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Supersimplicity and countable reducts of a unidimensional hypersimple theory

by

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Abstract. We show that a hypersimple unidimensional theory that, in the partial order of all countable reducts, has a club of reducts that are coordinatized in finite rank, is supersimple.

1. Introduction. In this paper we suggest an approach to the problem of supersimplicity of unidimensional hypersimple theories (T is said to be *hypersimple* if T is simple and eliminates hyperimaginaries). The problem has been answered in the affirmative in the following cases: in [H1], for any stable theory; in [S1], for any countable theory—this improved an earlier result for the case of a countable theory with the wnfcp (= weak nfcp), an analogue of the nfcp for simple theories [BPV]; and in [S2], for any (possibly uncountable) non-s-essentially 1-based theory (roughly, a theory that is far from being 1-based).

It is easy to see that supersimplicity of a theory is determined by (the supersimplicity of) the family of its countable reducts. Therefore, it is natural to try and reflect properties of a given unidimensional hypersimple theory to countable reducts. Clearly, unidimensionality is not preserved under reducts. On the other hand, any unidimensional hypersimple theory is coordinatized in finite rank (see Definition 5.1).

In this paper we show that supersimplicity of any (possibly uncountable) unidimensional hypersimple theory follows from coordinatization in finite rank of sufficiently many countable reducts of it.

We will assume basic knowledge of simple theories as in [K], [KP], [HKP]. A good textbook on simple theories that covers much more is [W]. The

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notation is standard, and throughout the paper we work in a large saturated model $\mathcal{C}$ of a complete first-order theory T in a language L.

2. Preliminaries. In this section, T is assumed to be simple. We quote several known facts that we will apply.

2.1. Basic facts. In this subsection we work in $\mathcal{C}$ with hyperimaginaries.

Fact 2.1 (Existence of canonical bases, [HKP]). *Let* $p \in S(a)$ *be an amalgamation base. Then there exists a type-definable* (*over* $\emptyset$) *equivalence relation* $E(x, x')$ *on tuples of the same sort as* a *such that* (*with* $\tilde{a} = a/E$):

1. *For every* $\sigma \in \mathrm{Aut}(\mathcal{C})$, *the parallelism class of* p *is fixed by* σ *iff* $\sigma(\tilde{a}) = \tilde{a}$.
2. p *does not fork over* $\tilde{a}$, *and if* p *does not fork over some hyperimaginary* $\tilde{b} \in \mathrm{bdd}(a)$, *then* $\tilde{a} \in \mathrm{bdd}(\tilde{b})$.

The following fact follows easily from Fact 2.1.

Fact 2.2. *Let* $p \in S(b)$ *be an amalgamation base,* $\langle a_i \mid i < \omega \rangle$ *a Morley sequence of* p, *and* $\tilde{b}$ *the canonical base of* p. *Then* $\tilde{b} \in \mathrm{dcl}(a_i \mid i < \omega)$.

2.2. Almost internality, analyzability and unidimensionality. In this subsection we work in $\mathcal{C}$ with hyperimaginaries unless otherwise stated; if T is hypersimple and we work in $\mathcal{C}^{\mathrm{eq}}$, we get equivalent definitions.

In this subsection, $\mathcal{P}$ denotes an A-invariant family of partial types and p a partial type over A. We say that p *is* (*almost*) $\mathcal{P}$*-internal* if for every realization a of p there exists b with $a \underset{A}{\perp} b$ such that for some tuple c of realizations of partial types in $\mathcal{P}$ over Ab we have $a \in \mathrm{dcl}(b, c)$ (respectively, $a \in \mathrm{acl}(b, c)$). We say that p *is analyzable in* $\mathcal{P}$ if for any $a \models p$ there exists a sequence $I = \langle a_i \mid i \le \alpha \rangle$ such that $a_\alpha = a$ and $\mathrm{tp}(a_i/\{a_j \mid j < i\} \cup A)$ is almost $\mathcal{P}$-internal for every $i \le \alpha$.

First, the following fact is well known and easy (a straightforward forking computation):

Fact 2.3.

1. *Assume that* $\mathrm{tp}(a_i/A)$ *is* [*almost*] $\mathcal{P}$*-internal for all* $i < \alpha$. *Then* $\mathrm{tp}(\langle a_i \mid i < \alpha \rangle/A)$ *is* [*almost*] $\mathcal{P}$*-internal. Thus, if* $\mathrm{tp}(a_i/A)$ *is analyzable in* $\mathcal{P}$ *for all* $i < \alpha$, *then so is* $\mathrm{tp}(\langle a_i \mid i < \alpha \rangle/A)$.
2. *If* $\mathrm{tp}(a/A)$ *is almost* $\mathcal{P}$*-internal, then so is* $\mathrm{tp}(a/B)$ *for any set* $B \supseteq A$.

The theory T is said to be *unidimensional* if whenever p and q are complete non-algebraic types, then p,q are non-orthogonal.

We will also need the following fact.

Fact 2.4 ([S4, Claim 5.4(2)]). *Work in* $\mathcal{C}^{\mathrm{eq}}$ (*without hyperimaginaries*). *Let* $p \in S(\emptyset)$ *and let* $\theta \in L$. *Assume* p *is analyzable in* θ *in* $\mathcal{C}^{\mathrm{eq}}$. *Then* p *is* *analyzable in θ in finitely many steps in $\mathcal{C}^{\mathrm{eq}}$. In particular, if T is a hypersimple unidimensional theory and there exists a non-algebraic supersimple definable set, then T has finite SU-rank, i.e. every complete type has finite SU-rank. In fact, for every given sort there is a finite bound on the SU-rank of all types in that sort, equivalently the global D-rank of any sort is finite.*

Another useful fact is the following.

FACT 2.5 ([S4, Theorem 5.12]). *Let T be any unidimensional simple theory. Then T eliminates $\exists^\infty$.*

2.3. The forking topology, EPFO and PCFT. The forking topology is introduced in [S0] and is a variant of Hrushovski's and Pillay's topologies from [H0] and [P1], respectively. In this section, T is assumed to be simple and we work in $\mathcal{C}$.

DEFINITION 2.6. Let $A \subseteq \mathcal{C}$ and let x be a finite tuple of variables.

An invariant set $\mathcal{U}$ over A is said to be a *basic τ^f-open set over A* if there is $\phi(x, y) \in L(A)$ such that

$$\mathcal{U} = \{a \mid \phi(a, y) \text{ forks over } A\}.$$

Note that the family of basic τ^f-open sets over A is closed under finite intersections, thus forms a basis for a unique topology on $S_x(A)$. An open set in this topology is called a *τ^f-open set over A* or a *forking-open set over A*.

An invariant set $\mathcal{U}$ over A is said to be *a basic τ^f_∞-open set over A* if $\mathcal{U}$ is a type-definable τ^f-open set over A. The family of basic τ^f_∞-open sets over A is a basis for a unique topology on $S_x(A)$. An open set in this topology is called a *τ^f_∞-open set over A*.

REMARK 2.7. The τ^f_∞-topology and in particular the τ^f-topology on $S_x(A)$ refine the Stone topology of $S_x(A)$ for all x, A.

We will apply the following fact.

FACT 2.8 ([S0, Lemma 2.6]). *Let $\mathcal{U}$ be a τ^f-open set over $\emptyset$ and let A be any set. Then $\mathcal{U}$ is τ^f-open over A.*

Recall the following definition from [S0].

DEFINITION 2.9. We say that *the τ^f-topologies over A are closed under projections* (T *is PCFT over* A) if for every τ^f-open set $\mathcal{U}(x, y)$ over A the set $\exists y\ \mathcal{U}(x, y)$ is a τ^f-open set over A. We say that *the τ^f-topologies are closed under projections* (T *is PCFT*) if they are so over every set A.

In [BPV, Proposition 4.5] the authors proved the following equivalence which, for convenience, we will use as a definition (their definition involves extension with respect to pairs of models of T).

Definition 2.10. We say that *the extension property is first-order in* T (or T *is EPFO*) if for any formulas $\phi(x,y), \psi(y,z) \in L$ the relation $Q_{\phi,\psi}$ defined by

$$Q_{\phi,\psi}(a) \quad \text{iff} \quad \phi(x,b) \text{ does not fork over } a \text{ for every } b \models \psi(y,a)$$

is type-definable (here a can be an infinite tuple from $\mathcal{C}$ whose sorts are fixed).

Fact 2.11 ([S1, Corollary 3.13]). *Suppose the extension property is first-order in* T. *Then* T *is PCFT.*

We say that an A-invariant set $\mathcal{U}$ *has finite SU-rank* if $SU(a/A) < \omega$ for all $a \in \mathcal{U}$, and *has bounded finite SU-rank* if there exists $n < \omega$ such that $SU(a/A) \le n$ for all $a \in \mathcal{U}$. The existence of a τ^f-open set of bounded finite SU-rank implies the existence of an SU-rank 1 formula (i.e. a weakly minimal formula):

Fact 2.12 ([S0, Proposition 2.13]). *Let* $\mathcal{U}$ *be an unbounded* τ^f*-open set over some set* A. *Assume* $\mathcal{U}$ *has bounded finite SU-rank. Then there exist a set* $B \supseteq A$ *with* $|B \setminus A| < \omega$ *and* $\theta(x) \in L(B)$ *of SU-rank* 1 *such that* $\theta^{\mathcal{C}} \subseteq \mathcal{U} \cup \operatorname{acl}(B)$.

Recall the following fact. First, let $\mathcal{P}^{SU\le 1}$ denote the class of complete real types over sets of size $\le |T|$, of SU-rank ≤ 1.

Fact 2.13 ([P2]). *Let* T *be a simple theory that eliminates* $\exists^\infty$. *Moreover, assume every type is analyzable in* $\mathcal{P}^{SU\le 1}$. *Then the extension property is first-order in* T.

A more general statement with a proof is presented in [S1, Lemma 3.7].

2.4. Stable independence and stable ***SU*****-rank.** In this subsection we recall some notions around the notion of stable dependence from [S1]. Here T is assumed to be simple and we work in $\mathcal{C}$.

First we recall the notion of stable independence.

Definition 2.14. Let $a \in \mathcal{C}$ and $A \subseteq B \subseteq \mathcal{C}$. We say that a *is stably independent from* B *over* A if for every stable $\phi(x,y) \in L$, if $\phi(x,b)$ is over B and $a' \models \phi(x,b)$ for some $a' \in \operatorname{dcl}(Aa)$, then $\phi(x,b)$ does not divide over A. In this case we write $a \underset{A}{\overset{s}{\smile\!\!\!\!\!\!|}} B$.

The notion of stable SU-rank is defined via stable dependence.

Definition 2.15. For $a \in \mathcal{C}$ and $A \subseteq \mathcal{C}$ the SU_s*-rank* is defined by induction on α: if $\alpha = \beta + 1$, then $SU_s(a/A) \ge \alpha$ if there exists $B \supseteq A$ such that $a \underset{A}{\overset{s}{\not\smile\!\!\!\!\!\!|}} B$ and $SU_s(a/B) \ge \beta$. For limit α, $SU_s(a/A) \ge \alpha$ if $SU_s(a/A) \ge \beta$ for all $\beta < \alpha$.

Let $\mathcal{U}$ be an A-invariant set. We write $SU_s(\mathcal{U}) = \alpha$ (the SU_s-rank of $\mathcal{U}$ is α) if $\max\{SU_s(p) \mid p \in S(A),\, p^{\mathcal{C}} \subseteq \mathcal{U}\} = \alpha$. We say that $\mathcal{U}$ *has bounded finite* SU_s*-rank* if $SU_s(\mathcal{U}) = n$ for some $n < \omega$. Note that the SU_s-rank of $\mathcal{U}$ might a priori depend on the choice of the set A over which $\mathcal{U}$ is invariant.

The following rank is a variation of stable SU-rank; it is non-increasing in extensions.

Definition 2.16. For $a \in \mathcal{C}$ and $A \subseteq \mathcal{C}$ the SU_{se}*-rank* is defined by induction on α: if $\alpha = \beta+1$, then $SU_{se}(a/A) \geq \alpha$ if there exist $B_1 \supseteq B_0 \supseteq A$ such that $a \underset{B_0}{\not\perp}^{s} B_1$ and $SU_{se}(a/B_1) \geq \beta$. For limit α, $SU_{se}(a/A) \geq \alpha$ if $SU_{se}(a/A) \geq \beta$ for all $\beta < \alpha$.

Let $\mathcal{U}$ be an A-invariant set. We write $SU_{se}(\mathcal{U}) = \alpha$ (the SU_{se}-rank of $\mathcal{U}$ is α) if $\max\{SU_{se}(p) \mid p \in S(A),\, p^{\mathcal{C}} \subseteq \mathcal{U}\} = \alpha$. We say that $\mathcal{U}$ *has bounded finite* SU_{se}*-rank* if $SU_{se}(\mathcal{U}) = n$ for some $n < \omega$.

Remark 2.17. Note that $SU_{se}(a/B) \leq SU_{se}(a/A)$ for all $a \in \mathcal{C}$ and all $A \subseteq B \subseteq \mathcal{C}$ (this is the reason for introducing SU_{se}). Also, clearly $SU_s(a/A) \leq SU_{se}(a/A) \leq SU(a/A)$ for all a, A. Clearly $SU_{se}(a/A) = 0$ iff $SU_s(a/A) = 0$ iff $a \in \operatorname{acl}(A)$ for all a, A.

We will apply the following fact.

Fact 2.18 ([S1, Lemma 7.3]). *For* $a \in \mathcal{C}$ *and* $A \subseteq B \subseteq \mathcal{C}$*, assume* $\operatorname{tp}(a/B)$ *does not fork over* $\operatorname{acl}(aA) \cap \operatorname{acl}(B)$ *and* $a \underset{A}{\not\perp} B$*. Then* $a \underset{A}{\not\perp}^{s} B$.

3. Elimination of hyperimaginaries in reducts. In this section we include a remark by Ehud Hrushovski that allowed us to remove the assumption that the reducts eliminate hyperimaginaries (in the main theorem). Here T denotes any complete theory in a language L, and we work in $\mathcal{C}$.

Definition 3.1. A reduct T^- of T to a sublanguage $L^- \subseteq L$ is said to be E*-closed* if for any sort S of L^-, for any L^--definable sets $D_1 \vdash D_2$ on S^2, there exists a definable equivalence relation $E^- \in L^-$ satisfying $D_1 \vdash E^- \vdash D_2$, provided that there exists such a relation in L.

For a partial order $(P, \leq)$, a subset $A \subseteq P$ is called a *club* in $(P, \leq)$ if A is *unbounded* in $(P, \leq)$, that is, above any element of P there is an element of A, and A is *closed* in $(P, \leq)$, that is, for any chain $C \subseteq A$, if $a \in P$ is the supremum of C (i.e. a is an upper bound of C and a is smaller than any other upper bound of C) then $a \in A$.

Notation 3.2. Let T^- be a reduct of T to L^-. The size of the reduct T^- is just $|T^-|$. Let λ be any infinite cardinal (or ∞). Let $(\mathcal{R}_T^{\lambda}, \leq_T)$ be the partial order of all reducts of T of size $\leq \lambda$, where the order is just inclusion (of the sublanguages of the reducts, i.e. of both the set of sorts and the set of formulas). It will be convenient to consider the isomorphic partial order $(\mathcal{R}^{\lambda}_{\mathcal{C}}, \leq_{\mathcal{C}})$ of all the (saturated) model reducts of $\mathcal{C}$ to a sublanguage of L size $\leq \lambda$.

Claim 3.3. *Let T be any complete L-theory that eliminates hyperimaginaries.*

1. *Let T^- be an E-closed reduct of T. Then T^- eliminates hyperimaginaries.*
2. *The set of E-closed reducts of T is a club in $(\mathcal{R}^{\infty}_{T}, \leq_T)$. Given any infinite $\lambda \leq |L|$, the set of E-closed reducts of T of size $\leq \lambda$ is a club in $(\mathcal{R}^{\lambda}_{T}, \leq_T)$.*

Proof. (1) Say T^- is the reduct of T to L^-, and so $\mathcal{C}|L^-$ is a saturated model of T^-. We claim that the hyperimaginaries of T^- are eliminated, namely: for every type-definable equivalence relation E^- of T^- on a complete type p^- of T^- over $\emptyset$, there are definable equivalence relations $E^-_i \in L^-$ such that E^- is equivalent to $\bigwedge_i E^-_i$ on p^-.

Indeed, let $E^- = E^-(x,x')$ and $p^- = p^-(x)$ be such. Let $\phi^-_i(x,x') \in L^-$ be such that $E^-(x,x') = \bigwedge_i \phi^-_i(x,x')$. Let p be any complete type of T over $\emptyset$ that extends p^-. By elimination of hyperimaginaries in T, there are $E_j(x,x') \in L$ such that $\bigwedge_j E_j(x,x')$ is equivalent to $E^-(x,x')$ on $p^{\mathcal{C}}$. By compactness, for any i there is $j(i)$ such that $E_{j(i)}(x,x') \vdash \phi^-_i(x,x')$ on $p^{\mathcal{C}}$, likewise for every j there exists $k(j)$ such that $\phi^-_{k(j)}(x,x') \vdash E_j(x,x')$ on $p^{\mathcal{C}}$. As T^- is an E-closed reduct of T, for every i there is a definable equivalence relation $E^-_i \in L^-$ such that $\phi^-_{k(j(i))}(x,x') \vdash E^-_i(x,x') \vdash \phi^-_i(x,x')$ on $p^{\mathcal{C}}$ (using compactness). We conclude that E^- is equivalent to $\bigwedge_i E^-_i(x,x')$ on $p^{\mathcal{C}}$, and thus on $p^-\mathcal{C}$ as well (as E^- and E^-_i are all invariant under automorphisms of $\mathcal{C}|L^-$).

(2) is immediate.

4. Dichotomies for $\emptyset$-invariant families of rank 1 types. In this section we prove an extension of the dichotomy theorem [S2, Corollary 2.13] which roughly says that the failure of a certain amount of 1-basedness around an *SU*-rank 1 type implies the existence a weakly minimal formula, almost internal in that *SU*-rank 1 type. Here T is assumed to be hypersimple and we work in $\mathcal{C}^{\mathrm{eq}}$.

We first recall some basic definitions from [S1].

Definition 4.1. A family

$$\Upsilon = \{\Upsilon_{x,A} \mid x \text{ is a finite sequence of variables and } A \subset \mathcal{C} \text{ is small}\}$$

is said to be a *projection closed family of topologies* if: each $\Upsilon_{x,A}$ is a topology on $S_x(A)$ that refines the Stone topology on $S_x(A)$; this family is invariant under automorphisms of $\mathcal{C}$ and change of variables to variables of the same sort; and the family is closed under product by the full Stone spaces $S_y(A)$ (where y is a disjoint tuple of variables) and closed under projections, namely whenever $\mathcal{U}(x,y) \in \Upsilon_{xy,A}$, then $\exists y\, \mathcal{U}(x,y) \in \Upsilon_{x,A}$.

We will be interested in the case $\Upsilon = \tau^f$, where T is a PCFT theory. From now on fix a general projection closed family Υ of topologies.

Definition 4.2. 1) A type $p \in S(A)$ is said to be *s-essentially* 1*-based over* $A_0 \subseteq A$ [*essentially* 1*-based over* $A_0 \subseteq A$] *by means of* Υ if for every finite tuple $\bar{c}$ from p and for every [type-definable] Υ-open set $\mathcal{U}$ over $A\bar{c}$ with the property that a is independent from A over A_0 for every $a \in \mathcal{U}$, the set $\{a \in \mathcal{U} \mid Cb(a/A\bar{c}) \not\subseteq \mathrm{bdd}(aA_0)\}$ is nowhere dense in the Stone topology of $\mathcal{U}$. We say $p \in S(A)$ is *s-essentially* 1*-based* [*essentially* 1*-based*] *by means of* Υ if p is s-essentially 1-based [essentially 1-based] over A by means of Υ.

2) Let V be an A_0-invariant set and let $p \in S(A_0)$. We say that p is *analyzable in* V *by s-essentially* 1*-based* [*essentially* 1*-based*] *types by means of* Υ if there exists $a \models p$ and there exists a sequence $(a_i \mid i \leq \alpha) \subseteq \mathrm{dcl}(A_0 a)$ with $a_\alpha = a$ such that $\mathrm{tp}(a_i/A_0 \cup \{a_j \mid j < i\})$ is V-internal and s-essentially 1-based [essentially 1-based] over A_0 by means of Υ for all $i \leq \alpha$.

We start with a generalization of [S2, Theorem 2.11] to a $\emptyset$-invariant family of *SU*-rank 1 types. This generalization assumes the language is countable; a version for a possibly uncountable language is phrased at the end of this section.

Theorem 4.3. *Let* T *be any countable hypersimple theory. Let* Υ *be a projection-closed family of topologies with* $\{a \in \mathcal{C}^x \mid a \notin \mathrm{acl}(A)\} \in \Upsilon_{x,A}$ *for all* x *and every set* A. *Let* $\mathcal{P}_0$ *be an* $\emptyset$*-invariant family of SU-rank* 1 *types. Then either there exists an unbounded type-definable* Υ*-open set over some small set that is almost* $\mathcal{P}_0$*-internal and has* ***bounded*** *finite SU-rank, or every complete type* $p \in S(A)$ *that is internal in* $\mathcal{P}_0$ *is essentially* 1*-based over* $\emptyset$ *by means of* Υ. *In particular, either there exists an unbounded type-definable* Υ*-open set that is almost* $\mathcal{P}_0$*-internal and has* ***bounded*** *finite SU-rank, or whenever* $p \in S(A)$, *where* A *is countable, and* p *is analyzable in* $\mathcal{P}_0$, p *is analyzable in* $\mathcal{P}_0$ *by essentially* 1*-based types by means of* Υ.

Proof. Υ will be fixed and we will freely omit the phrase "by means of Υ". To see the "in particular" part, work over a countable A and assume that every $p' \in S(A')$ with countable $A' \supseteq A$ that is internal in $\mathcal{P}_0$, is essentially 1-based over A. Moreover, assume $p \in S(A)$ is non-algebraic and every non-algebraic extension of p is non-foreign to $\mathcal{P}_0$. Then for $a \models p$ there exists $a' \in \mathrm{dcl}(Aa) \setminus \mathrm{acl}(A)$ such that $\mathrm{tp}(a'/A)$ is $\mathcal{P}_0$-internal and thus essentially 1-based over A by our assumption. By repeating this process we conclude that p is analyzable in $\mathcal{P}_0$ by essentially 1-based types.

We now prove the main part. Assume there exists $p \in S(A)$ that is internal in $\mathcal{P}_0$ and is not essentially 1-based over $\emptyset$. By the definition, there exist a finite tuple d of realizations of p and b that is independent from d over A and a finite tuple $\bar{c}$ of realizations of types from $\mathcal{P}_0$ over Ab such that $d \in \mathrm{dcl}(Ab\bar{c})$, and there exists a type-definable Υ-open set $\mathcal{U}$ over Ad such that a is independent from A for all $a \in \mathcal{U}$ and $\{a \in \mathcal{U} \mid Cb(a/Ad) \not\subseteq \mathrm{acl}(a)\}$ is not nowhere dense in the Stone topology of $\mathcal{U}$. So, since Υ refines the Stone topology, by intersecting $\mathcal{U}$ with a definable set we may assume that $\{a \in \mathcal{U} \mid Cb(a/Ad) \not\subseteq \mathrm{acl}(a)\}$ is dense in the Stone topology of $\mathcal{U}$. Now, for each (finite) subsequence $\bar{c}_0$ of $\bar{c}$, let

$$F_{\bar{c}_0} = \{a \in \mathcal{U} \mid \exists b', \bar{c}'_0, \bar{c}'_1 \ \mathrm{tp}(b'\bar{c}'_0\bar{c}'_1/Ad) = \mathrm{tp}(b\bar{c}_0(\bar{c} \setminus \bar{c}_0)/Ad) \text{ and } a \perp Ab'\bar{c}'_0\}.$$

Note that since d is independent from b over A, any $a \in \mathcal{U}$ is independent from Ab' whenever $\mathrm{tp}(b'/Ad) = \mathrm{tp}(b/Ad)$ and $a \underset{Ad}{\perp} b'$. Thus $F_{\langle\rangle} = \mathcal{U}$. Let $\bar{c}^*_0$ be a maximal subsequence (with respect to inclusion) of $\bar{c}$ such that $F_{\bar{c}^*_0}$ has non-empty Stone interior in $\mathcal{U}$ over Ad (note that $F_{\bar{c}}$ has no Stone interior relative to $\mathcal{U}$). Let $\mathcal{U}^* = \bigcap_{\bar{c}^*_0 \subset \bar{c}' \subseteq \bar{c}} (\mathcal{U} \setminus F_{\bar{c}'})$. Note that each $F_{\bar{c}'}$ is relatively Stone-closed in $\mathcal{U}$. Thus $\mathcal{U}^*$ is Stone-dense and Stone-open in $\mathcal{U}$ and therefore there exists a non-empty relatively Stone-open (in $\mathcal{U}$) set $W^* \subseteq F_{\bar{c}^*_0} \cap \mathcal{U}^*$. As $\mathcal{U}$ is type-definable, we may assume W^* is type-definable.

Claim 4.4. *W^* is a non-empty Υ-open set over Ad such that $\{a \in W^* \mid Cb(a/Ad) \not\subseteq \mathrm{acl}(a)\}$ is dense in the Stone topology of W^* and for every $a \in W^*$ there exists $b'\bar{c}'_0\bar{c}'_1 \models \mathrm{tp}(b\bar{c}^*_0(\bar{c} \setminus \bar{c}^*_0)/Ad)$ such that a is independent from $Ab'\bar{c}'_0$ over $\emptyset$, and moreover for every $b'\bar{c}'_0\bar{c}'_1 \models \mathrm{tp}(b\bar{c}^*_0(\bar{c} \setminus \bar{c}^*_0)/Ad)$ such that a is independent from $Ab'\bar{c}'_0$ we necessarily have $\bar{c}'_1 \in \mathrm{acl}(aAb'\bar{c}'_0)$.*

Proof. The first part is immediate from the fact that $W^* \subseteq F_{\bar{c}^*_0}$. For the "moreover" part, note that since $a \in W^* \subseteq \mathcal{U}^*$, by the definition of $\mathcal{U}^*$ we have $c' \in \mathrm{acl}(aAb'\bar{c}'_0)$ for all $b'\bar{c}'_0\bar{c}'_1 \models \mathrm{tp}(b\bar{c}^*_0(\bar{c} \setminus \bar{c}^*_0)/Ad)$ and $c' \in \bar{c}'_1$ (as $SU(c'/Ab') \leq 1$ for every $c' \in \bar{c}'_1$). ■

Let us now define a set V over Ad by

$$V = \{(e', b', \bar{c}'_0, \bar{c}'_1, a') \mid \text{if } \mathrm{tp}(b'\bar{c}'_0\bar{c}'_1/Ad) = \mathrm{tp}(b\bar{c}^*_0(\bar{c} \setminus \bar{c}^*_0)/Ad) \text{ and } a' \perp Ab'\bar{c}'_0 \text{ then } e' \in \mathrm{acl}(Cb(Ab'\bar{c}'_0\bar{c}'_1/a'))\}.$$

Let $V^* = \{e' \mid \exists a' \in W^* \ \forall b', \bar{c}'_0, \bar{c}'_1 \ V(e', b', \bar{c}'_0, \bar{c}'_1, a')\}$.

Claim 4.5. *V^* is a Υ-open set over Ad.*

Proof. Recall the following fact [S2, Proposition 2.4].

FACT 4.6. *Let* $q(x, y) \in S(\emptyset)$ *and let* $\chi(x, y, z)$ *be an* $\emptyset$*-invariant set such that* $b \trianglerighteq_a bc$ *for all* $(c, b, a) \models \chi(x, y, z)$*. Then the set*

$$\mathcal{U} = \{(e, c, b, a) \mid e \in \operatorname{acl}(Cb(cb/a))\}$$

is relatively Stone-open inside the set

$$F = \{(e, c, b, a) \mid b \mathop{\smile\!\!\!\!\!\!\,|}\, a, \models \chi(c, b, a), \operatorname{tp}(cb) = q\}$$

(*where e is taken from a fixed sort too*).

By Fact 4.6 and Claim 4.4, there exists a Stone-open set V' over Ad such that for all $a' \in W^*$ and for all $e', b', \bar{c}'_0, \bar{c}'_1$ we have $V'(e', b', \bar{c}'_0, \bar{c}'_1, a')$ if and only if $V(e', b', \bar{c}'_0, \bar{c}'_1, a')$. Thus we may replace V by V' in the definition of V^*. As Stone-open sets are closed under the $\forall$ quantifier, and the Υ topology refines the Stone topology and is closed under product by full Stone spaces and closed under projections, we conclude that V^* is a Υ-open set. ■

CLAIM 4.7. *For an appropriate sort for* e'*, the set* V^* *is unbounded and is almost* $\mathcal{P}_0$*-internal* (*over* Ad) *and thus has finite SU-rank over* Ad*.*

Proof. First, note the following general observation (following easily from Fact 2.1).

FACT 4.8. *Assume* $d \in \operatorname{dcl}(c)$*. Then* $Cb(d/a) \in \operatorname{dcl}(Cb(c/a))$ *for all* a*.*

Let $a^* \in W^*$ be such that $Cb(a^*/Ad) \not\subseteq \operatorname{acl}(a^*)$. Then $Cb(Ad/a^*) \not\subseteq \operatorname{acl}(Ad)$. Fact 4.8 implies that there exists $e^* \notin \operatorname{acl}(Ad)$ such that $e^* \in \operatorname{acl}(Cb(Ab'\bar{c}'_0\bar{c}'_1/a^*))$ for all $b'\bar{c}'_0\bar{c}'_1 \models \operatorname{tp}(b\bar{c}^*_0(\bar{c} \setminus \bar{c}^*_0)/Ad)$. In particular, $e^* \in V^*$. Thus, if we fix the sort for e' in the definition of V^* to be the sort of e^*, then V^* is unbounded. Now, let $e' \in V^*$. Then for some $a' \in W^*$, we have $\models V(e', \bar{c}'_0, \bar{c}'_1, b', a')$ for all $b', \bar{c}'_0, \bar{c}'_1$. By Claim 4.4, there exists $b'\bar{c}'_0\bar{c}'_1 \models \operatorname{tp}(b\bar{c}^*_0(\bar{c} \setminus \bar{c}^*_0)/Ad)$ such that a' is independent from $Ab'\bar{c}'_0$ over $\emptyset$. Thus, by the definition of V^* and V, we see that $e' \in \operatorname{acl}(Cb(Ab'\bar{c}'_0\bar{c}'_1/a'))$. Since Ab' is independent from a' over $\emptyset$, $\operatorname{tp}(e')$ is almost $\mathcal{P}_0$-internal (as $Cb(Ab'\bar{c}'_0\bar{c}'_1/a')$ is in the definable closure of any Morley sequence of $\operatorname{Lstp}(Ab'\bar{c}'_0\bar{c}'_1/a')$ by Fact 2.2). In particular, $\operatorname{tp}(e'/Ad)$ is almost $\mathcal{P}_0$-internal by Fact 2.3, and therefore $\operatorname{tp}(e'/Ad)$ has finite SU-rank. ■

CLAIM 4.9. *There exists* $V^{**} \subseteq V^*$ *that is unbounded, type-definable and* Υ*-open over* Ad*.*

Proof. By the definition of V^* and the proof of Claim 4.5 there exists a Stone-open set V_0 over Ad such that $V^* = \{e' \mid \exists a' \in W^* \ V_0(e', a')\}$. By replacing V_0 by a definable set and using the fact that W^* is type-definable and that Υ is a projection-closed family of topologies, we get the required set V^{**}. ■

Now, by the proof of Claim 4.7 we know that for all $e' \in V^{**}$ we have $e' \in \operatorname{acl}(Cb(Ab'\bar{c}'_0\bar{c}'_1/a'))$ for some $a' \in W^*$ and some $b', \bar{c}'_0, \bar{c}'_1$ such that a' is independent from $Ab'\bar{c}'_0$ over $\emptyset$ and $b'\bar{c}'_0\bar{c}'_1 \models \operatorname{tp}(b\bar{c}^*_0(\bar{c} \setminus \bar{c}^*_0)/Ad)$. Let $q = \operatorname{tp}(Ab)$. For every $\chi = \chi(x, y_0, \ldots, y_n, \bar{z}_0, \bar{z}_1, \ldots \bar{z}_n) \in L$ (for some $n < \omega$) such that $\forall y_0 y_1 \ldots y_n \bar{z}_0 \bar{z}_1 \ldots \bar{z}_n \, \exists^{<\infty} x \, \chi(x, y_0, y_1, \ldots, y_n, \bar{z}_0, \bar{z}_1, \ldots, \bar{z}_n)$, let

$$\begin{aligned} F_\chi = \{e \in V^{**} \mid \, \models \chi(e, C_0, C_1, \ldots, C_n, \bar{c}_0, \bar{c}_1, \ldots \bar{c}_n) &\text{ for some } \bar{c}_0, \ldots, \bar{c}_n \text{ and} \\ &\text{some } \emptyset\text{-independent sequence } (C_i \mid i \leq n) \text{ of realizations} \\ &\text{of } q \text{ such that } \operatorname{tp}(C_i, \bar{c}_i) = \operatorname{tp}(Ab, \bar{c}) \text{ and } e \perp (C_i \mid i \leq n)\}. \end{aligned}$$

Note that each F_χ is type-definable. By the above, we find that $V^{**} \subseteq \bigcup_\chi F_\chi$ (the union is over all χ as above). By the Baire category theorem applied to the Stone topology of the Stone-closed set $V^{**} \setminus \operatorname{acl}(Ad)$, there exists $\theta \in L(Ad)$ such that

$$\tilde{V} \equiv \theta^{\mathcal{C}} \cap (V^{**} \setminus \operatorname{acl}(Ad)) \neq \emptyset \quad \text{and} \quad \tilde{V} \subseteq F_{\chi^*}$$

for some χ^* as above. Clearly, $\tilde{V}$ is unbounded, type-definable and Υ-open (by the assumptions on Υ). Now, there exists a fixed $m^* < \omega$ such that for every $a \in \tilde{V}$, $SU(a/Ad) \leq m^*$ and $\operatorname{tp}(a/Ad)$ is almost $\mathcal{P}_0$-internal (as $\operatorname{tp}(a)$ is almost $\mathcal{P}_0$-internal). This completes the proof of the main part of the theorem. ∎

We now state the main theorem of this section.

THEOREM 4.10. *Let T be any countable hypersimple theory with PCFT. Let $\mathcal{P}_0$ be an $\emptyset$-invariant family of SU-rank 1 types. Then either there exists a weakly minimal formula that is almost $\mathcal{P}_0$-internal, or every complete type $p \in S(A)$ that is internal in $\mathcal{P}_0$ is essentially 1-based over $\emptyset$ by means of τ^f. In particular, either there exists a weakly minimal formula that is almost $\mathcal{P}_0$-internal, or whenever $p \in S(A)$, where A is countable, and p is analyzable in $\mathcal{P}_0$, p is analyzable in $\mathcal{P}_0$ by essentially 1-based types by means of τ^f.*

Proof. Our assumptions are clearly a special case of the assumptions of Theorem 4.3, thus we only need to prove the first part. By the conclusion of Theorem 4.3, we may assume that there exists a τ^f-open set $\mathcal{U}$ of bounded finite *SU*-rank over some small set A that is almost $\mathcal{P}_0$-internal. By Fact 2.12, there exists a weakly minimal $\theta(x, b) \in L(B)$ for some small set $B \supseteq A$ such that $\theta^{\mathcal{C}} \subseteq \mathcal{U} \cup \operatorname{acl}(B)$. Now, $\operatorname{tp}(a/B)$ is almost $\mathcal{P}_0$-internal for every $a \in \theta^{\mathcal{C}}$, and so $\operatorname{tp}(a/b)$ (b is the parameter of $\theta(x, b)$) is almost $\mathcal{P}_0$-internal for every $a \in \theta^{\mathcal{C}}$ (by taking non-forking extensions). ∎

Finally, we state a version of Theorem 4.3 for a possibly uncountable language; it generalizes [S2, Theorem 2.3] to any $\emptyset$-invariant family of *SU*-rank 1 types. The proof is almost identical to the proof of [S2, Theorem 2.3] (the necessary modifications are essentially included in the proof of Theorem 4.3).

Theorem 4.11. *Let T be any hypersimple theory. Let Υ be a projection-closed family of topologies. Let $\mathcal{P}_0$ be an $\emptyset$-invariant family of SU-rank* 1 *types. Then either there exists an unbounded Υ-open set* (*over some small set A*) *that is almost $\mathcal{P}_0$-internal* (*and in particular has finite SU-rank*), *or every complete type $p \in S(A)$ that is internal in $\mathcal{P}_0$ is s-essentially* 1*-based over $\emptyset$ by means of Υ. In particular, either there exists an unbounded Υ-open set that is almost $\mathcal{P}_0$-internal, or whenever $p \in S(A)$ and p is analyzable in $\mathcal{P}_0$, p is analyzable in $\mathcal{P}_0$ by s-essentially* 1*-based types by means of Υ.*

5. Main result.

From now on we assume T is an arbitrary simple theory with elimination of imaginaries unless stated otherwise. We work in $\mathcal{C}$.

Definition 5.1. We say that T is *analyzable in SU-rank* 1 *types* if every type is analyzable in the family of SU-rank 1 types.

We say that T is *coordinatized in finite rank* if for every $a \in \mathcal{C}$ and $A \subseteq \mathcal{C}$ such that $a \notin \operatorname{acl}(A)$ there exists $a' \in \operatorname{acl}(aA) \setminus \operatorname{acl}(A)$ with $SU(a'/A) < \omega$.

Lemma 5.2. *Assume T is hypersimple. Then T is coordinatized in finite rank iff T is analyzable in SU-rank* 1 *types.*

Proof. If T is analyzable in SU-rank 1 types, then clearly T is coordinatized in finite rank. Assume now that T is coordinatized in finite rank. We first note the following.

Claim 5.3. *Let T be any simple theory. Let $a \in \mathcal{C}$ and $C \subseteq \mathcal{C}$ be such that $SU(a/C) = n < \omega$ and $SU(a/bC) = n-1$ for some $b \in \mathcal{C}$ with $SU(b/C) < \infty$. Then* $\operatorname{tp}(a/C)$ *is non-orthogonal to some SU-rank* 1 *hyperimaginary type.*

Proof. Let $e = Cb(\operatorname{LStp}(a/bC))$ (e is a hyperimaginary). Since $SU(e/C) < \infty$ (as we assume $SU(b/C) < \infty$), there exists a set $A \supseteq C$ such that $SU(e/A) = 1$. By extension we may clearly assume $a \mathop{\smile\!\!\!\!\!\!\mid}_{eC} A$, so by the definition of e, $a \mathop{\smile\!\!\!\!\!\!\mid}_{e} A$. We claim that

$$(*) \qquad e \in \operatorname{bdd}(aA).$$

Indeed, otherwise $e \mathop{\smile\!\!\!\!\!\!\mid}_{A} a$ and so $e \in \operatorname{bdd}(A)$ (as $\operatorname{tp}(a/e)$ is canonical), contrary to $SU(e/A) = 1$.

Now, $SU(a/eA) = SU(a/e) = n-1$. By ($*$), $SU(a/A) = SU(ae/A) \geq SU(a/eA) + SU(e/A) = n$. Thus $a \mathop{\smile\!\!\!\!\!\!\mid}_{C} A$ and so $\operatorname{tp}(a/C)$ is non-orthogonal to $\operatorname{tp}(e/A)$. ∎

Now, let a, A be such that $a \notin \operatorname{acl}(A)$. By our assumption, there exists $a' \in \operatorname{acl}(aA) \setminus \operatorname{acl}(A)$ with $SU(a'/A) = n$ for some $n < \omega$. Let $b \in \mathcal{C}$ be such that $SU(a/Ab) = n-1$ and let $(b_i \mid i < \alpha)$ be such that $b_i \in \operatorname{acl}(bA)$ and $0 < SU(b_i/Ab_{<i}) < \omega$ for all $i < \alpha$ and $\operatorname{acl}(Ab) = \operatorname{acl}(A \cup \{b_i \mid i < \alpha\})$.

As $a' \underset{A}{\not\perp} b$, there exists a minimal $i^* < \alpha$ such that $a' \underset{A}{\not\perp} \{b_i \mid i \leq i^*\}$. Then $a' \underset{A}{\perp} \{b_i \mid i < i^*\}$. Now $a', A \cup \{b_i \mid i < i^*\}, b_{i^*}$ satisfies the assumptions of Claim 5.3 (take $a = a'$, $C = A \cup \{b_i \mid i < i^*\}$, $b = b_{i^*}$ in that claim). Thus $\mathrm{tp}(a'/A \cup \{b_i \mid i < i^*\})$ is non-orthogonal to some SU-rank 1 type which we may clearly assume to be a type of an imaginary. Thus $\mathrm{tp}(a'/A)$ is non-orthogonal to an SU-rank 1 imaginary type, and so also is $\mathrm{tp}(a/A)$ (as $a' \in \mathrm{acl}(aA)$). ∎

We start with the following proposition that generalizes the main result in [S1]; the proof is similar to that of [S1] but uses Theorem 4.10.

Proposition 5.4. *Assume T is a countable hypersimple theory that is coordinatized in finite rank and eliminates $\exists^\infty$. Then there exists a weakly minimal formula.*

Proof. By Lemma 5.2, T is analyzable in SU-rank 1 types. As T eliminates $\exists^\infty$, T is EPFO by Fact 2.13. By Fact 2.11, T is PCFT. Let $\mathcal{P}_0$ be the family of all SU-rank 1 types. By Theorem 4.10, we may assume that every complete finitary type over a countable set is analyzable in $\mathcal{P}_0$ by essentially 1-based types by means of τ^f. The following fact is a special case of [S3, Corollary 4.5]. In fact, [S3, Corollary 4.5] says the statement below is valid for any $\mathcal{U}_0$ that is a $\tilde{\tau}^f_{\mathrm{low}}$-set (see [S3]).

Fact 5.5 ([S3, Corollary 4.5, a special case]). *Let T be a countable simple theory with EPFO. Let $\mathcal{U}_0 = \mathcal{C}^s \setminus \mathrm{acl}(\emptyset)$ for some non-algebraic sort s. Assume for every $a \in \mathcal{U}_0$ there exists $a' \in \mathrm{acl}(a) \setminus \mathrm{acl}(\emptyset)$ such that $SU_{se}(a') < \omega$. Then there exists an unbounded τ^f_∞-open set $\mathcal{U}$ over a finite set such that $\mathcal{U}$ has bounded finite SU_{se}-rank.*

Since T satisfies the assumptions of Fact 5.5, let $\mathcal{U}$ be a set as in its conclusion. In particular, $SU_s(\mathcal{U}) = n$ for some $n < \omega$. Recall now the following easy lemma.

Fact 5.6 ([S1, Lemma 7.4]). *Assume $\mathcal{U}$ is an unbounded τ^f_∞-open set of bounded finite SU_s-rank over some finite set A. Then there exists a τ^f_∞-open set $\mathcal{U}^* \subseteq \mathcal{U}$ over some finite set $B^* \supseteq A$ of SU_s-rank 1.*

By Fact 5.6, we may assume $SU_s(\mathcal{U}) = 1$ and $\mathcal{U}$ is a type-definable τ^f-open set over a finite set A_0. We claim $SU(\mathcal{U}) = 1$. Indeed, otherwise there exists a and $d \in \mathcal{U}$ such that $d \underset{A_0}{\not\perp} a$ and $d \notin \mathrm{acl}(aA_0)$. As every finitary type over a countable set is analyzable in $\mathcal{P}_0$, there exists $(a_i \mid i \leq \alpha) \subseteq \mathrm{dcl}(aA_0)$ with $a_\alpha = a$ (where $\alpha < \omega_1$) such that $\mathrm{tp}(a_i/A_0 \cup \{a_j \mid j < i\})$ is essentially 1-based over A_0 by means of τ^f for all $i \leq \alpha$. Now, let $i^* \leq \alpha$ be minimal such that there exists $d' \in \mathcal{U}$ satisfying $d' \underset{A_0}{\not\perp} \{a_i \mid i \leq i^*\}$ and

$d' \notin \mathrm{acl}(A_0 \cup \{a_i \mid i \leq i^*\})$. Pick $\phi(x, a') \in L(A_0 \cup \{a_i \mid i \leq i^*\})$ that forks over A_0 and such that $\phi(d', a')$. Let

$$V = \bigl\{d \in \mathcal{U} \mid \phi(d, a') \text{ and } d \notin \mathrm{acl}(A_0 \cup \{a_i \mid i \leq i^*\})\bigr\}.$$

By minimality of i^*, d is independent from $\{a_i \mid i < i^*\}$ over A_0 for all $d \in V$. Clearly V is type-definable, and by Fact 2.8, V is a τ^f-open set over $A_0 \cup \{a_i \mid i \leq i^*\}$. Now, since $\mathrm{tp}(a_{i^*}/A_0 \cup \{a_i \mid i < i^*\})$ is essentially 1-based over A_0 by means of τ^f, the set

$$\bigl\{d \in V \mid Cb(d/A_0 \cup \{a_i \mid i \leq i^*\}) \in \mathrm{bdd}(dA_0)\bigr\}$$

contains a relatively Stone-open and Stone-dense subset of V. In particular, there exists $d^* \in V$ such that $\mathrm{tp}(d^*/A_0 \cup \{a_i \mid i \leq i^*\}$ does not fork over $\mathrm{acl}(A_0 d^*) \cap \mathrm{acl}(A_0 \cup \{a_i \mid i \leq i^*\})$. Since we know that

$$d^* \mathop{\not\perp}_{A_0} A_0 \cup \{a_i \mid i \leq i^*\},$$

Fact 2.18 implies $d^* \mathop{\not\perp}^{s}_{A_0} A_0 \cup \{a_i \mid i \leq i^*\}$. Hence $d^* \in V$ yields $SU_s(d^*/A_0) \geq 2$, which contradicts $SU_s(\mathcal{U}) = 1$. Thus we have proved $SU(\mathcal{U}) = 1$. By Fact 2.12 there exists a definable set of SU-rank 1. ∎

Before stating the main theorem, we give some terminology and easy remarks. From now on we work in $\mathcal{C} = \mathcal{C}^{\mathrm{eq}}$. Recall that $(\mathcal{R}^{\lambda}_{\mathcal{C}}, \leq_{\mathcal{C}})$ is the partial order of reducts of $\mathcal{C}$ of size $\leq \lambda$.

Definition 5.7. Let $\mathcal{C}|L^- \in \mathcal{R}^{\lambda}_{\mathcal{C}}$. We will say that $\mathcal{C}|L^-$ is *eq-closed* if $T^- = \mathrm{Th}(\mathcal{C}|L^-)$ has uniform elimination of imaginaries, i.e. for every definable equivalence relation $E \in L^-$ on $S_0 \times S_1 \times \cdots \times S_k$, where S_i are sorts of L^-, there is a definable function $f_E \in L^-$ whose domain is $(S_0 \times S_1 \times \cdots \times S_k)^{\mathcal{C}}$ such that for all $\bar{a}, \bar{b}$, we have $f_E(\bar{a}) = f_E(\bar{b})$ iff $E(\bar{a}, \bar{b})$.

Remark 5.8. *For every reduct $\mathcal{C}|L^- \in \mathcal{R}^{\lambda}_{\mathcal{C}}$ there exists a reduct $\mathcal{C}|L^* \in \mathcal{R}^{\lambda}_{\mathcal{C}}$ that is eq-closed and is an expansion of $\mathcal{C}|L^-$. Thus for every infinite cardinal λ, the set of reducts in $\mathcal{R}^{\lambda}_{\mathcal{C}}$ that are eq-closed is a club in $(\mathcal{R}^{\lambda}_{\mathcal{C}}, \leq_{\mathcal{C}})$.*

Proof. Expand the reduct $\mathcal{C}|L^-$ of $\mathcal{C} = \mathcal{C}^{\mathrm{eq}}$ by adding for every definable equivalence relation E on $S_0 \times S_1 \times \cdots \times S_k$, where S_i are sorts of L^- and $E \in L^-$, a definable function $f_E \in L$ (together with its image) whose domain is $(S_0 \times S_1 \times \cdots \times S_k)^{\mathcal{C}}$ and is onto the interpretation of some sort of L such that $f_E(\bar{a}) = f_E(\bar{b})$ iff $E(\bar{a}, \bar{b})$. Now, the resulting expansion will have uniform elimination of imaginaries. It is immediate that the set of eq-closed reducts in $\mathcal{R}^{\lambda}_{\mathcal{C}}$ is closed in $(\mathcal{R}^{\lambda}_{\mathcal{C}}, \leq_{\mathcal{C}})$. ∎

Now, note the following easy general remark on clubs.

Remark 5.9. Let $(P, \leq)$ be a directed partial order that is ω-closed (i.e. any increasing sequence $(a_i \mid i < \omega)$ has a supremum). Then the intersection of finitely many clubs in $(P, \leq)$ is a club.

In the proof we will refer to the following notion.

Definition 5.10. We say that T is *strongly non-supersimple* if $D(\phi(x,a)) = \infty$ for every non-algebraic $\phi(x,a) \in L(\mathcal{C})$.

Remark 5.11. Note that T is strongly non-supersimple iff for every non-algebraic $\phi(x,a) \in L(\mathcal{C})$ there exists a non-algebraic $\psi(x,b) \in L(\mathcal{C})$ such that $\psi(x,b) \vdash \phi(x,a)$ and $\psi(x,b)$ forks over a iff there does not exist a weakly minimal formula.

We now state the main result of this paper.

Theorem 5.12. *Let* $T = T^{\mathrm{eq}}$ *be a hypersimple unidimensional theory. Assume there is a club of countable reducts of* T *in* $(\mathcal{R}_{\mathcal{C}}^{\aleph_0}, \leq_{\mathcal{C}})$ *that are coordinatized in finite rank. Then* T *is supersimple.*

Proof. First, if T is not strongly non-supersimple then we are done by Fact 2.4. Therefore we may assume T is strongly non-supersimple. By Fact 2.5, T eliminates $\exists^{\infty}$, thus every reduct of T eliminates $\exists^{\infty}$.

Claim 5.13. *The set* $\tilde{\mathcal{C}_1}$ *of countable strongly non-supersimple reducts of* $\mathcal{C}$ *is a club in* $(\mathcal{R}_{\mathcal{C}}^{\aleph_0}, \leq_{\mathcal{C}})$.

Proof. First, we prove that $\tilde{\mathcal{C}_1}$ is unbounded in $(\mathcal{R}_{\mathcal{C}}^{\aleph_0}, \leq_{\mathcal{C}})$. Let $\mathcal{C}|L^- \in \mathcal{R}_{\mathcal{C}}^{\aleph_0}$. We construct by induction an increasing sequence $(\mathcal{C}_n \mid n < \omega)$ of reducts, $\mathcal{C}_n \in \mathcal{R}^{\aleph_0}$, where $\mathcal{C}_n = \mathcal{C}|L_n^-$ for some countable sublanguage L_n^- of L, $T_n^- = \mathrm{Th}(\mathcal{C}_n)$ in the following way. Let $\mathcal{C}_0 = \mathcal{C}|L_0^-$, $L_0^- = L^-$ and assume L_k^- have already been defined for $k \leq n$. We define L_{n+1}^-. For any fixed $\phi(x,y) \in L_n^- \setminus L_{n-1}^-$ we define a finite set of formulas $\Delta_\phi = \{\psi_i \mid i \leq n(\phi)\}$, where $\psi_i = \psi_i(x, y_i) \in L = L(T)$, $n(\phi) < \omega$ in the following way. Since T is strongly non-supersimple, for every $a \in \mathcal{C}$ such that $\exists^{\infty} x\ \phi(x,a)$, there exists $\psi_a(x,z) \in L$ and some $b \in \mathcal{C}$ such that $\psi_a(x,b) \vdash \phi(x,a)$, $\psi_a(x,b)$ forks over a and

$$(*) \qquad \exists^{\infty} x\ \psi_a(x,b).$$

For every $\psi(x,z) \in L$ let

$$\theta_{\phi,\psi}(z,y) = \exists^{\infty} x\ \psi(x,z) \wedge \forall x\ (\psi(x,z) \to \phi(x,y)).$$

By elimination of $\exists^{\infty}$ (in $\mathcal{C}$), $\theta_{\phi,\psi}(z,y)$ is definable. Now, let $Q_{\psi,\theta_{\phi,\psi}}(y)$ be the relation in Fact 2.10 defined for $\theta_{\phi,\psi}, \psi$. So, for every $a \in \mathcal{C}$, $\neg Q_{\psi,\theta_{\phi,\psi}}(a)$ iff there exists b such that $\psi(x,b)$ is not algebraic, $\psi(x,b) \vdash \phi(x,a)$ and $\psi(x,b)$ forks over a. Since T is EPFO, we know that each $\neg Q_{\psi,\theta_{\phi,\psi}}$ is Stone-open. By $(*)$, in $\mathcal{C}$,

$$\exists^{\infty} x\ \phi(x,y) \vdash \bigvee_{\psi \in L} \neg Q_{\psi,\theta_{\phi,\psi}}(y).$$

By compactness, there are $\psi_0 = \psi_0(\phi), \ldots, \psi_{n(\phi)} = \psi_{n(\phi)}(\phi) \in L$ such that in $\mathcal{C}$,

$$\exists^\infty x\ \phi(x,y) \vdash \bigvee_{i<n(\phi)} \neg Q_{\psi_i,\theta_{\phi,\psi_i}}(y). \tag{$**$}$$

Now, let $\Delta_\phi = \{\psi_i(\phi) \mid i \le n(\phi)\}$. Let L^-_{n+1} be the set of formulas generated by the set

$$\nu_{n+1} = L^-_n \cup \bigcup \{\Delta_\phi \mid \phi(x,y) \in L^-_n \setminus L^-_{n-1}\},$$

where the set of sorts attached to L^-_{n+1} is the set of all sorts of variables that appear in ν_{n+1}. Let $\mathcal{C}_{n+1} = \mathcal{C}|L^-_{n+1}$.

Now, let $L^-_\omega = \bigcup_{n<\omega} L^-_n$ and let $\mathcal{C}_\omega = \mathcal{C}|L^-_\omega$, $T^-_\omega = \mathrm{Th}(\mathcal{C}_\omega)$. We claim that T^-_ω is strongly non-supersimple. Indeed, given a formula $\phi_\omega(x,y) \in L^-_\omega$, let $a \in \mathcal{C}_\omega$ be such that $\models \exists^\infty x\ \phi_\omega(x,a)$. Then, by $(**)$ there exists $\psi(x,z) \in \Delta_{\phi_\omega} \subseteq L^-_\omega$ such that for some b we have $\psi(x,b) \vdash \phi_\omega(x,a)$ and $\psi(x,b)$ forks over a in $\mathcal{C}$, and thus in particular $\psi(x,b)$ forks over a in $\mathcal{C}_\omega$. Thus T^-_ω is strongly non-supersimple.

Now, to show that $\tilde{\mathcal{C}}_1$ is closed in $(\mathcal{R}^{\aleph_0}_{\mathcal{C}}, \le_{\mathcal{C}})$, let $\tilde{\mathcal{C}}$ be a chain in $\tilde{\mathcal{C}}_1$. We claim that $\mathcal{C}^* = \bigcup \tilde{\mathcal{C}}$ (the universe of $\mathcal{C}^*$ is the union of the interpretations of the sorts of all members of $\tilde{\mathcal{C}}$, and likewise for the definable sets of $\mathcal{C}^*$) is strongly non-supersimple. Indeed, let $\phi(x,a) \in L(\mathcal{C}^*)$ be non-algebraic. Then there exists $\mathcal{C}_0 = \mathcal{C}|L_0 \in \tilde{\mathcal{C}}$ for some countable sublanguage L_0 of L such that $\phi(x,a) \in L_0(\mathcal{C}_0)$. Since $\mathrm{Th}(\mathcal{C}_0)$ is strongly non-supersimple, there exists a non-algebraic $\psi(x,b) \in L_0(\mathcal{C}_0)$ such that $\psi(x,b) \vdash \phi(x,a)$ and $\psi(x,b)$ forks over a in $\mathcal{C}_0$. By Ramsey and compactness, there exists a formula $\psi(x,b')$ that is a-conjugate to $\psi(x,b)$ in $\mathcal{C}_0$ and that forks over a in the sense of $\mathcal{C}^*$. Thus $\mathrm{Th}(\mathcal{C}^*)$ is strongly non-supersimple. ■

By Claim 5.13, Claim 3.3, Remark 5.8, the assumptions of the theorem and Remark 5.9, there is a club of reducts in $\mathcal{R}^{\aleph_0}_{\mathcal{C}}$ that are strongly non-supersimple, hypersimple, eq-closed and coordinatized in finite rank. Any such reduct contradicts Proposition 5.4. This completes the proof of Theorem 5.12. ■

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