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ON THE k-FOLD ITERATE OF THE SUM OF DIVISORS FUNCTION

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Abstract. Let $\gamma(n)$ stand for the product of the prime factors of n. The index of composition $\lambda(n)$ of an integer $n \geq 2$ is defined as $\lambda(n) = \log n/\log \gamma(n)$ with $\lambda(1) = 1$. Given an arbitrary integer $k \geq 0$ and letting $\sigma_k(n)$ be the k-fold iterate of the sum of divisors function, we show that, given any real number $\varepsilon > 0$, $\lambda(\sigma_k(n)) < 1+\varepsilon$ for almost all positive integers n.

1. Introduction and notation. Let $\gamma(n)$ stand for the product of the prime factors of the positive integer n. In 2003, De Koninck and Doyon [DD] studied the mean value and various other properties of the *index of composition* of an integer, defined for $n \geq 2$ by $\lambda(n) := \log n/\log \gamma(n)$, with $\lambda(1) = 1$. Later, others (see [DK], [DKS], [ZZ]) further studied the behaviour of this function. Of particular interest is the result of De Koninck and Luca [DL] who showed that the normal order of $\lambda(\sigma(n))$ is 1, where $\sigma(n)$ stands for the sum of divisors function.

Given an arbitrary integer $k \geq 0$, let $\sigma_k(n)$ stand for the k-fold iterate of $\sigma(n)$, that is, $\sigma_0(n) = n$, $\sigma_1(n) = \sigma(n)$, $\sigma_2(n) = \sigma(\sigma_1(n))$, and so on. Here, for any integer $k \geq 0$ and any real $\varepsilon > 0$, we show that $\lambda(\sigma_k(n)) < 1+\varepsilon$ for almost all positive integers n.

We denote by $p(n)$ and $P(n)$ the smallest and largest prime factors of n, respectively. We write $\Pi(n)$ for the largest prime power dividing n. The letters p, q, π and Q, with or without subscript, will stand exclusively for primes. On the other hand, the letters c and C, with or without subscript, will stand for absolute constants but not necessarily the same at each occurrence. Moreover, we shall use the abbreviations $x_1 = \log x$, $x_2 = \log\log x$, and so on. Finally, given any real number $x \geq 1$, we let $\mathcal{N}_x = \{1, 2, \ldots, \lfloor x \rfloor\}$.

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2. Main results

THEOREM 2.1. *For any fixed integer $k \geq 0$ and real $\varepsilon > 0$,*

$$\frac{1}{x}\#\{n \leq x : \lambda(\sigma_k(n)) \geq 1+\varepsilon\} \to 0 \quad (x \to \infty). \tag{2.1}$$

REMARK 2.2. The case $k = 0$, namely that the normal order of $\lambda(n)$ is one, was proved by De Koninck and Doyon [DD]. The case $k = 1$ was settled by De Koninck and Luca [DL], who actually proved more, namely that

$$\frac{1}{x}\sum_{n\leq x}\lambda(\sigma(n)) \to 1 \quad (x \to \infty).$$

We could not generalize the approach used in [DL] to prove (2.1) for any $k \geq 2$. We will therefore use a totally different approach.

On the other hand, if ϕ stands for the Euler totient function and $\phi_k(n)$ for the k-fold iterate of $\phi(n)$, it turns out that the next theorem is much easier to prove than Theorem 2.1.

THEOREM 2.3. *For any fixed integer $k \geq 0$ and real $\varepsilon > 0$,*

$$\frac{1}{x}\#\{n \leq x : \lambda(\phi_k(n)) \geq 1+\varepsilon\} \to 0 \quad (x \to \infty).$$

Finally, let $\sigma^*(n)$ stand for the sum of the unitary divisors of n, and $\sigma_k^*(n)$ for its k-fold iterate. We can then prove the following.

THEOREM 2.4. *For any fixed integer $k \geq 0$ and real $\varepsilon > 0$,*

$$\frac{1}{x}\#\{n \leq x : \lambda(\sigma_k^*(n)) \geq 1+\varepsilon\} \to 0 \quad (x \to \infty).$$

3. Preliminary lemmas

LEMMA 3.1. *For all integers k and ℓ, let*

$$\delta(x,k,\ell) := \sum_{\substack{p\leq x\\ p\equiv \ell \,(\mathrm{mod}\, k)}} \frac{1}{p}.$$

Then, for $\ell = 1$ or -1, $k \leq x$, and $x \geq 3$, we have

$$\delta(x,k,\ell) \leq \frac{C_1 x_2}{\phi(k)},$$

where $C_1 > 0$ is an absolute constant.

Proof. This is Lemma 2.5 in Bassily, Kátai and Wijsmuller [BKW]. ∎

We say that a k+1-tuple of primes $(q_0, q_1, \ldots, q_k)$ is a *k-chain* if $q_{i-1} \mid q_i + 1$ for $i = 1, \ldots, k$, in which case we write $q_0 \to q_1 \to \cdots \to q_k$. We shall need the following result.

LEMMA 3.2. *For any fixed prime q_0 and integer $k \geq 1$, there exist absolute constants $c_1, \ldots, c_k$ such that*

$$\sum_{\substack{q_1 \leq x \\ q_0 \to q_1}} \frac{1}{q_1} \leq \frac{c_1 x_2}{q_0}, \qquad \sum_{\substack{q_2 \leq x \\ q_0 \to q_1 \to q_2}} \frac{1}{q_2} \leq \frac{c_2 x_2^2}{q_0}, \quad \ldots, \quad \sum_{\substack{q_k \leq x \\ q_0 \to q_1 \to \cdots \to q_k}} \frac{1}{q_k} \leq \frac{c_k x_2^k}{q_0}.$$

Proof. Using Lemma 3.1, we get, for some constant $c_1 > 0$,

$$\sum_{\substack{q_1 \leq x \\ q_0 \to q_1}} \frac{1}{q_1} \leq \frac{C_1 x_2}{\phi(q_0)} = \frac{C_1 x_2}{q_0 - 1} \leq \frac{c_1 x_2}{q_0}, \tag{3.1}$$

which proves the first inequality. To obtain the second one, observe that, by (3.1), for some constant $c_2 > 0$,

$$\sum_{\substack{q_2 \leq x \\ q_0 \to q_1 \to q_2}} \frac{1}{q_2} = \sum_{\substack{q_1 \leq x \\ q_0 \to q_1}} \sum_{\substack{q_2 \leq x \\ q_1 \to q_2}} \frac{1}{q_2} \leq \sum_{\substack{q_1 \leq x \\ q_0 \to q_1}} \frac{c_1 x_2}{q_1} = c_1 x_2 \sum_{\substack{q_1 \leq x \\ q_0 \to q_1}} \frac{1}{q_1}$$
$$\leq c_1 x_2 \frac{c_1 x_2}{q_0} = \frac{c_2 x_2^2}{q_0},$$

thus establishing the second inequality. Proceeding in the same manner, we deduce the other inequalities. ∎

4. Proof of the theorems. We only prove Theorem 2.1 since the proofs of Theorems 2.3 and 2.4 are similar.

We first introduce the sequence $(w_k)_{k \geq 0} = (w_k(x))_{k \geq 0}$ of real functions satisfying

$$\log w_k(x) = x_2^{m_k}, \tag{4.1}$$

where $0 < m_0 < m_1 < \cdots$ is a suitable sequence of integers, to be determined later.

Our plan is to introduce our approach in the cases $k = 0$ and $k = 1$, and then to use induction on k.

For $k = 0$, we first write each positive integer $n \leq x$ as

$$\sigma_0(n) = n = A_0(n) B_0(n),$$

where $B_0(n) := \prod_{q \mid n,\, q > w_0} q$ and $A_0(n) = n / B_0(n)$. Then, let $Y_x \to \infty$ as $x \to \infty$ with $Y_x \leq x_5$ and consider the set

$$\mathcal{U}_x^{(0)} := \{n \in \mathcal{N}_x : \mu(B_0(n)) = 0 \text{ or } \Pi(A_0(n)) > Y_x^{Y_x} \text{ or } P(A_0(n)) \geq Y_x\},$$

where μ stands for the Möbius function; observe that

$$\# \mathcal{U}_x^{(0)} = o(x) \quad (x \to \infty). \tag{4.2}$$

Now setting

$$\mathcal{N}_x^{(1)} := \mathcal{N}_x \setminus \mathcal{U}_x^{(0)},$$

we find that, for $n \in \mathcal{N}_x^{(1)}$, $B_0(n)$ is squarefree and $(A_0(n), B_0(n)) = 1$, thus allowing us to write

$$\sigma(n) = \sigma(A_0(n))\sigma(B_0(n)) \quad (n \in \mathcal{N}_x^{(1)}).$$

To each prime number q, we associate the strongly additive function f_q defined on primes p by

$$f_q(p) = \begin{cases} k & \text{if } q^k \parallel p+1, \\ 0 & \text{if } q \nmid p+1. \end{cases}$$

Then, we set

$$s(n) := \prod_{q \le x_2^2} q^{f_q(n)} \quad \text{and} \quad E(x) := \sum_{n \in \mathcal{N}_x^{(1)}} \log s(n) = \sum_{n \in \mathcal{N}_x^{(1)}} \sum_{q \le x_2^2} (\log q) f_q(n).$$

We have, in light of Lemma 3.1,

$$E(x) \le \sum_{q \le x_2^2} (\log q) \sum_{q^k \le x} \sum_{q^k | p+1} \frac{1}{p} \le C_1 x x_2 \sum_{\substack{q \le x_2^2 \\ q^k \le x}} \frac{\log q}{\phi(q^k)} \le C_2 x x_2 x_3,$$

so that

$$s(n) < \exp(x_2 x_3 x_4) \quad \text{for } n \le x \text{ with at most } o(x) \text{ exceptions.} \tag{4.3}$$

Letting $\mathcal{U}_x^{(1)}$ be the set of those integers $n \in \mathcal{N}_x^{(1)}$ for which $q^2 \mid \sigma(B_0(n))$ for at least one prime $q > x_2^2$, we have, using Lemma 3.2,

$$\sum_{n \in \mathcal{N}_x^{(1)}} \sum_{\substack{q^2 | \sigma(B_0(n)) \\ q > x_2^2}} 1 \le \sum_{q > x_2^2} \sum_{\substack{p \le x \\ p+1 \equiv 0 \, (\mathrm{mod}\, q^2)}} \left\lfloor \frac{x}{p} \right\rfloor + \sum_{q > x_2^2} \sum_{\substack{p_1, p_2 \le x \\ q \to p_1,\, q \to p_2 \\ p_1 \ne p_2}} \left\lfloor \frac{x}{p_1 p_2} \right\rfloor$$
$$\le C_3 x x_2^2 \sum_{q > x_2^2} \frac{1}{q^2} \le C_4 \frac{x}{x_3},$$

implying that

$$\#\mathcal{U}_x^{(1)} = o(x) \quad (x \to \infty). \tag{4.4}$$

Letting $w_1 = w_1(x)$ be such that $\log w_1(x) = x_2^2$ (that is, choosing $m_1 = 2$ in (4.1)) and setting

$$r(n) := \prod_{\substack{q | \sigma(B_0(n)) \\ x_2^2 < q \le w_1}} q,$$

we deduce, again using Lemma 3.2, that

$$\sum_{n \le x} \log r(n) \le \sum_{x_2^2 < q \le w_1} (\log q) \sum_{\substack{p \le x \\ q \to p}} \frac{x}{p} \le C_5 x x_2 \sum_{q \le w_1} \frac{\log q}{q} \le C_6 x x_2^3. \tag{4.5}$$

We now set $\mathcal{N}_x^{(2)} := \mathcal{N}_x^{(1)} \setminus \mathcal{U}_x^{(1)}$ and

$$\mathcal{U}_x^{(2)} := \{n \in \mathcal{N}_x^{(2)} : s(n) > \exp(x_2x_3x_4) \text{ or } r(n) > \exp(x_2^3x_3)\}.$$

Thus, in light of (4.2)–(4.5), we have

$$\#\mathcal{U}_x^{(2)} = o(x) \quad (x \to \infty).$$

This motivates the definition

$$\mathcal{N}_x^{(3)} := \mathcal{N}_x^{(2)} \setminus \mathcal{U}_x^{(2)}.$$

Writing

$$A_1(n) = \sigma(A_0(n))s(n)r(n), \qquad B_1(n) = \prod_{\substack{q|\sigma(B_0(n))\\ q>w_1}} q,$$

we then obtain

$$A_1(n) \le \sigma(A_0(n)) \exp(2x_2^3x_3) \quad (n \in \mathcal{N}_x^{(3)}). \tag{4.6}$$

On the other hand,

$$\sigma(A_0(n)) \le CY_x^{Y_x} \log Y_x^{Y_x} \le x_3 \quad (n \in \mathcal{N}_x^{(3)}), \tag{4.7}$$

which implies that $(\sigma(A_0(n)), B_1(n)) = 1$, and since we obviously have $(s(n)r(n), B_1(n)) = 1$, we may conclude that

$$\sigma(n) = A_1(n)B_1(n),$$

where

$$(A_1(n), B_1(n)) = 1, \qquad B_1(n) \text{ is squarefree}, \qquad B_1(n) \mid \gamma(\sigma(n)).$$

Consequently, in light of (4.6) and (4.7),

$$\frac{\sigma(n)}{\gamma(\sigma(n))} \le A_1(n) \le x_3 \exp(2x_2^3x_3) \quad (n \in \mathcal{N}_x^{(3)}). \tag{4.8}$$

Now, write

$$\lambda(\sigma(n)) = \frac{\log \sigma(n)}{\log \gamma(\sigma(n))} = 1 + \frac{\log\big(\sigma(n)/\gamma(\sigma(n))\big)}{\log \gamma(\sigma(n))} = 1 + \theta_n, \tag{4.9}$$

say. Using (4.6) and (4.8), we find that

$$\theta_n \le \frac{\log\big(\sigma(n)/\gamma(\sigma(n))\big)}{\log(\sigma(n)/A_1(n))} \le \frac{x_4 + 2x_2^3x_3}{\log \sigma(n) - (x_4 + 2x_2^3x_3)}. \tag{4.10}$$

Since, for $n \in [x/x_1, x]$, we have $\log \sigma(n) > x_1 - x_2$, it follows from (4.10) that

$$\theta_n \le \frac{3x_2^3x_3}{x_1} \quad (n \in \mathcal{N}_x^{(3)}). \tag{4.11}$$

Using (4.11) in (4.9) proves (2.1) for the case $k = 1$.

Having proved our result for the cases $k = 0$ and $k = 1$, we now use induction. Assume that (2.1) is true for $j = 1, \ldots, k$. For $j = 1, \ldots, k$, let $\mathcal{N}_x^{(j)}$ be the sets with $\mathcal{N}_x \supseteq \mathcal{N}_x^{(1)} \supseteq \mathcal{N}_x^{(2)} \supseteq \cdots$ and

$$\sigma_j(n) = A_j(n)B_j(n) \quad (n \in \mathcal{N}_x^{(j)}),$$

where

$$B_j(n) = \prod_{\substack{q|\sigma(B_{j-1}(n))\\ q>w_j}} q \quad \text{and} \quad A_j(n) = \frac{\sigma_j(n)}{B_j(n)},$$

with $w_j = w_j(x)$ as in (4.1) and

$$\sigma(A_j(n)) < w_{j+1} \quad (n \in \mathcal{N}_x^{(j)}),$$

with

$$(A_j(n), B_j(n)) = 1, \quad B_j(n) \text{ is squarefree}, \quad p(B_j(n)) > w_j.$$

We can therefore write

$$\sigma_k(n) = A_k(n)B_k(n),$$

where

$$\begin{aligned} &\sigma(A_k(n)) < w_{k+1}, && p(B_k(n)) > w_k,\\ &B_k(n) \text{ is squarefree}, && (A_k(n), B_k(n)) = 1. \end{aligned}$$

Hence, $B_k(n)$ is a divisor of $\gamma(\sigma_k(n))$, and following the same argument as in the case $k = 1$, we conclude that (2.1) holds for k.

For the case $k + 1$, we first write

$$\sigma_{k+1}(n) = \sigma(A_k(n))\sigma(B_k(n)) \tag{4.12}$$

and set

$$s_k(n) = \prod_{\substack{\pi|\sigma(B_k(n))\\ \pi\le x_2^{2(k+1)}}} \pi^{f_\pi(\sigma(B_k(n)))}, \quad r_k(n) = \prod_{\substack{\pi|\sigma(B_k(n))\\ x_2^{2(k+1)}<\pi\le w_{k+1}}} \pi,$$

$$t_k(n) = \prod_{\substack{\pi|\sigma(B_k(n))\\ \pi>w_{k+1}}} \pi,$$

where, in each of the above products, π runs over primes. First observe that

$$\sum_{n\in\mathcal{N}_x^{(k)}} \log s_k(n) = \sum_{n\in\mathcal{N}_x^{(k)}} \sum_{\pi\le x_2^{2(k+1)}} f_\pi(\sigma(B_k(n)))\log\pi$$

$$= \sum_{\pi\le x_2^{2(k+1)}} (\log\pi) \sum_{\substack{\pi\to p_1\to\cdots\to p_{k+1}\\ p_1>w_k}} f_\pi(p_1)\left\lfloor\frac{x}{p_{k+1}}\right\rfloor$$

$$\le C_{k+1}xx_2^{2(k+1)} \sum_{\pi\le x_2^{2(k+1)}} \frac{\log\pi}{\pi} \le C_{k+1}xx_2^{2(k+1)}x_3.$$

It follows from this estimate that

$$\frac{1}{x}\#\{n\in\mathcal{N}_x^{(k)} : s_k(n) > e^{\kappa_x x_2^{2(k+1)}x_3}\} \to 0 \qquad (x\to\infty), \tag{4.13}$$

provided κ_x is any function such that $\kappa_x\to\infty$ arbitrarily slowly as $x\to\infty$.

We will now prove that, as $x\to\infty$,

$$\frac{1}{x}\#\{n\in\mathcal{N}_x^{(k)} : \text{there exists } \pi > x_2^{2(k+1)} \text{ such that } \pi^2 \mid \sigma(B_k(n))\}\to 0. \tag{4.14}$$

Indeed, if $n\in\mathcal{N}_x^{(k)}$ and $\pi^2\mid\sigma(B_k(n))$, then there exist two chains of primes, namely

$$\pi\to Q_1\to\cdots\to Q_{k+1}, \qquad Q_{k+1}\mid n,$$
$$\pi\to p_1\to\cdots\to p_{k+1}, \qquad p_{k+1}\mid n,$$

from which it follows, using Lemma 3.2, that

$$\sum_{\pi>x_2^{2(k+1)}} \sum_{\substack{\pi^2\mid\sigma(B_k(n))\\ n\in\mathcal{N}_x^{(k)}}} 1 \le \sum_{\substack{\pi>x_2^{2(k+1)}\\ \pi\to Q_1\to\cdots\to Q_{k+1}\\ \pi\to p_1\to\cdots\to p_{k+1}}} \left\lfloor\frac{x}{Q_{k+1}p_{k+1}}\right\rfloor$$

$$\le Cxx_2^{2(k+1)} \sum_{\pi>x_2^{2(k+1)}} \frac{1}{\pi^2} \ll \frac{x}{x_3},$$

thus establishing (4.14).

On the other hand,

$$\sum_{n\in\mathcal{N}_x^{(k)}} \log r_k(n) = \sum_{x_2^{2(k+1)}<\pi\le w_{k+1}} (\log\pi) \sum_{\pi\to p_1\to\cdots\to p_{k+1}} \left\lfloor\frac{x}{p_{k+1}}\right\rfloor$$

$$\le Cxx_2^{2(k+1)} \sum_{\pi\le w_{k+1}} \frac{\log\pi}{\pi} \le Cxx_2^{2(k+1)}\log w_{k+1}$$

$$= Cxx_2^{2(k+1)+m_{k+1}},$$

which proves that

$$\frac{1}{x}\#\{n \in \mathcal{N}_x^{(k)} : r_k(n) > e^{Cx_2^{2(k+1)+m_{k+1}}}\} \to 0 \qquad (x \to \infty). \tag{4.15}$$

Replace (4.12) by

$$\sigma_{k+1}(n) = \sigma(A_k(n))s_k(n)r_k(n)t_k(n);$$

then, since for all $n \in \mathcal{N}_x^{(k)}$, we have $\sigma(A_k(n)) < w_{k+1}$ while $p(t_k(n)) > w_{k+1}$ and $P(s_k(n)r_k(n)) < w_{k+1}$, it follows that, choosing

$$A_{k+1}(n) = \sigma(A_k(n))s_k(n)r_k(n), \qquad B_{k+1}(n) = t_k(n),$$

we have

$$\sigma_{k+1}(n) = A_{k+1}(n)B_{k+1}(n).$$

In light of (4.13)–(4.15), we can now say that, with the possible exception of $o(x)$ integers $n \le x$ as $x \to \infty$,

$$\sigma(A_{k+1}(n)) < w_{k+2} \qquad \text{for a corresponding suitable large integer } m_{k+2}.$$

Moreover, since $B_{k+1}(n)$ is squarefree, we observe that

$$B_{k+1}(n) \mid \gamma(\sigma_{k+1}(n)),$$

and we may then conclude the proof similarly to the case of k, thus proving (2.1) for $k+1$ and thereby completing the proof of Theorem 2.1.

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