$,

$$\|T(f_i \cdot b)(s)\| > (r-h)/\|T^{-1}\|.$$

Let $i_1 \geq i_0$ be such that $k \in U_{i_1} \subset V$ and

$$\|T(f_{i_1} \cdot b)(s)\| > (r-h)/\|T^{-1}\|.$$

From the convergence of $(s_\gamma)_{\gamma\in\Gamma}$ to s, we can find $\gamma_{i_1} \in \Gamma$ such that if $\gamma \geq \gamma_{i_1}$ then

$$\|T(f_{i_1} \cdot b)(s_\gamma)\| > (r-h)/\|T^{-1}\|.$$

Fix $\gamma \geq \gamma_{i_1}$ satisfying

$$\|T(f_{i_1} \cdot b)(s_\gamma)\| > (r-h)/\|T^{-1}\| \tag{8.1}$$

and $k_\gamma \notin V$. Let V_2 be an open subset of K with $k_\gamma \in V_2$ and $V_2 \cap V = \emptyset$. Now, we know that $\rho(s_\gamma) = k_\gamma$, thus there is $b_\gamma \in S_X^+$ such that

$$\|T_{s_\gamma}(\{k_\gamma\})(b_\gamma)\| = \left\|\lim_i T(f'_i \cdot b_\gamma)(s_\gamma)\right\| \geq r/\|T^{-1}\| > (r-h)/\|T^{-1}\|,$$

where $\{U'_i, f'_i\}_{i\in I_{k_\gamma}} \leftrightarrow \{k_\gamma\}$. Hence for some $i_2 \in I_{k_\gamma}$, if $i \geq i_2$ then

$$\|T(f'_i \cdot b_\gamma)(s_\gamma)\| > (r-h)/\|T^{-1}\|.$$

Take $i_3 \geq i_2$ with $k_\gamma \in U'_{i_3} \subset V_2$ and

$$\|T(f'_{i_3} \cdot b_\gamma)(s_\gamma)\| > (r-h)/\|T^{-1}\|. \tag{8.2}$$

In view of (8.1) and (8.2), by Lemma 5.1 there exist $\beta_1, \beta_2 \in \mathbb{R}$ with $|\beta_1|, |\beta_2| \leq 1$ such that

$$(r-h)\lambda^+(X)/\|T^{-1}\| \leq \|\beta_1 T(f_{i_1} \cdot b)(s_\gamma) + \beta_2 T(f'_{i_3} \cdot b_\gamma)(s_\gamma)\|.$$

Since $\|f_{i_1}\| = \|f'_{i_3}\| = 1$ and $f_{i_1} \cdot f'_{i_3} = 0$, we have

$$\|\beta_1(f_{i_1} \cdot b) + \beta_2(f'_{i_3} \cdot b_\gamma)\| \leq 1.$$

Thus

$$\begin{aligned}(r-h)\lambda^+(X)/\|T^{-1}\| &\leq \|\beta_1 T(f_{i_1} \cdot b)(s_\gamma) + \beta_2 T(f'_{i_3} \cdot b_\gamma)(s_\gamma)\| \\ &\leq \|T(\beta_1(f_{i_1} \cdot b) + \beta_2(f'_{i_3} \cdot b_\gamma))\| \leq \|T\|.\end{aligned}$$

When $h \to 0$, we obtain

$$\|T\| \geq r\lambda^+(X)/\|T^{-1}\|,$$

which contradicts (5.1). This proves the continuity of ρ, and the continuity of τ follows similarly. ∎

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