COLLOQUIUM MATHEMATICUM

VOL. 148 2017 NO. 1

SEMI-SYMMETRIC KÄHLER SURFACES

BY

WŁODZIMIERZ JELONEK (Kraków)

Abstract. The aim of this paper is to describe Kähler surfaces which admit an opposite almost Hermitian structure satisfying the first Gray condition.

1. Introduction. The aim of the present paper is to describe connected Kähler surfaces (M, g, J) admitting a negative almost Hermitian structure $\overline{J}$ satisfying the first Gray condition $R(\overline{J}X, \overline{J}Y, Z, U) = R(X, Y, Z, U)$.

Such surfaces are QCH Kähler surfaces (see [J-4]), i.e. surfaces admitting a global, 2-dimensional, J-invariant distribution $\mathcal{D}$ having the following property: The holomorphic curvature $K(\pi) = R(X, JX, JX, X)$ of any J-invariant 2-plane $\pi \subset T_xM$, where $X \in \pi$ and $g(X, X) = 1$, depends only on the point x and the number $|X_{\mathcal{D}}| = \sqrt{g(X_{\mathcal{D}}, X_{\mathcal{D}})}$, where $X_{\mathcal{D}}$ is the orthogonal projection of X onto $\mathcal{D}$. In this case we have

$$R(X, JX, JX, X) = \phi(x, |X_{\mathcal{D}}|)$$

where $\phi(x, t) = a(x) + b(x)t^2 + c(x)t^4$ and a, b, c are smooth functions on M. Also $R = a\Pi + b\Phi + c\Psi$ for certain curvature tensors $\Pi, \Phi, \Psi \in \bigotimes^4 \mathfrak{X}^*(M)$ of Kähler type. The investigation of such manifolds, called QCH Kähler manifolds, was started by G. Ganchev and V. Mihova [G-M-1], [G-M-2]. Every QCH Kähler surface is holomorphically pseudosymmetric and $R.R = \frac{1}{6}(\tau - \kappa)\Pi.R$ (see [J-4], [O]). In [J-2] we used the local results to obtain a global classification of such manifolds under the assumption that $\dim M = 2n \geq 6$.

In the present paper we show that a Kähler surface (M, g, J) is semi-symmetric if and only if either it is locally symmetric, or it admits a negative almost Hermitian structure $\overline{J}$ which satisfies the first Gray condition $R(\overline{J}X, \overline{J}Y, Z, U) = R(X, Y, Z, U)$. We also prove that a semi-symmetric Kähler surface (M, g, J) either is a QCH Kähler surface or is locally isometric to

2010 *Mathematics Subject Classification*: 53C55, 53C25, 53B35.

Key words and phrases: Kähler surface, holomorphic sectional curvature, quasi constant holomorphic sectional curvature, QCH manifold, ambi-Kähler manifold, semi-symmetric Kähler surface.

Received 2 September 2015; revised 5 June 2016.

Published online 23 January 2017.

a space form. In [J-4] we have proved that (M,g,J) is a QCH Kähler surface if and only if it admits a negative almost complex structure $\overline{J}$ satisfying the second Gray condition

$$R(X,Y,Z,W)-R(\overline{J}X,\overline{J}Y,Z,W)=R(\overline{J}X,Y,\overline{J}Z,W)+R(\overline{J}X,Y,Z,\overline{J}W).$$

Apostolov, Calderbank and Gauduchon [A-C-G] have classified weakly self-dual Kähler surfaces, extending the result of Bryant who classified self-dual Kähler surfaces [B]. Weakly self-dual Kähler surfaces turn out to be of Calabi type and of orthotoric type or surfaces with parallel Ricci tensor. Any Calabi type Kähler surface and every orthotoric Kähler surface is a QCH manifold. In both cases the opposite complex structure $\overline{J}$ is conformally Kähler.

2. The first Gray condition. Let (M,g,J) be a 4-dimensional Kähler manifold with a negative almost Hermitian structure $\overline{J}$. Then $\mathcal{D}=\ker(J\overline{J}-\mathrm{Id})$ is a J-invariant distribution. Let $\mathfrak{X}(M)$ denote the Lie algebra of all differentiable vector fields on M. If $X\in\mathfrak{X}(M)$ then we denote by $X^\flat$ the 1-form $\phi\in\mathfrak{X}^*(M)$ dual to X with respect to g, i.e. $\phi(Y)=X^\flat(Y)=g(X,Y)$, and by ω the Kähler form of (M,g,J), i.e. $\omega(X,Y)=g(JX,Y)$. Let (M,g,J) be a QCH Kähler surface with respect to a J-invariant 2-dimensional distribution $\mathcal{D}$. Let $\mathcal{E}=\mathcal{D}^\perp$, which is also a J-invariant 2-dimensional distribution. Then $\mathcal{E}=\ker(J\overline{J}+\mathrm{Id})$. We set $h=g\circ(p_{\mathcal{D}}\times p_{\mathcal{D}})$ and $m=g\circ(p_{\mathcal{E}}\times p_{\mathcal{E}})$, where $p_{\mathcal{D}},p_{\mathcal{E}}$ are the orthogonal projections onto $\mathcal{D},\mathcal{E}$ respectively. It follows that $g=h+m$.

For every almost Hermitian manifold (M,g,J) the self-dual Weyl tensor W^+ decomposes under the action of the unitary group $U(2)$. We have (see [A-A-D]) $\bigwedge^+M=\mathbb{R}\omega\oplus LM$ where $LM=[[\bigwedge^{(0,2)}M]]$, and we can write W^+ as a matrix with respect to this block decomposition,

$$W^+=\begin{pmatrix}\kappa/6 & W_2^+\\ (W_2^+)^* & W_3^+-(\kappa/12)\mathrm{Id}_{|LM}\end{pmatrix},$$

where κ is the conformal scalar curvature of (M,g,J) (see [A-A-D]). We denote by τ the scalar curvature of (M,g,J). The self-dual Weyl tensor W^+ of (M,g,J) is called *degenerate* if $W_2=0$ and $W_3=0$. In general, the self-dual Weyl tensor of an oriented 4-manifold (M,g) is called degenerate if it has at most two eigenvalues as an endomorphism $W^+:\bigwedge^+M\to\bigwedge^+M$.

We say that an almost Hermitian structure J satisfies the *first Gray condition* if

$$(\mathrm{G}_1)\qquad R(X,Y,Z,W)=R(JX,JY,Z,W),$$

and the *second Gray condition* if

$$(\mathrm{G}_2)\qquad R(X,Y,Z,W)-R(JX,JY,Z,W)=R(JX,Y,JZ,W)+R(JX,Y,Z,JW),$$

which is equivalent to $\mathrm{Ric}(J,J) = \mathrm{Ric}$ and $W_2^+ = W_3^+ = 0$. Condition (G_1) implies (G_2). (M, g, J) satisfies the second Gray condition if J preserves the Ricci tensor and W^+ is degenerate. A Kähler surface is QCH if and only if it admits a negative almost Hermitian structure satisfying the second Gray condition. Every QCH Kähler surface is holomorphically pseudosymmetric and $R.R = \frac{1}{6}(\tau - \kappa)\Pi.R$ (see [J-4], [O]). We shall denote by Ric_0 and ρ_0 the trace free part of the Ricci tensor Ric and of the Ricci form ρ respectively. An *ambi-Kähler structure* on a real 4-manifold consists of a pair of Kähler metrics (g_+, J_+, ω_+) and (g_-, J_-, ω_-) such that g_+ and g_- are conformal metrics and J_+ gives an opposite orientation to that given by J_- (i.e. the volume elements $\frac{1}{2}\omega_+ \wedge \omega_+$ and $\frac{1}{2}\omega_- \wedge \omega_-$ have opposite signs). A foliation $\mathcal{F}$ on a Riemannian manifold (M, g) is called *conformal* if for every $V \in \Gamma(T\mathcal{F})$,

$$L_V g = \alpha(V) g$$

on $T\mathcal{F}^\perp$, where α is a one-form vanishing on $T\mathcal{F}^\perp$. A foliation $\mathcal{F}$ is called *homothetic* if it is conformal and $d\alpha = 0$ (see [Ch-N]). A foliation $\mathcal{F}$ on a complex manifold (M, J) is called *complex* if $JT\mathcal{F} \subset T\mathcal{F}$, and *holomorphic* if $L_X J(TM) \subset T\mathcal{F}$ for any $X \in \Gamma(T\mathcal{F})$. Complex homothetic foliations by curves on Kähler manifolds were recently classified locally in [Ch-N] (see also [J-5]).

3. Curvature tensor of a QCH Kähler surface. We shall recall some results from [G-M-1]. Let

$$R(X,Y)Z = ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})Z \tag{1}$$

and let us write

$$R(X,Y,Z,W) = g(R(X,Y)Z, W).$$

If R is the curvature tensor of a QCH Kähler manifold (M, g, J), then there exist functions $a, b, c \in C^\infty(M)$ such that

$$R = a\Pi + b\Phi + c\Psi, \tag{2}$$

where Π is the standard Kähler tensor of constant holomorphic curvature, i.e.

$$\begin{aligned} \Pi(X,Y,Z,U) = \tfrac{1}{4}\big(&g(Y,Z)g(X,U) - g(X,Z)g(Y,U) \\ &+ g(JY,Z)g(JX,U) - g(JX,Z)g(JY,U) - 2g(JX,Y)g(JZ,U)\big), \end{aligned} \tag{3}$$

the tensor Φ is defined by the following relation:

$$\begin{aligned} \Phi(X,Y,Z,U) = \tfrac{1}{8}\big(&g(Y,Z)h(X,U) - g(X,Z)h(Y,U) + g(X,U)h(Y,Z) \\ &- g(Y,U)h(X,Z) + g(JY,Z)h(JX,U) - g(JX,Z)h(JY,U) \\ &+ g(JX,U)h(JY,Z) - g(JY,U)h(JX,Z) - 2g(JX,Y)h(JZ,U) \\ &- 2g(JZ,U)h(JX,Y)\big), \end{aligned} \tag{4}$$

and finally

$$\Psi(X,Y,Z,U) = -h(JX,Y)h(JZ,U) = -(h_J \otimes h_J)(X,Y,Z,U), \tag{5}$$

where $h_J(X,Y) = h(JX,Y)$. Let $V = (V, g, J)$ be a real $2n$-dimensional vector space with complex structure J which is skew-symmetric with respect to the scalar product g on V. Assume further that $V = D \oplus E$, where D is a 2-dimensional, J-invariant subspace of V, and E denotes its orthogonal complement in V. Note that the tensors Π, Φ, Ψ given above are of Kähler type. It is easy to check that for a unit vector $X \in V$, we have $\Pi(X,JX,JX,X) = 1$, $\Phi(X,JX,JX,X) = |X_D|^2$, $\Psi(X,JX,JX,X) = |X_D|^4$, where X_D denotes the orthogonal projection of a vector X onto the subspace D and $|X| = \sqrt{g(X,X)}$. It follows that for a tensor (2) defined on V we have

$$R(X,JX,JX,X) = \phi(|X_D|)$$

where $\phi(t) = a + bt^2 + ct^4$. If $\mathrm{Ric}_0 = \delta(h - m)$ then (see [J-4])

$$R = (\tau/6 - \delta + \kappa/12)\Pi + (2\delta - \kappa/2)\Phi + (\kappa/2)\Psi. \tag{6}$$

Let $J, \overline{J}$ be Hermitian, opposite orthogonal structures on a Riemannian 4-manifold (M, g) such that J is a positive almost complex structure. Let $\mathcal{E} = \ker(J\overline{J} - \mathrm{Id})$, $\mathcal{D} = \ker(J\overline{J} + \mathrm{Id})$ and let the tensors Π, Φ, Ψ be defined as above, where $h = g(p_{\mathcal{D}}, p_{\mathcal{D}})$. Set $K = \frac{1}{6}\Pi - \Phi + \Psi$. Then K is a curvature tensor, $b(K) = 0$, $c(K) = 0$, where b is the Bianchi operator and c is the Ricci contraction. For a QCH Kähler surface (M, g, J) we have $W^- = (\kappa/2)K$ (see [J-4]).

Let (M, g, J) be a Kähler surface which is a QCH manifold with respect to the distribution $\mathcal{D}$. Then (M, g, J) is also a QCH manifold with respect to the distribution $\mathcal{E} = \mathcal{D}^{\perp}$, and if Φ', Ψ' are the above tensors with respect to $\mathcal{E}$ then

$$R = (a + b + c)\Pi - (b + 2c)\Phi' + c\Psi'. \tag{7}$$

If (M, g, J) is a QCH Kähler surface then one can show that the Ricci tensor ρ of (M, g, J) satisfies the equation

$$\rho(X,Y) = \lambda m(X,Y) + \mu h(X,Y) \tag{8}$$

where $\lambda = \frac{3}{2}a + \frac{b}{4}, \mu = \frac{3}{2}a + \frac{5}{4}b + c$ are the eigenvalues of ρ (see [G-M-1, Corollary 2.1 and Remark 2.1]). In particular, the distributions $\mathcal{E}, \mathcal{D}$ are eigendistributions of the tensor ρ corresponding to the eigenvalues λ, μ of ρ.

Lemma 3.1. *The tensors $\Pi.\Phi$ and $\Pi.\Psi$ are linearly independent.*

Proof. Let $\{e_1, e_2, e_3, e_4\}$ be an orthogonal basis in TM such that

$$\mathcal{D} = \mathrm{span}\{e_1, e_2\}, \quad \mathcal{E} = \mathrm{span}\{e_3, e_4\}$$

and $e_2 = Je_1$, $e_4 = Je_3$. Then

$$\begin{aligned} \Pi(e_1,e_3).\Phi(e_1,e_4,e_3,e_4) &= \Phi(e_3,e_4,e_3,e_4) - \Phi(e_1,e_2,e_3,e_4) \\ &\quad - \Phi(e_1,e_4,e_1,e_4) - \Phi(e_1,e_4,e_3,e_2) = 3/16, \\ \Pi(e_1,e_3).\Psi(e_1,e_4,e_3,e_4) &= 0. \end{aligned} \tag{9}$$

■

Proposition 3.1. *If a QCH Kähler surface satisfies at a point $x \in M$ the condition $R.R = 0$ then at x we have $R = a\Pi$ or $2a + b = \frac{1}{6}(\tau - \kappa) = 0$.*

Proof. We have $R.R = (a+b/2)\Pi.R$. On the other hand, $\Pi.R = b\Pi.\Phi + c\Pi.\Psi$. From Lemma 3.1, $\Pi.R = 0$ if and only if $b = c = 0$ if and only if $R = a\Pi$. If $R \neq a\Pi$ then $R.R = 0$ implies $2a + b = 0$. ■

Lemma 3.2. *The tensor $R = a\Pi + b\Phi + c\Psi$ satisfies the first Gray condition with respect to $\overline{J}$ if and only if $2a + b = 0$.*

Proof. Note that $\Psi(\overline{J}X, \overline{J}Y, Z, U) = \Psi(X,Y,Z,U)$. We also have $(I = \overline{J})$

$$\begin{aligned} &\Pi(\overline{J}X, \overline{J}Y, Z, U) - \Pi(X,Y,Z,U) \\ &\quad = \tfrac{1}{2}\big(m(IY,Z)h(IX,U) + m(IX,U)h(IY,Z) - m(IX,Z)h(IY,U) \\ &\quad - h(IX,Z)m(IY,U) - h(Y,Z)m(X,V) - m(Y,Z)h(X,U) \\ &\quad + h(X,Z)m(Y,U) + m(X,Z)h(Y,U)\big) \end{aligned} \tag{10}$$

and

$$\begin{aligned} &\Phi(\overline{J}X, \overline{J}Y, Z, U) - \Phi(X,Y,Z,U) \\ &\quad = \tfrac{1}{4}\big(m(IY,Z)h(IX,U) + m(IX,U)h(IY,Z) - m(IX,Z)h(IY,U) \\ &\quad - h(IX,Z)m(IY,U) - h(Y,Z)m(X,V) - m(Y,Z)h(X,U) \\ &\quad + h(X,Z)m(Y,U) + m(X,Z)h(Y,U)\big). \end{aligned} \tag{11}$$

Hence

$$\begin{aligned} &R(\overline{J}X, \overline{J}Y, Z, U) - R(X,Y,Z,U) \\ &= (2a+b)\big(m(IY,Z)h(IX,U) + m(IX,U)h(IY,Z) - m(IX,Z)h(IY,U) \\ &\quad - h(IX,Z)m(IY,U) - h(Y,Z)m(X,V) - m(Y,Z)h(X,U) \\ &\quad + h(X,Z)m(Y,U) + m(X,Z)h(Y,U)\big). \end{aligned} \tag{12}$$

Consequently, $R(\overline{J}X, \overline{J}Y, Z, U) - R(X,Y,Z,U) = 0$ if and only if $2a+b = 0$. ■

Proposition 3.2. *Assume that a Kähler surface (M,g,J) admits a negative almost Hermitian structure $\overline{J}$ satisfying the first Gray condition. Then (M,g,J) is a QCH semi-symmetric surface and $\tau = \kappa$ where κ is the conformal scalar curvature of $\overline{J}$.*

Proof. Since (M,g,J) satisfies the first Gray condition with respect to $\overline{J}$, it clearly satisfies the second condition and is a QCH surface. On the other hand, $R.R = (a + b/2)\Pi.R = 0$ since $a + b/2 = \frac{1}{6}(\tau - \kappa) = 0$. ■

Lemma 3.3. *Assume that a product $M = \mathbb{R}\times N$ is a Kähler surface where N is a 3-dimensional Riemannian manifold. Then M is locally a product of Riemannian surfaces. If M is simply connected and complete then M is a product of Riemannian surfaces.*

Proof. Let H be a unit vector field tangent to $\mathbb{R}$. Then $\nabla H = 0$. Thus if $X = JH$ then X is a unit covariantly constant vector field on N. The distribution $\mathcal{D} = \{Y \in TN : g(X, Z) = 0\}$ is parallel and $J\mathcal{D} = \mathcal{D}$. Hence locally $N = \mathbb{R} \times \Sigma$ and M is a product of Riemannian surfaces. If M is simply connected and complete then N is simply connected and complete and from the de Rham theorem $N = \mathbb{R} \times \Sigma$ and $M = \mathbb{C} \times \Sigma$. ∎

Proposition 3.3. *Let (M, g, J) be a semi-symmetric Kähler surface. Then locally (M, g, J) is a space form or a QCH Kähler surface.*

Proof. We use a classification result of Szabó [Sz] and Lumiste [L] and Lemma 3.3. Note that $JV^0 = V^0$ in the Szabó decomposition since $R(X, Y) \circ J = J \circ R(X, Y)$. Hence $\dim V^0 = 0, 2, 4$. Note also that for elliptic, hyperbolic and Euclidean cones we have $\dim V^0 = 1$. Hence locally (M, g, J) is a symmetric space, a product of two Riemannian surfaces and a space foliated by 2-dimensional Euclidean spaces. A space foliated by 2-dimensional Euclidean spaces is a QCH Kähler surface with respect to $\mathcal{E} = V^0$ or $\mathcal{D} = V^1$. In fact $R(X, JX, JX, X) = R(X_{\mathcal{D}}, JX_{\mathcal{D}}, JX_{\mathcal{D}}, X_{\mathcal{D}})$ where $X_{\mathcal{D}}$ is the orthogonal projection of X onto $\mathcal{D}$. Hence $R = c\Psi$ where Ψ is the tensor with respect to $\mathcal{D}$. It is also clear that a product $M = \Sigma_1 \times \Sigma_2$ of Riemannian surfaces is a QCH Kähler surface with respect to $\mathcal{D} = T\Sigma_1$ or $\mathcal{E} = T\Sigma_2$. Note also that a locally symmetric irreducible Kähler surface is self-dual, and hence it is a space form (see [D]). ∎

Proposition 3.4. *Assume that (M, g, J) is a simply connected, complete, real analytic Kähler surface admitting a negative almost Hermitian structure satisfying the first Gray condition. Then M is a product of two Riemannian surfaces or $M = (\mathbb{C}^2, can)$ with standard flat Kähler metric.*

Proof. We use the classification result of Szabó [Sz, Th. 4.5, p. 103]. Since (M, g, J) is a semi-symmetric space, it is a direct product of symmetric spaces and Riemannian surfaces, 3-dimensional spaces which are hyperbolically foliated on an everywhere dense open subset and k-dimensional spaces which are parabolically foliated on an open dense subset. From Lemma 3.3 and the fact that $\dim H^1_3, \dim PF^n_{n-2} \geq 3$ it follows that M is symmetric, and hence it is a space form or a product of Riemannian surfaces (note that P^4_2 cannot be Kähler since $\dim S = 1$—see [Sz]). The space form of nonzero holomorphic curvature does not admit a negative almost Hermitian structure satisfying the first Gray condition (see Lemma 3.2). Hence M is a product of Riemannian surfaces or $M = (\mathbb{C}^2, can)$ with the standard flat Kähler metric *can.* ∎

COROLLARY 3.1. *Assume that (M, g, J) is a complete, real analytic Kähler surface admitting a negative almost Hermitian structure satisfying the first Gray condition. Then M is locally a product of two Riemannian surfaces or $M = \mathbb{C}^2/\Gamma$.*

Note that in the case of a product of Riemannian surfaces the opposite almost Hermitian structure $\overline{J}$ is Kähler. In case $M = (\mathbb{C}^2, can)$ any opposite almost Hermitian structure $\overline{J}$ satisfies the first Gray condition.

PROPOSITION 3.5. *Let (M, g, J) be a compact Kähler surface admitting an opposite Hermitian structure $\overline{J}$ satisfying the first Gray condition. Then $\overline{J}$ is a Kähler structure. If (M, g, J) is additionally simply connected then (M, g, J) is a product of Riemannian surfaces.*

Proof. Since $\overline{J}$ is Hermitian, we have $\kappa = \tau - \frac{3}{2}(|\theta|^2 + 2\delta\theta)$ where θ is the Lee form of $(M, g, \overline{J})$ (see [G]). Hence if $\overline{J}$ satisfies (G_1) then $\tau = \kappa$ and $|\theta|^2 + 2\delta\theta = 0$. Thus

$$\int_M (|\theta|^2 + 2\delta\theta) = 0,$$

and consequently $\int_M |\theta|^2 = 0$, which gives $\theta = 0$. Thus $(M, g, \overline{J})$ is Kähler and the result follows from the de Rham theorem. ∎

Now we give examples of Kähler surfaces foliated by 2-dimensional Euclidean spaces which are not products of two Riemannian surfaces. Hence they admit a negative almost Hermitian structure satisfying the first Gray condition (G_1), in fact this structure is Hermitian. These manifolds are of Calabi type and hence are ambi-Kähler. They are not complete.

Let (Σ, h) be a compact Riemannian surface with Kähler form ω such that $\frac{1}{2\pi}\omega$ is an integral form corresponding to $1 \in H^2(\Sigma, \mathbb{Z}) = \mathbb{Z}$. Let P_k be an S^1-bundle over Σ corresponding to the integral class $k\frac{1}{2\pi}\omega$ where $k \in \mathbb{N}$. Let θ be a connection form on the S^1-bundle P_k such that $d\theta = k\omega$. Consider the manifold $M = \mathbb{R}_+ \times P_k$, where $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$ with the metric $g = dt^2 + (2gg'/k)^2\theta^2 + g^2 p^*h$, where $g = g(t)$ is a function on $\mathbb{R}$ and t is the natural coordinate on $\mathbb{R}_+$. The metric g is Kähler and admits a negative Hermitian structure I. The fundamental vector field ξ of the action of S^1 on P_k is a holomorphic Killing vector field for M (see [J-2]). Furthermore, M is a Calabi type manifold and $a + b/2 = -4g''/g$ (see [J-2], [J-3]). Hence M is semi-symmetric if $g(t) = t$. Moreover, M is a fibre bundle over Σ with totally geodesic fibres $\mathbb{C}^*$ with a flat metric. Hence V^0 is the distribution tangent to the fibres $\mathbb{C}^*$. The metric is $g = dt^2 + (4t^2/k^2)\theta^2 + t^2 p^*h$.

We shall now show that QCH surfaces with nonvanishing Bochner tensor $W = W^-$ have the property that the condition $R.W = 0$ implies $R.R = 0$.

Proposition 3.6. *Let (M,g,J) be a Kähler QCH surface with $R.W^-=0$ and $\kappa\neq 0$ on M. Then (M,g,J) is semi-symmetric.*

Proof. Since $W^-=\frac{\kappa}{2}(\frac{1}{6}\Pi-\Phi+\Psi)$ and $\kappa\neq 0$, it follows that $R.W^-=0$ if and only if $R.K=0$ where $K=\frac{1}{6}\Pi-\Phi+\Psi$. Note that $\Phi-\Phi'=\Psi-\Psi'$ and $\Phi+\Phi'=\Pi$. Hence $\nabla\Phi=-\nabla\Phi'$ and $2\nabla\Phi=\nabla\Psi-\nabla\Psi'$. Consequently, $2\nabla K=\nabla\Psi+\nabla\Psi'$. Let us write $\omega_1=h_J$ and $\omega_2=m_J$. Then $\omega=\omega_1+\omega_2$ and $\Psi=-\omega_1\otimes\omega_1$, $\Psi'=-\omega_2\otimes\omega_2$. Thus

$$\begin{aligned}\nabla_X\Psi+\nabla_X\Psi' &\\ &=\nabla_X\omega_1\otimes\omega_1+\omega_1\otimes\nabla_X\omega_1+\nabla_X\omega_2\otimes\omega_2+\omega_2\otimes\nabla_X\omega_2\\ &=\nabla_X\omega_1\otimes(\omega_1-\omega_2)+(\omega_1-\omega_2)\otimes\nabla_X\omega_1\\ &=\tfrac{1}{2}\nabla_X\omega'\otimes\omega'+\tfrac{1}{2}\omega'\otimes\nabla_X\omega'=\tfrac{1}{2}\nabla_X(\omega'\otimes\omega').\end{aligned}\tag{13}$$

Hence $R.K=R.(\omega'\otimes\omega')$. Note that $R.(\omega'\otimes\omega')=0$ if and only if $R.\omega'=0$. In fact $R.(\omega'\otimes\omega')=R.\omega'\otimes\omega'+\omega'\otimes R.\omega'$. Suppose $\omega'(X,Y)=0$. Then

$$0=R.\omega'(U,W)\otimes\omega'(X,Y)+\omega'(U,W)\otimes R.\omega'(X,Y)$$

for all U,W. Taking U,W such that $\omega'(U,W)\neq 0$ we get $R.\omega'(X,Y)=0$. If $\omega'(X,Y)\neq 0$ then

$$0=R.(\omega'\otimes\omega')(X,Y,X,Y)=2R.\omega'(X,Y)\omega'(X,Y),$$

hence again $R.\omega'(X,Y)=0$. Note that $R.\omega'=0$ if and only if $\overline{J}$ satisfies the first Gray condition, and hence $\kappa=\tau$ and $R.R=0$.

We also give another proof of this fact. Note that (see [J-1])

$$\begin{aligned}R.K&=a\Pi.K+b\Phi.K+c\Psi.K'\\ &=-a\Pi.\Phi+a.\Pi.\Psi+\tfrac{1}{6}b\Phi.\Pi-b\Phi.\Phi+b\Phi.\Psi+\tfrac{1}{6}c\Psi.\Pi-c\Psi.\Phi+c\Psi.\Psi\\ &=-a\Pi.\Phi+a\Pi.\Psi-b\Phi.\Phi+b\Phi.\Psi\\ &=-a\Pi.\Phi+a\Pi.\Psi-b\tfrac{1}{2}\Pi.\Phi+b\tfrac{1}{2}\Pi.\Psi.\end{aligned}\tag{14}$$

Since the tensors $\Pi.\Phi,\Pi.\Psi$ are linearly independent, it follows that $a+\frac{1}{2}b=0$. ∎

Now we consider semi-symmetric Kähler surfaces foliated by 2-dimensional Euclidean spaces. Let $\mathcal{D}=V^0=\{X: R(U,V)X=0$ for all U,V in $TM\}$. Then $\mathcal{D}$ is a totally geodesic foliation. Define I by $IX=JX$ if $X\in\mathcal{D}$, and $IX=-JX$ if $X\in\mathcal{E}=\mathcal{D}^\perp$. Note that $R=c\Psi$ with respect to $\mathcal{E}$ and $\tau=2c$, where τ is the scalar curvature of (M,g,J). We have

$$\begin{aligned}\nabla_XR(Y,Z,W,T)=-Xc\omega(Y,Z)\omega(W,T)-c\nabla_X\omega(Y,Z)\omega(W,T)\\ -c\omega(Y,Z)\nabla_X\omega(W,T),\end{aligned}$$

where $\omega=m_J\in\bigwedge^2\mathcal{E}$, and hence from the Bianchi identity we obtain

$$-dc\wedge\omega\otimes\omega(W,T)-cd\omega\otimes\omega(W,T)-c\omega\wedge\nabla_{.}\omega(W,T)=0.\tag{15}$$

Since $\rho = c\omega$ where ρ is the Ricci form of (M, g, J), we get $d\omega = -d\ln \wedge c\omega$. Hence from (15) we get

$$\omega \wedge \nabla_{\cdot}\omega(W, T) = 0.$$

Consequently, $\nabla_X \omega = 0$ for $X \in \mathcal{D}$. Note that the Kähler form corresponding to J is $\Omega = \omega_1 + \omega$ and corresponding to I is $\Omega_1 = \omega_1 - \omega$, and consequently $\nabla_X I = 0$ for $X \in \mathcal{D}$ where $\omega_1 = h_J \in \bigwedge^2 \mathcal{D}$. We also have $d\Omega_1 = 2d\omega_1 = 2d\ln c \wedge \omega = 2\theta \wedge \Omega_1$, where $\theta(X) = g(\nabla \ln c_{|\mathcal{D}}, X)$ is the Lee form of I. Assume that $\zeta \in \Gamma(\mathcal{D})$. Then

$$L_\zeta \rho = \zeta \lrcorner d\rho + d(\zeta \lrcorner \rho) = 0.$$

We also have $\nabla_\zeta \rho = \zeta c\omega$ since $\nabla_\zeta \omega = 0$. Thus $(L_\zeta - \nabla_\zeta).\rho(X, Y) = -\nabla\zeta.\rho(X, Y) = c\omega(\nabla_X \zeta, Y) + c\omega(X, \nabla_Y \zeta)$. Hence $\omega(\nabla_X \zeta, Y) + \omega(X, \nabla_Y \zeta) = -\theta(\zeta)\omega(X, Y)$. Assume that $X, Y \in \Gamma(\mathcal{E})$. Then

$$g(\nabla_X J\zeta, Y) - g(X, \nabla_Y J\zeta) = -\theta(\zeta)\omega(X, Y),$$

and for any $\xi \in \Gamma(\mathcal{D})$ we obtain

$$g(\nabla_X \xi, Y) - g(X, \nabla_Y \xi) = \theta(J\xi)\omega(X, Y). \tag{16}$$

Note that since $\mathcal{D}$ is totally geodesic, the foliation $\mathcal{D}$ is holomorphic if and only if $g(\nabla_{JX}\xi, Y) = g(J\nabla_X \xi, Y)$ for any $X, Y \in \mathcal{E}$. Hence if $\mathcal{D}$ is holomorphic, we get

$$\begin{aligned} g(\nabla_{JX}\xi, Y) - g(JX, \nabla_Y \xi) &= -\theta(J\xi)g(X, Y), \\ g(\nabla_X J\xi, Y) + g(X, \nabla_Y J\xi) &= -\theta(J\xi)g(X, Y), \\ g(\nabla_X \zeta, Y) + g(X, \nabla_Y \zeta) &= -\theta(\zeta)g(X, Y) \end{aligned} \tag{17}$$

for any $\zeta \in \mathcal{D}$. From (16) and (17) we obtain

$$2g(\nabla_X \zeta, Y) = -J\theta(\zeta)\omega(X, Y) - \theta(\zeta)g(X, Y). \tag{18}$$

Hence

$$2(\nabla_X Y)_{|\mathcal{D}} = \theta^\sharp g(X, Y) + J\theta^\sharp \omega(X, Y) \tag{19}$$

for all $X, Y \in \Gamma(\mathcal{E})$. From (17) it follows that if $\mathcal{D}$ is holomorphic then it is conformal. Conversely, assume that $\mathcal{D}$ is conformal, i.e. $g(\nabla_X \zeta, Y) + g(X, \nabla_Y \zeta) = \phi(\zeta)g(X, Y)$ for some form $\phi \in \bigwedge^1 \mathcal{D}$. Then

$$2g(\nabla_X \zeta, Y) = -J\theta(\zeta)\omega(X, Y) + \phi(\zeta)g(X, Y). \tag{20}$$

From (19) it is clear that $g(\nabla_{JX}\zeta, JY) = g(\nabla_X \zeta, Y)$ for all $X, Y \in \mathcal{E}$, and thus $\nabla_{JX}\zeta - J\nabla_X \zeta \in \mathcal{D}$, which means that $\mathcal{D}$ is holomorphic. It follows that $\phi = -\theta$.

Thus we get

THEOREM 3.1. *Let (M, g, J) be a Kähler semi-symmetric surface foliated by 2-dimensional Euclidean spaces. Then the following conditions are equivalent:*

(a) *$\mathcal{D}$ is a holomorphic foliation.*
(b) *$\mathcal{D}$ is a conformal foliation.*
(c) $2g(\nabla_X\zeta, Y) = -J\theta(\zeta)\omega(X,Y) - \theta(\zeta)g(X,Y)$ *for all* $\zeta \in \Gamma(\mathcal{D})$ *and all* $X, Y \in \Gamma(\mathcal{E})$.
(d) *The almost Hermitian structure I is Hermitian.*

Proof. The equivalence of (a)–(c) has been proved above. We shall show that conditions (a) and (d) are equivalent. Note that if $Y \in \Gamma(\mathcal{D})$ then $IY = JY$, and consequently $\nabla_X IY = 2J(\nabla_X Y)_{|\mathcal{E}}$. Similarly, if $Y \in \Gamma(\mathcal{E})$ then $\nabla_X IY = -2J(\nabla_X Y)_{|\mathcal{D}}$.

Assume that (a) holds. Then (c) holds. To prove that I is integrable we have to show that $\nabla_X IY = \nabla_{IX} IIY$ for $X \in \Gamma(\mathcal{E})$ and $Y \in \mathfrak{X}(M)$. Let $\{E_1, E_2, E_3, E_4\}$ be a local orthonormal basis of (M, g) such that E_1, E_2 span $\mathcal{D}$ and E_3, E_4 span $\mathcal{E}$. We assume further that $JE_1 = IE_1 = E_2$ and $JE_3 = -IE_3 = E_4$. We have $\nabla_{E_3} IE_1 = 2J(\nabla_{E_3} E_1)_{|\mathcal{E}}$ and $\nabla_{IE_3} IIE_4 = -\nabla_{E_4} IE_2 = -2J(\nabla_{E_4} E_2)_{|\mathcal{E}}$. Using (c) it is clear that $\nabla_{E_3} IE_1 = \nabla_{IE_3} IIE_4$. If $X, Y \in \Gamma(E)$ then $\nabla_X IY = -2J(\nabla_X Y)_{|\mathcal{D}}$ and $\nabla_{IX} IIY = -2J(\nabla_{JX} JY)_{|\mathcal{D}}$. From (c) it is clear that $\nabla_X IY = \nabla_{IX} IIY$ also in this case.

Conversely, if I is integrable then $\nabla_X Y_{|\mathcal{D}} = (\nabla_{JX} JY)_{|\mathcal{D}}$ for all $X, Y \in \Gamma(\mathcal{E})$, which is equivalent to $\mathcal{D}$ being holomorphic. ∎

Since the Weyl tensor W^- is degenerate, the form $d\theta$ is self-dual and $d\theta(J, J) = -d\theta$ if I is integrable.

Proposition 3.7. *Let (M, g, J) be a Kähler semi-symmetric surface foliated by 2-dimensional Euclidean spaces. Then the following conditions are equivalent:*

(a) $d\theta = 0$.
(b) *$J\theta^\sharp$ is a Hamiltonian vector field for (M, g, J).*

Moreover, if I is integrable then each of these conditions is equivalent to any of the following:

(c) $Y|\theta|^2 = 0$ *for* $Y \in \mathcal{E}$.
(d) *I is locally conformally Kähler.*

Proof. Note that $L_{J\theta}\Omega = dJ\theta \lrcorner \Omega = -d\theta$. Hence (a) is equivalent to (b). It is easy to see that $d\theta(X, Y) = 0$ if X, Y are both in $\mathcal{D}$ or both in $\mathcal{E}$. Now $d\theta(Y, \xi) = \frac{1}{2} Y|\theta|^2$ for $Y \in \mathcal{E}$. Since $d\theta(J, J) = -d\theta$ if J is integrable, this proves that (a) is equivalent to (c) in this case. ∎

In the following theorem we use the description of Calabi type Kähler surfaces from [A-C-G].

Theorem 3.2. *Let (M, g, J) be a Kähler surface admitting an opposite Hermitian structure I satisfying the first Gray condition* (G_1) *which is locally* *conformally Kähler. Then locally*

$$g = zg_{\Sigma} + \frac{1}{Cz}dz^2 + Cz(dt + \alpha)^2 \tag{21}$$

where (Σ, g_{Σ}) *is a Riemannian surface with area form* ω_{Σ} *and* $d\alpha = \omega_{\Sigma}$ *or* (M, g, J) *is a product of Riemannian surfaces or a space form with zero holomorphic sectional curvature. The Kähler form of* (M, g, J) *is* $\Omega = z\omega_{\Sigma} + dz \wedge (dt + \alpha)$.

Proof. First assume that (M, g, J) is a Kähler semi-symmetric surface foliated by 2-dimensional Euclidean spaces. Then $\mathcal{D}$ is a totally geodesic homothetic foliation. Such foliations were classified locally in [Ch-N]. Thus (M, g, J) is a Kähler surface of Calabi type. From [A-C-G] it follows that $\kappa = \tau$ if $V(z) = Cz^2$ where

$$g = zg_{\Sigma} + \frac{z}{V(z)}dz^2 + \frac{V(z)}{z}(dt + \alpha)^2$$

is a general Calabi type metric which is not a Kähler product.

For the general case, note that QCH Kähler surfaces for which I is Hermitian and locally conformally Kähler are of Calabi type, or are orthotoric surfaces, or $W = 0$ (see [J-4]). Semi-symmetric surfaces with $W = 0$ are products of Riemannian surfaces of constant opposite scalar curvatures (see [B]). One can easily check that an orthotoric surface can be semi-symmetric only if $W = 0$, which finishes the proof. ∎

REMARK 3.1. Note that the examples with the metric $g = dt^2 + (2gg'/k)^2\theta^2 + g^2p^*h$ given before are a special kind of (21), and these classes of manifolds coincide locally.

Acknowledgements. The author would like to thank Professor Andrzej Derdziński for his suggestions which improved the exposition of the paper. The paper was supported by NCN grant 2011/01/B/ST1/02643.

REFERENCES

[A-A-D] V. Apostolov, J. Armstrong and T. Draghici, *Local ridigity of certain classes of almost Kähler 4-manifolds*, Ann. Global Anal. Geom. 21 (2005), 151–167.

[A-C-G] V. Apostolov, D. M. J. Calderbank and P. Gauduchon, *The geometry of weakly self-dual Kähler surfaces*, Compos. Math. 135 (2003), 279–322.

[B] R. Bryant, *Bochner–Kähler metrics*, J. Amer. Math. Soc. 14 (2001), 623–715.

[Ch-N] S. G. Chiossi and P.-A. Nagy, *Complex homothetic foliations on Kähler manifolds*, Bull. London Math. Soc. 44 (2012), 113–124.

[D] A. Derdziński, *Self-dual Kähler manifolds and Einstein manifolds of dimension four*, Compos. Math. 49 (1983), 405–433.

[G-M-1] G. Ganchev and V. Mihova, *Kähler manifolds of quasi-constant holomorphic sectional curvatures*, Cent. Eur. J. Math. 6 (2008), 43–75.

[G-M-2] G. Ganchev and V. Mihova, *Warped product Kähler manifolds and Bochner–Kähler metrics*, J. Geom. Phys. 58 (2008), 803–824.

[G] P. Gauduchon, *La 1-forme de torsion d'une variété hermitienne compacte*, Math. Ann. 267 (1984), 495–518.

[J-1] W. Jelonek, *Compact holomorphically pseudosymmetric Kähler manifolds*, Colloq. Math. 117 (2009), 243–249.

[J-2] W. Jelonek, *Kähler manifolds with quasi-constant holomorphic curvature*, Ann. Global Anal. Geom. 36 (2009), 143–159.

[J-3] W. Jelonek, *Holomorphically pseudosymmetric Kähler metrics on* $\mathbb{CP}^n$, Colloq. Math. 127 (2012), 127–131.

[J-4] W. Jelonek, *Kähler surfaces with quasi-constant holomorphic curvature*, Glasgow Math. J. 58 (2016), 503–512.

[J-5] W. Jelonek, *Kähler manifolds with homothetic foliation by curves*, Differential Geom. Appl. 46 (2016), 119–131.

[K-N] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. 2, Interscience, New York, 1963.

[L] Ü. Lumiste, *Semisymmetric curvature operators and Riemannian 4-spaces elementarily classified*, Algebras Groups Geom. 13 (1996), 371–388.

[O] Z. Olszak, *Bochner flat Kählerian manifolds with a certain condition on the Ricci tensor*, Simon Stevin 63 (1989), 295–303.

[Sz] Z. I. Szabó, *Structure theorems on Riemannian spaces satisfying* $R(X,Y).R = 0$ *II. Global version*, Geom. Dedicata 19 (1985), 65–108.

Włodzimierz Jelonek
Institute of Mathematics
Cracow University of Technology
Warszawska 24
31-155 Kraków, Poland
E-mail: wjelon@pk.edu.pl