C O L L O Q U I U M   M A T H E M A T I C U M

VOL. 147 2017 NO. 2

MAPPINGS OF DEGREE 5, PART II

BY

M. MACIEJEWSKI and A. PRÓSZYŃSKI (Bydgoszcz)

Abstract. We continue the investigation started in [1]. The purpose of the present paper is to find the second (last) part of generating relations (namely, all the 2-covering relations) satisfied by homogeneous polynomial mappings of degree 5.

1. Preliminaries. Let R be a commutative ring with 1, and let X, Y be R-modules. If a mapping $f : X \to Y$ satisfies $f(0) = 0$ then we define $\Delta^n f : X^n \to Y$ $(n = 1, 2, \ldots)$ by induction as follows: $\Delta^1 f = f$ and

$$\begin{aligned}(\Delta^{n+1} f)(x_0, \ldots, x_n) = (\Delta^n f)&(x_0 + x_1, x_2, \ldots, x_n) \\ &- (\Delta^n f)(x_0, x_2, \ldots, x_n) - (\Delta^n f)(x_1, x_2, \ldots, x_n).\end{aligned}$$

In the following, we will abbreviate $(\Delta^n f)(x_1, \ldots, x_n)$ to $(x_1, \ldots, x_n)$. A mapping $f : X \to Y$ is called a *regular m-application* if it satisfies the following conditions:

(A1) $f(rx) = r^m f(x)$ for $r \in R$, $x \in X$,

(A2) $\Delta^m f : X^m \to Y$ is m-linear,

(A) $(rx, sy, -) - r(x, sy, -) - s(rx, y, -) + rs(x, y, -) = 0$ for $r, s \in R$, $x, y \in X$,

where $() = \Delta^{m-1} f$ and $-$ stands for the remaining $m - 3$ variables. The functor of regular m-applications on X is represented by the module $\overline{\Delta}^m(X)$ generated by the elements $\overline{\delta}^m(x)$, $x \in X$, where $\overline{\delta}^m : X \to \overline{\Delta}^m(X)$ is the standard regular m-application (for details, we refer to [1]).

We consider the mth divided power $\Gamma^m(X)$ of X and the homomorphism

$$\overline{h}^m = \overline{h}^m(X) : \overline{\Delta}^m(X) \to \Gamma^m(X), \quad \overline{h}^m(\overline{\delta}^m(x)) = x^{(m)},$$

defining a natural transformation of functors. Let $\{x_1, \ldots, x_k\}$ be the stan-

---

2010 *Mathematics Subject Classification*: 11E76, 13C13.

Key words and phrases: higher degree forms and mappings, modules over commutative rings, generators and relations.

Received 27 January 2016.

DOI: 10.4064/cm6877-3-2016

dard basis of the module R^k, $k = 1, 2, \ldots$. Define

$$\begin{aligned}
\Gamma^{m,k}(R) &= R\Big\{x_1^{(i_1)}\ldots x_k^{(i_k)};\ \sum i_j = m,\ i_j \geq 1\Big\} \subset \Gamma^m(R^k),\\
\overline{\Delta}^{m,k}(R) &= R\{(r_1x_1,\ldots,r_kx_k); r_1,\ldots,r_k \in R\} \subset \overline{\Delta}^m(R^k),\\
\overline{h}^{m,k} &= \overline{h}^m|_{\overline{\Delta}^{m,k}(R)} : \overline{\Delta}^{m,k}(R) \to \Gamma^{m,k}(R),\\
\overline{h}^{m,k}(r_1x_1,\ldots,r_kx_k) &= \sum_{(i)}(r_1x_1)^{(i_1)}\ldots(r_kx_k)^{(i_k)}\\
&= \sum_{(i)} r_1^{i_1}\ldots r_k^{i_k}x_1^{(i_1)}\ldots x_k^{(i_k)}
\end{aligned}$$

where $() = \Delta^k\overline{\delta}^m$ and (i) runs over all sequences of positive integers $i_1,\ldots,i_k$ satisfying $i_1 + \cdots + i_k = m$.

Our goal is to determine the kernels of $\overline{h}^{m,k}$ for $m = 5$. It is known that they are zero except for $k = 2$ and $k = 3$. In [1], we found generators of $\mathrm{Ker}(\overline{h}^{5,3})$, and in this paper we find generators of $\mathrm{Ker}(\overline{h}^{5,2})$. Both systems of generators determine a full system of covering relations of the functor Hom_R^5 of homogeneous polynomial mappings of degree 5 (see [1]), composed of 3-covering relations ([1, Theorem 8]) and 2-covering relations (Theorem 4 of the present paper).

2. Computation of $\mathrm{Ker}(\overline{h}^{5,2})$.

The purpose of this section is to find generators of the kernel of the homomorphism $\overline{h}^{5,2} : \overline{\Delta}^{5,2}(R) \to \Gamma^{5,2}(R)$,

$$\overline{h}^{5,2}(rx, sy) = r^4s((4,1)) + r^3s^2((3,2)) + r^2s^3((2,3)) + rs^4((1,4)), \tag{1}$$

where $x = x_1$, $y = x_2$, $(rx, sy) = (\Delta^2\overline{\delta}^5)(rx, sy)$ and $((i,j)) = x^{(i)}y^{(j)}$.

Our main tool will be the following

Theorem 1 ([3, Theorem 5.9]). *The submodule* $\overline{\Gamma}^{5,2}(R) = \mathrm{Im}(\overline{h}^{5,2})$ *admits the following decomposition:*

$$\overline{\Gamma}^{5,2}(R) = R\sigma \oplus I_3(R)((3,2)) \oplus I_3(R)((2,3)) \oplus I_4(R)(((4,1)) + ((2,3)))$$

where σ *denotes the sum of all base elements of* $\Gamma^{5,2}(R)$, *i.e.*

$$\sigma = ((4,1)) + ((3,2)) + ((2,3)) + ((1,4)) = \overline{h}^{5,2}(x,y)$$

and $I_n(R) = (r^n - r;\ r \in R) = (r^ns - rs^n;\ r, s \in R)$. *In this decomposition, a generator* $\overline{h}^{5,2}(rx, sy)$ *can be expressed as follows:*

$$\begin{aligned}
\overline{h}^{5,2}(rx, sy) &= rs^4\sigma + s(r^3s - rs^3)((3,2))\\
&\quad + r(rs^3 - r^3s)((2,3)) + (r^4s - rs^4)(((2,3)) + ((4,1))).
\end{aligned} \tag{2}$$

Observe that there exist (not uniquely determined) elements $W_1(r)$, $W_2(r), T(r) \in \overline{\Delta}^{5,2}(R)$ such that

$$\overline{h}^{5,2}(W_1(r)) = (r^3 - r)((3,2)), \tag{3}$$

$$\overline{h}^{5,2}(W_2(r)) = (r^3 - r)((2,3)), \tag{4}$$

$$\overline{h}^{5,2}(T(r)) = (r^4 - r)(((2,3)) + ((4,1))). \tag{5}$$

We assume in this section that we have fixed such elements; we determine them in the next section. Let us denote

$$\begin{aligned} D(r,s) = (rx, sy) - rs^4(x,y) - s(sW_1(r) - rW_1(s)) \\ - r(rW_2(s) - sW_2(r)) - (sT(r) - rT(s)). \end{aligned}$$

Corollary 1. $\overline{h}^{5,2}(D(r,s)) = 0$.

Proof. This follows from (2)–(5) and the following computations:

$$\begin{aligned} \overline{h}^{5,2}(sW_1(r) - rW_1(s)) &= (s(r^3 - r) - r(s^3 - s))((3,2)) \\ &= (r^3 s - rs^3)((3,2)), \\ \overline{h}^{5,2}(rW_2(s) - sW_2(r)) &= (r(s^3 - s) - s(r^3 - r))((2,3)) \\ &= (rs^3 - r^3 s)((2,3)), \\ \overline{h}^{5,2}(sT(r) - rT(s)) &= (s(r^4 - r) - r(s^4 - s))(((2,3)) + ((4,1))) \\ &= (r^4 s - rs^4)(((2,3)) + ((4,1))). \ \blacksquare \end{aligned}$$

Since any generator (rx, sy) of $\overline{\Delta}^{5,2}(R)$ is a linear combination of $D(r,s)$, (x,y) and values of W_1, W_2, T, we obtain the following

Corollary 2. *If we denote* $D(R,R) = R\{D(r,s);\ r,s \in R\}$, $W_i(R) = R\{W_i(r);\ r \in R\}$, $i = 1,2$, *and* $T(R) = R\{T(r);\ r \in R\}$ *then*

$$\overline{\Delta}^{5,2}(R) = D(R,R) + R(x,y) + W_1(R) + W_2(R) + T(R).$$

The next step is based on the following

Theorem 2 ([2, Theorem 1]). *The generating relations between generators* $[r] = r^3 - r$ *of* $I_3(R)$ *are the following:*

(D) $[rs] = r^3[s] + s[r], \quad r,s \in R,$

(C1) $3s[r] - 3r[s] = (r - s)([r+s] - [r] - [s]), \quad r,s \in R,$

(C2) $([ar^3 + bs^3] - [ar^3] - [bs^3]) - ([ar + bs] - [ar] - [bs])$
$= 3a^2b[r^2 s] + 3ab^2[rs^2], \quad a,b,r,s \in R,$

(C3) $[r+s+t] - [r+s] - [s+t] - [r+t] + [r] + [s] + [t]$
$= rst[2], \quad r,s,t \in R,$

and for generators $[r] = r^4 - r$ *of* $I_4(R)$ *the generating relations are*

(D) $\quad [rs] = r^4[s] + s[r], \qquad r, s \in R,$

(C) $\quad [r+s] = [r] + [s] + (2r^3s + 3r^2s^2 + 2rs^2)[-1], \qquad r, s \in R.$

We are ready to prove the main result of this section.

Theorem 3. $\mathrm{Ker}(\overline{h}^{5,2}) = K$ *where* K *is a submodule of* $\overline{\Delta}^{5,2}(R)$ *generated by the following elements:*

1. $D(r,s)$,
2. $W_i(rs) - r^3W_i(s) - sW_i(r)$, $i = 1,2$,
3. $3sW_i(r) - 3rW_i(s) - (r-s)(W_i(r+s) - W_i(r) - W_i(s))$, $i = 1,2$,
4. $W_i(ar^3+bs^3) - W_i(ar^3) - W_i(bs^3) - W_i(ar+bs) + W_i(ar) + W_i(bs) - 3a^2bW_i(r^2s) - 3ab^2W_i(rs^2)$, $i = 1,2$,
5. $W_i(r+s+t) - W_i(r+s) - W_i(s+t) - W_i(r+t) + W_i(r) + W_i(s) + W_i(t) - rstW_i(2)$, $i = 1,2$,
6. $T(rs) - r^4T(s) - sT(r)$,
7. $T(r+s) - T(r) - T(s) - (2r^3s + 3r^2s^2 + 2rs^3)T(-1)$,

where $a, b, r, s \in R$.

Proof. Observe that all generators of K belong to $\mathrm{Ker}(\overline{h}^{5,2})$. In fact, for $D(r,s)$ this follows from Corollary 1, and for elements (2)–(7) from Theorem 2. Hence $\overline{h}^{5,2}$ induces a homomorphism

$$h' : \overline{\Delta}^{5,2}(R)/K \to \overline{\Gamma}^{5,2}(R) \subset \Gamma^{5,2}(R).$$

Let us denote by $\overline{W_1(R)}$ (resp. $\overline{W_2(R)}$, $\overline{T(R)}$) the reduction of $W_1(R)$ (resp. $W_2(R)$, $T(R)$) modulo K. Then

$$h'(\overline{W_1(R)}) = I_3(R)((3,2)), \qquad h'(\overline{W_2(R)}) = I_3(R)((2,3)),$$
$$h'(\overline{T(R)}) = I_4(R)(((4,1)) + ((2,3)))$$

and the restrictions $h'|_{\overline{W_1(R)}}$, $h'|_{\overline{W_2(R)}}$, $h'|_{\overline{T(R)}}$ are mono by Theorem 2. Let now $x \in \mathrm{Ker}(\overline{h}^{5,2})$. Then $\overline{x} \in \mathrm{Ker}(h')$. Since $D(r,s) \in K$, Corollary 2 gives $\overline{x} = r\overline{(x,y)} + \overline{w_1} + \overline{w_2} + \overline{t}$ where $r \in R$, $w_i \in W_i(R)$, $i = 1,2$ and $t \in T(R)$. Therefore,

$$0 = h'(\overline{x}) = r\sigma + h'(\overline{w_1}) + h'(\overline{w_2}) + h'(\overline{t}).$$

Since the summands of the right hand side belong to different direct components of $\overline{\Gamma}^{5,2}(R)$, we get $r = 0$ and $h'(\overline{w_1}) = h'(\overline{w_2}) = h'(\overline{t}) = 0$. But the restrictions of h' are mono, so $\overline{v_1} = \overline{v_2} = \overline{w} = 0$. Hence $\overline{x} = 0$, and consequently $x \in K$. ∎

3. Description of $W_1(r)$, $W_2(r)$, $T(r)$. We present a construction of the elements $W_1(r)$, $W_2(r)$, $T(r)$, independent of the choice of a base ring R.

We will assume that $W_1(1) = W_2(1) = T(1) = 0$. This gives

$$D(r,1) = (rx,y) - r(x,y) - W_1(r) + rW_2(r) - T(r).$$

Since $D(r,1) \in \operatorname{Ker}(\overline{h}^{5,2})$, we can assume that $D(r,1) = 0$, or equivalently

$$T(r) = (rx,y) - r(x,y) - W_1(r) + rW_2(r),$$

and assuming (3) and (4) of Section 2 we get (5). Hence we need to determine $W_1(r)$ and $W_2(r)$ only.

In [1], we defined for any $r \in R$ the following elements of $\overline{\Delta}^{5,3}(R)$:

$$\begin{aligned} C_3(r) &= 3(rx,y,z) - 3r(x,y,z) + (1-r)(rx,x,y,z), \\ [r] &= (rx,x,y,z) + (x,ry,y,z) + (x,y,rz,z) \\ &\quad - r^2\big((x,x,y,z) + (x,y,y,z) + (x,y,z,z)\big) - 3(r-r^2)(x,y,z), \end{aligned}$$

where $x = x_1$, $y = x_2$, $z = x_3$ form the standard basis of R^3. Set

$$U_1(r) = C_3(r) + [r].$$

It follows from [1, p. 239] that

$$\overline{h}^5(U_1(r)) = \overline{h}^{5,3}(U_1(r)) = (r-r^2)((1,2,2)).$$

Consider a homomorphism $f : Rx \oplus Ry \oplus Rz \to Rx \oplus Ry$ defined on the basis as follows: $f(x) = x$, $f(y) = x$, $f(z) = y$. Set $W_1'(r) = (\overline{\Delta}^5(f))(U_1(r))$ and observe that

$$\begin{aligned} W_1'(r) &= 3(rx,x,y) - 3r(x,x,y) + (1-r)(rx,x,x,y) + (rx,x,x,y) \\ &\quad + (x,rx,x,y) + (x,x,ry,y) \\ &\quad - r^2\big((x,x,x,y) + (x,x,x,y) + (x,x,y,y)\big) - 3(r-r^2)(x,x,y) \\ &= 3(rx,x,y) + (3-r)(rx,x,x,y) + (x,x,ry,y) \\ &\quad - r^2\big(2(x,x,x,y) + (x,x,y,y)\big) - 3(2r-r^2)(x,x,y) \in \overline{\Delta}^{5,2}(R). \end{aligned}$$

Since $\overline{h}^5(U_1(r)) = (r-r^2)((1,2,2))$ and $\overline{h}^5$ is a natural transformation, we obtain

$$\begin{aligned} \overline{h}^{5,2}(W_1'(r)) &= \overline{h}^5(W_1'(r)) = \overline{h}^5\big((\overline{\Delta}^5(f))(U_1(r))\big) \\ &= \Gamma^5(f)(\overline{h}^5(U_1(r))) = \Gamma^5(f)((r-r^2)((1,2,2))) \\ &= (r-r^2)\Gamma^5(f)(((1,2,2))) = (r-r^2)\Gamma^5(f)(xy^{(2)}z^{(2)}). \end{aligned}$$

Observe that

$$\Gamma^5(f)(xy^{(2)}z^{(2)}) = xx^{(2)}y^{(2)} = (1,2)x^{(3)}y^{(2)} = 3((3,2)).$$

Hence $\overline{h}^{5,2}(W_1'(r)) = 3(r-r^2)((3,2))$. By symmetry we get an element

$$\begin{aligned} W_2'(r) &= 3(x,ry,y) + (3-r)(x,ry,y,y) + (x,x,ry,y) \\ &\quad - r^2((x,x,y,y) + 2(x,y,y,y)) - 3(2r-r^2)(x,y,y) \end{aligned}$$

such that $\overline{h}^{5,2}(W_2'(r)) = 3(r-r^2)((2,3))$. This gives

Corollary 3. *The following equalities hold:*

$$\begin{aligned}\overline{h}^{5,2}((r+1)W_1'(r)) &= 3(r-r^3)((3,2)),\\ \overline{h}^{5,2}((r+1)W_2'(r)) &= 3(r-r^3)((2,3)).\end{aligned}$$

Using (1) we obtain

$$\begin{aligned}\overline{h}^{5,2}(rx,y) &= r^4((4,1)) + r^3((3,2)) + r^2((2,3)) + r((1,4)),\\ \overline{h}^{5,2}(-rx,y) &= r^4((4,1)) - r^3((3,2)) + r^2((2,3)) - r((1,4)).\end{aligned}$$

Hence

$$\overline{h}^{5,2}((rx,y)-(-rx,y)) = 2r^3((3,2)) + 2r((1,4))$$

and in particular $\overline{h}^{5,2}((x,y)-(-x,y)) = 2((3,2)) + 2((1,4))$.

If we denote

$$W_1''(r) = (rx,y) - (-rx,y) - r((x,y)-(-x,y))$$

then $\overline{h}^{5,2}(W_1''(r)) = 2(r^3-r)((3,2))$.

By symmetry we get an element

$$W_2''(r) = (x,ry) + (x,-ry) - r((x,y)+(x,-y))$$

such that $\overline{h}^{5,2}(W_2''(r)) = 2(r^3-r)((2,3))$.

Then Corollary 3 gives

Corollary 4. *Denote*

$$\begin{aligned}W_1(r) &= -(r+1)W_1'(r) - W_1''(r),\\ W_2(r) &= -(r+1)W_2'(r) - W_2''(r),\\ T(r) &= (rx,y) - r(x,y) - W_1(r) + rW_2(r).\end{aligned}$$

Then

$$\begin{aligned}\overline{h}^{5,2}(W_1(r)) &= (r^3-r)((3,2)),\\ \overline{h}^{5,2}(W_2(r)) &= (r^3-r)((2,3)),\\ \overline{h}^{5,2}(T(r)) &= (r^4-r)(((2,3)) + ((4,1))).\end{aligned}$$

4. The main theorem. Let now $f : X \to Y$ be a mapping with $f(0) = 0$, let $(\ ,\ ) = \Delta^2 f$, $(\ ,\ ,\ ) = \Delta^3 f$, and let x, y denote any elements of X. For any $r, s \in R$ consider the following values:

$$\begin{aligned}
W(r) = W(r,x,y) &:= -(r+1)\big(3(rx,x,y) + (3-r)(rx,x,x,y) \\
&\quad + (x,x,ry,y) - r^2(2(x,x,x,y) + (x,x,y,y)) - 3(2r-r^2)(x,x,y)\big) \\
&\quad - \big((rx,y) - (-rx,y) - r((x,y) - (-x,y))\big), \\
\overline{W}(r) = \overline{W}(r,x,y) &:= W(r,y,x), \\
T(r) = T(r,x,y) &:= (rx,y) - r(x,y) - W(r) + r\overline{W}(r), \\
D(r,s) = D(r,s,x,y) &:= (rx,sy) - rs^4(x,y) - s(sW(r) - rW(s)) \\
&\quad - r(r\overline{W}(s) - s\overline{W}(r)) - (sT(r) - rT(s)).
\end{aligned}$$

As in [1], Theorem 3 yields the main result of the paper. Observe that the symmetric versions of the following relations can obviously be omitted.

THEOREM 4. *The following relations constitute a complete* 2*-covering system for the functor* Hom^5: (A1), (A2), (A) *and*

(1) $D(r,s) = 0$,
(2) $W(rs) = r^3W(s) + sW(r)$,
(3) $3sW(r) - 3rW(s) = (r-s)(W(r+s) - W(r) - W(s))$,
(4) $W(ar^3+bs^3) - W(ar^3) - W(bs^3) - W(ar+bs) + W(ar) + W(bs) = 3a^2bW(r^2s) + 3ab^2W(rs^2)$,
(5) $W(r+s+t) - W(r+s) - W(s+t) - W(r+t) + W(r) + W(s) + W(t) - rstW(2) = 0$,
(6) $T(rs) = r^4T(s) + sT(r)$,
(7) $T(r+s) = T(r) + T(s) + (2r^3s + 3r^2s^2 + 2rs^2)T(-1)$,

where $a,b,r,s,t \in R$. *Together with the relations* (B), (B1), (B2), (S) *of* [1, *Theorem* 8], *they constitute a full system of covering relations of the functor* Hom^5.

Eliminating T we can simplify some relations. For example

COROLLARY 5. *The relations* (1) *and* (6) *are equivalent to*

$$(1') \qquad (rx,sy) + r(sx,y) - s(rx,y) - rs^4(x,y) - (s^2-s)W(r) + (rs-r)W(s) - (r^2-rs)\overline{W}(s) = 0,$$

$$(6') \qquad (rsx,y) - r^4(sx,y) - s(rx,y) + r^4s(x,y) + (r^4-r^3)W(s) + r(s^2-s)\overline{W}(r) = 0.$$

REFERENCES

[1] M. Maciejewski and A. Prószyński, *Mappings of degree 5, part I*, Colloq. Math. 117 (2009), 223–242.
[2] M. Maciejewski and A. Prószyński, *On n-derivations and relations between elements $r^n - r$ for some n*, Bull. Polish Acad. Sci. Math. 62 (2014), 29–42.

[3] A. Prószyński, *Forms and mappings. I: Generalities*, Fund. Math. 122 (1984), 219–235.

M. Maciejewski, A. Prószyński
Kazimierz Wielki University
85-072 Bydgoszcz, Poland
E-mail: maciejm@ukw.edu.pl
apmat@ukw.edu.pl