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ON n-ABSORBING RINGS AND IDEALS

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Abstract. A proper ideal I of a commutative ring R is n-absorbing (resp. strongly n-absorbing) if for all elements (resp. ideals) $a_1, \ldots, a_{n+1}$ of R/I, $a_1 \cdots a_{n+1} = 0$ implies that the product of some n of the a_i is 0. It was conjectured by Anderson and Badawi that if I is an n-absorbing ideal of R then (1) I is strongly n-absorbing, (2) $I[x]$ is an n-absorbing ideal of $R[x]$, and (3) $\mathrm{Rad}(I)^n \subseteq I$. We prove that these conjectures hold in various classes of rings, thus extending several known results on n-absorbing ideals. As a by-product, we show that (2) implies (1).

1. Introduction. Throughout, R is a commutative ring with $1 \neq 0$ and n is a positive integer. Following Badawi [B], a proper ideal I of a ring R is called *n-absorbing* (resp. *strongly n-absorbing*) if for all elements (resp. ideals) $a_1, \ldots, a_{n+1}$ of R such that $a_1 \cdots a_{n+1}$ is in I, there exists an n-element subset S of $\{1, \ldots, n+1\}$ such that $\prod_{i \in S} a_i$ is in I. The ring R is *n-absorbing* (resp. *strongly n-absorbing*) if for all elements (resp. ideals) $a_1, \ldots, a_{n+1}$ of R, $a_1 \cdots a_{n+1} = 0$ implies that the product of some n of the a_i is 0 (see Darani and Puczyłowski [DP]). Clearly, a proper ideal I of R is [strongly] n-absorbing if and only if the ring R/I is [strongly] n-absorbing. It is equally clear that a strongly n-absorbing ideal I is n-absorbing, and the converse is true for $n = 1$, since I is 1-absorbing precisely when it is prime. Three outstanding conjectures on n-absorbing ideals are the following (see Anderson and Badawi [AB] and also Cahen et al. [CFFG, Problem 30]):

- (C1) If an ideal of R is n-absorbing then it is strongly n-absorbing.
- (C2) If an ideal I of R is n-absorbing then $I[x]$ is an n-absorbing ideal of $R[x]$.
- (C3) If an ideal I of R is n-absorbing then $\mathrm{Rad}(I)^n \subseteq I$.

The case of $n = 2$ was settled for arbitrary rings by Badawi [B] for (C1) and (C3), and by Anderson and Badawi [AB] for (C2). Although these conjectures are still open for all $n \geq 3$ (however, see Proposition 2.8 below

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for (C3) when $n = 3$), affirmative answers for (C1) have been obtained for some classes of rings and ideals: Anderson and Badawi [AB] proved that (C1) holds in all Prüfer domains, and Darani and Puczyłowski [DP] proved that it holds for any proper ideal I of R provided the additive group R/I is torsion-free. Moreover, it is proved in [AB] that (C1) implies (C3).

In this note, we show that these conjectures hold for wider classes of ideals. Among other results, we show that (C2) implies (C1), that (C2) (and hence (C1) and (C3)) holds in all arithmetical rings, and that if an ideal I is 2-absorbing then $I[[x]]$ is a 2-absorbing ideal of the power series ring $R[[x]]$. The arguments we use also provide alternative proofs of the conjectures for the case $n = 2$. Throughout, the cardinality of a set X is denoted $|X|$.

2. Results

Definition 2.1. An ideal A of the ring R is an *n-ideal* if for any family $\{A_i\}_{1\le i\le n}$ of ideals of R such that $A \subseteq \bigcup_{1\le i\le n} \operatorname{ann}_R A_i$ we have $A \subseteq \operatorname{ann}_R A_i$ for some i. If all ideals of R are n-ideals, we say R is an *n-ring*.

It is clear that all rings are 2-rings, that n-rings are k-rings for each positive integer $k \le n$, and that it is enough, in the above definition, to assume that the ideal A is n-generated.

Our first result is the following theorem.

Theorem 2.2. *If R is an n-absorbing n-ring, then it is strongly n-absorbing. In particular, if I is an n-absorbing ideal of R and the ring R/I is an n-ring, then I is strongly n-absorbing.*

Proof. Let $A_1, \dots, A_{n+1}$ be ideals of R with $A_1A_2\cdots A_{n+1} = 0$. Let $a_1 \in A_1$. Then $a_1a_2\cdots a_{n+1} = 0$ for all $a_i \in A_i$ $(2 \le i \le n+1)$, so that either $a_1 \in \bigcup_{2\le j\le n+1} \operatorname{ann}_R(\prod_{i\ne j} a_i)$, and then $A_1 \subseteq \bigcup_{2\le j\le n+1} \operatorname{ann}_R(\prod_{i\ne j} a_i)$, or $a_2\cdots a_{n+1} = 0$. Hence, either $a_2\cdots a_{n+1} = 0$, or, as R is an n-ring, $A_1\prod_{i\ne j} a_i = 0$ for some j $(2 \le j \le n+1)$. We thus find that either $A_1\prod_{i\ge 3} a_i = 0$, or $a_2 \in \operatorname{ann}_R(\prod_{i\ge 3} a_i) \cup \bigcup_{3\le j\le n+1} \operatorname{ann}_R(A_1\prod_{i\ge 3,\, i\ne j} a_i)$ for all a_2 in A_2. Since R is an n-ring, this in turn means that for all $a_i \in A_i$ $(2 < i \le n+1)$,

$$
\begin{aligned}
&A_1\prod_{i\ge 3} a_i = 0 \quad\text{or}\quad A_2\prod_{i\ge 3} a_i = 0 \quad\text{or}\\
&A_1A_2\prod_{i\in S} a_i = 0 \quad\text{for some } (n-2)\text{-element subset } S \text{ of } \{3,\dots,n+1\}.
\end{aligned}
\tag{2.1}
$$

We now prove, by induction, that for each integer m with $2 \le m \le n$ and for all $a_i \in A_i$, we have the following possibilities only:

(i) $(\prod_{i\in S} A_i)\prod_{m+1\le j\le n+1} a_j = 0$ for some $(m-1)$-element subset S of $\{1,\dots,m\}$, or
(ii) $(\prod_{1\le i\le m} A_i)\prod_{j\in S} a_j = 0$ for some $(n-m)$-element subset S of $\{m+1,\dots,n+1\}$ (with $S=\emptyset$ and $\prod_{j\in S} a_j = 1$, the empty product, when $m=n$).

For $m=2$, this is true by (2.1). Suppose it is true for $m=N$. We have

(a) $A_{i_1}\cdots A_{i_{N-1}} a_{N+1}\cdots a_{n+1} = 0$ for some $1\le i_1<\cdots<i_{N-1}\le N$, or
(b) $A_1\cdots A_N a_{N+1} a_{j_2}\cdots a_{j_{n-N}} = 0$ for some $N+1<j_2<\cdots<j_{n-N}\le n+1$, or
(c) $A_1\cdots A_N a_{N+2} a_{N+3}\cdots a_{n+1} = 0$.

Hence, for all $a_i\in A_i$ $(N+1\le i\le n+1)$,

$$a_{N+1}\in \operatorname{ann}_R(A_{i_1}\cdots A_{i_{N-1}} a_{N+2}\cdots a_{n+1}) \cup \bigcup_{N+1<j_2<\cdots<j_{n-N}\le n+1} \operatorname{ann}_R(A_1\cdots A_N a_{j_2}\cdots a_{j_{n-N}}) \quad \text{or}$$

$$A_1\cdots A_N a_{N+2}a_{N+3}\cdots a_{n+1} = 0.$$

Since R is an n-ring, this means

$$A_{i_1}\cdots A_{i_{N-1}} A_{N+1} a_{N+2}\cdots a_{n+1} = 0, \quad \text{or} \tag{2.2}$$

$$A_1\cdots A_N A_{N+1} a_{j_2}\cdots a_{j_{n-N}} = 0, \quad \text{or} \tag{2.3}$$

$$A_1\cdots A_N a_{N+2} a_{N+3}\cdots a_{n+1} = 0. \tag{2.4}$$

Clearly, (2.2) and (2.4) mean $A_{i_1}\cdots A_{i_{N-1}} A_{i_N}\prod_{N+2\le i\le n+1} a_i = 0$, where $1\le i_1<\cdots<i_N\le N+1$, and (2.3) means $(\prod_{1\le i\le N+1} A_i) a_{j_1}\cdots a_{j_{n-N-1}} = 0$, where $N+2\le j_1<\cdots<j_{n-N-1}\le n+1$, completing the induction. In particular, for $m=n$ and all $a_{n+1}\in A_{n+1}$, we obtain $(\prod_{i\in S} A_i)a_{n+1} = 0$ for some $(n-1)$-element subset S of $\{1,\dots,n\}$, i.e. $(\prod_{i\in S} A_i)A_{n+1} = 0$ or $\prod_{1\le i\le n} A_i = 0$. This proves that R is n-absorbing. ∎

Since every ring is easily shown to be a 2-ring, Theorem 2.2 implies the following result. It was first proved by Badawi [B] using a different approach.

COROLLARY 2.3. *Every* 2*-absorbing ideal is strongly* 2*-absorbing.*

The following lemma says more than we need but may be of independent interest. It is a refinement of a result (and proof) in [QB]. Recall that for an R-module M, an element r of the ring R is *M-regular* if $\operatorname{ann}_M(r) = 0$.

LEMMA 2.4. *Let $A, A_1,\dots,A_n$ be ideals of R and suppose that R contains n elements $r_1,\dots,r_n$ such that for all but possibly one k, $r_i - r_j$ is R/A_k-regular whenever $r_i\ne r_j$. If $A\subseteq A_1\cup\cdots\cup A_n$ then A is contained in one of the A_i.*

Proof. Suppose the lemma is false and that n is the smallest positive integer for which the conclusion does not hold (it is easy to see that $n\ge 3$).

Assume, without loss of generality, that $r_i - r_j$ is R/A_k-regular for all $k \geq 2$, let $a_1 \in A \setminus (A_2 \cup A_3 \cup \cdots \cup A_n)$, $a_2 \in A \setminus (A_1 \cup A_3 \cup \cdots \cup A_n)$, and S be the set $\{r_i a_1 + a_2 : 1 \leq i \leq n\}$. Clearly, $a_j \in A_j$ $(j = 1, 2)$ and there is no r_i such that $r_i a_1 + a_2 \in A_1$. Hence two elements of S, say $r_1 a_1 + a_2$ and $r_2 a_1 + a_2$, must lie in the same A_k for some $2 \leq k \leq n$. We therefore obtain $(r_1 - r_2) a_1 \in A_k$, and thus $a_1 \in A_k$, a contradiction. ■

PROPOSITION 2.5. *Let R be a ring containing n elements $r_1, \ldots, r_n$ such that $r_i - r_j$ is not a zero-divisor when $r_i \neq r_j$. Then R is an n-ring. In particular, if R is n-absorbing, then it is strongly n-absorbing.*

Proof. Let A and A_k $(1 \leq k \leq n)$ be ideals of R such that $A \subseteq \bigcup_{1 \leq k \leq n} \operatorname{ann}_R A_k$. Clearly, for each k, $r_i - r_j$ is $R/\operatorname{ann}_R A_k$-regular whenever $r_i \neq r_j$. By Lemma 2.4, A is contained in some A_k. This implies that R is an n-ring, and the proof is complete by Theorem 2.2. ■

Quartararo and Butts [QB] call an ideal A of R a *u-ideal* if, for all positive integers n and all ideals A_k of R, $A \subseteq \bigcup_{1 \leq k \leq n} A_k$ implies that A is contained in some A_k. The ring R is a *u-ring* if all its ideals are u-ideals. It is proved in [QB] that R is a u-ring if and only if for each maximal ideal P of R, either R/P is infinite or R_P is Bézout (all its f.g. ideals are principal). A closer look at the argument in [QB], along with appropriate modifications, shows that if either R is a locally Bézout ring or each of its residue fields has at least n elements, then, for all ideals A and A_k of R $(1 \leq k \leq n)$, $A \subseteq \bigcup_{1 \leq k \leq n} A_k$ implies that A is contained in some A_k. In particular, R is an n-ring under these hypotheses.

We can now provide some classes of rings in which conjecture (C1) holds.

PROPOSITION 2.6. *Let I be an n-absorbing ideal of R and let S be the ring R/I. Then I is strongly n-absorbing in each of the following cases:*

(i) *S is locally Bézout* (*e.g. S is arithmetical*).
(ii) *For each maximal ideal P of R containing I, $|R/P| \geq n$.*
(iii) *S is $(n-1)!$-torsion-free as an additive group.*
(iv) *S is* (*isomorphic to*) *a polynomial ring or a power series ring* (*in any nonempty set of indeterminates*).

Proof. Parts (i) and (ii) follow from the preceding paragraph. To prove (iii), observe that if S is $(n-1)!$-torsion-free as an additive group, then the elements $s_i = i \cdot 1$ $(i = 0, 1, \ldots, n-1)$ of S are such that $s_i - s_j$ is not a zero-divisor when $s_i \neq s_j$, and use Proposition 2.5. To prove (iv), let $S = T[X]$ or $T[[X]]$, where X is a nonempty set of indeterminates, let $x \in X$, and let $p_k = x^k$ $(k \in \mathbb{N})$. Then, whenever $i < j$, $p_i - p_j = x^i(1 - x^{j-i})$ is not a zero-divisor in S, and hence (iv) follows from Proposition 2.5. ■

REMARK 2.7. It is worth noting the following facts concerning the above proposition:

(1) Part (i) has [AB, Corollary 6.9] as a special case since factor rings of Prüfer domains (and more generally of arithmetical rings) are arithmetical. (See also Corollary 2.12 below.)
(2) Part (ii) (or (iii)) has [B, Theorem 2.13] as a special case since fields always contain 0 and 1 (and every ring with nonzero identity is 1-torsion-free).
(3) Part (iii) is a refinement of a result first proved in [DP, Theorem 4.2] by a different approach.

By [AB, Theorem 6.1], conjecture (C1) implies conjecture (C3), so that, by Proposition 2.6(iii), (C3) holds for all 3-absorbing ideals I of the ring R such that R/I is 2-torsion-free. However, as our next result shows, the 2-torsion-free condition is redundant in this case.

PROPOSITION 2.8. *Let I be a 3-absorbing ideal of the ring R. Then* $\mathrm{rad}(I)^3 \subseteq I$.

Proof. It is clearly enough to assume that the ring R is 3-absorbing with $x_1, x_2, x_3 \in R$ such that $x_1^3 = x_2^3 = x_3^3 = 0$ and to show that $x_1x_2x_3 = 0$. From $(x_1^2 + x_2)x_1x_2^2 = 0$, we obtain $(x_1^2 + x_2)x_1x_2 = 0$ or $(x_1^2 + x_2)x_2^2 = 0$ or $x_1x_2^2 = 0$, which implies $x_1^2x_2 = 0$ or $x_1x_2^2 = 0$. Also, $(x_1 + x_2)x_1^2x_2 = 0$ gives $(x_1 + x_2)x_1x_2 = 0$ or $(x_1 + x_2)x_1^2 = 0$ or $x_1^2x_2 = 0$. Suppose that $x_1^2x_2 \neq 0$; then $x_1x_2^2 = 0$ and $(x_1 + x_2)x_1x_2 = 0$ imply $x_1^2x_2 = 0$, a contradiction. Thus we must have $x_1^2x_2 = 0$, and by symmetry $x_i^2x_j = 0$ for all $i, j \in \{1, 2, 3\}$. Finally, from $(x_1 + x_2 + x_3)x_1x_2x_3 = 0$ we obtain $x_1x_2x_3 = 0$ or $(x_1 + x_2 + x_3)x_ix_j = 0$ for some $i \neq j$ (in $\{1, 2, 3\}$). Clearly, in all cases, we obtain $x_1x_2x_3 = 0$, as required. ∎

Another consequence of Proposition 2.6 is the following result. Its proof is based on an argument given in [KO].

COROLLARY 2.9. *Let R be a ring and κ be an infinite cardinal such that $|R| > 2^\kappa$. Suppose that each maximal ideal P of R is κ-generated and $\bigcap_{r\in\mathbb{N}} P^r = 0$. Then every n-absorbing ideal of R is strongly n-absorbing (for each positive integer n). In particular, this is true if $|R| > 2^{\aleph_0}$ and R is a Noetherian domain or a Noetherian local ring.*

Proof. Assume that $|R/P|$ is finite for some maximal ideal P of R. For each $m \in \mathbb{N}$, P^m/P^{m+1} is κ-generated as an R/P-space, so that $|P^m/P^{m+1}| \leq |R/P|^\kappa$. Hence $|R/P^m| = |R/P| \cdot |P/P^2| \cdots |P^{m-1}/P^m| \leq (|R/P|^\kappa)^m$. Since $\bigcap_{r\in\mathbb{N}} P^r = 0$, we have a monomorphism $R \to \prod_{m\in\mathbb{N}} R/P^m$, and therefore $|R| \leq \prod_{m\in\mathbb{N}}(|R/P|^\kappa)^m$, i.e. $|R| \leq 2^{\kappa\aleph_0} = 2^\kappa$, a contradiction. Hence $|R/P|$ must be infinite and the first part of the conclusion follows from Proposition 2.6. The second part follows by the Krull intersection theorem. ∎

The next result sheds some light on the relationship between the two conjectures (C1) and (C2) and shows, in particular, that (C2) is stronger than (C1), and hence than (C3) as well. Recall that the *content* $c(p)$ of a polynomial $p \in R[x]$ (or more generally of a power series p in $R[[x]]$) is the ideal of R generated by the coefficients of p, and that R is *Gaussian* if for all $f, g \in R[x]$, $c(fg) = c(f)c(g)$. If $c(f)c(g) = 0$ for all $f, g \in R[x]$ such that $fg = 0$, then R is said to be *Armendariz*. By [AC], Gaussian rings are precisely those whose homomorphic images are Armendariz.

PROPOSITION 2.10. *Let I be an ideal of the ring R.*

(i) *If $I[x]$ is an n-absorbing ideal of the ring $R[x]$, then I is strongly n-absorbing.*
(ii) *If R/I is Armendariz and I is strongly n-absorbing, then $I[x]$ is an n-absorbing ideal of $R[x]$.*

Proof. (i) Assume $I[x]$ is an n-absorbing ideal of $R[x]$, i.e. $R[x]/I[x]$ is an n-absorbing ring. Since $R[x]/I[x] \cong (R/I)[x]$, we infer from Proposition 2.6 that $R[x]/I[x]$ is strongly n-absorbing. Now let $A_1, \ldots, A_{n+1}$ be ideals of R/I with $A_1 \cdots A_{n+1} = 0$. Clearly, $A_1[x] \cdots A_{n+1}[x] = 0$, and so there exist distinct $i_1, \ldots, i_n$ in $\{1, \ldots, n+1\}$ such that $A_{i_1} \cdots A_{i_n} \subseteq A_{i_1}[x] \cdots A_{i_n}[x] = 0$.

(ii) Assume I is strongly n-absorbing, let $S := R/I$ and let $f_1, \ldots, f_{n+1} \in S[x]$ be such that $f_1 \cdots f_{n+1} = 0$. Since S is Armendariz, we have

$$c(f_1) \cdots c(f_{n+1}) = 0$$

in S (see [AC, Proposition 1]), and we may assume without loss of generality that $c(f_1) \cdots c(f_n) = 0$. This clearly implies $f_1 \cdots f_n = 0$, as required. ∎

PROPOSITION 2.11. *Let I be an n-absorbing ideal of R and let $S = R/I$. Then $I[x]$ is an n-absorbing ideal of $R[x]$ in each of the following cases:*

(i) *S is Armendariz and $|R/P| \geq n$ for each maximal ideal P of R containing I.*
(ii) *S is Armendariz and is $(n-1)!$-torsion-free as an additive group.*
(iii) *S is locally Bézout.*
(iv) *S contains infinitely many elements s_i such that $s_i - s_j$ is not a zero-divisor when $s_i \neq s_j$. In particular, this is true if S is torsion-free as an additive group.*

Proof. Parts (i)–(iii) follow from Propositions 2.6 and 2.10 (since locally Bézout rings are Armendariz).

To prove (iv), it is enough to show that if $f_1(x), \ldots, f_{n+1}(x)$ are polynomials in $S[x]$ such that $f_1(x) \cdots f_{n+1}(x) = 0$, then the product of some n of the $f_i(x)$ is zero. We have $f_1(s_i) \cdots f_{n+1}(s_i) = 0$ for each s_i, so there exists an n-element subset $E(s_i) \subset \{1, \ldots, n+1\}$ such that $\prod_{r \in E(s_i)} f_r(s_i) = 0$. Clearly, there are infinitely many s_i, say $\sigma_1, \sigma_2, \ldots$, with the same set $E(\sigma_i)$, and we may, without loss of generality, assume such a set to be $\{1, \ldots, n\}$, i.e. $f_1(\sigma_h) \cdots f_n(\sigma_h) = 0$ for $h \in \mathbb{N}$. Let $f_1(x) \cdots f_n(x) = g_0 + g_1 x + \cdots + g_m x^m$. Then clearly

$$\begin{bmatrix} 1 & \sigma_1 & \cdots & \sigma_1^m \\ 1 & \sigma_2 & \cdots & \sigma_2^m \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \sigma_m & \cdots & \sigma_m^m \end{bmatrix} \begin{bmatrix} g_0 \\ g_1 \\ \cdots \\ g_m \end{bmatrix} = 0,$$

and since

$$\begin{vmatrix} 1 & \sigma_1 & \cdots & \sigma_1^m \\ 1 & \sigma_2 & \cdots & \sigma_2^m \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \sigma_m & \cdots & \sigma_m^m \end{vmatrix} = \prod_{1 \le h < k \le m} (\sigma_k - \sigma_h),$$

we obtain $f_1(x) \cdots f_n(x) \prod_{1 \le h < k \le m} (\sigma_k - \sigma_h) = 0$. Since none of the factors $\sigma_k - \sigma_h$ is a zero-divisor, we obtain $f_1(x) \cdots f_n(x) = 0$, as required. The second part of (iv) follows from the fact that when S is torsion-free, the elements $s_i = i \cdot 1$ ($i \in \mathbb{N}$) of S are such that $s_i - s_j$ is not a zero-divisor when $s_i \neq s_j$. ∎

Since homomorphic images of arithmetical rings are locally Bézout, an immediate consequence of Proposition 2.11 (and Proposition 2.10) is the following generalization of [AB, Corollary 6.9].

COROLLARY 2.12. *Let I be an n-absorbing ideal of an arithmetical ring R. Then $I[x]$ is an n-absorbing ideal of $R[x]$. In particular, I is strongly n-absorbing.*

COROLLARY 2.13. *Let $R[x]$ be an n-absorbing ring. Then so too is $R[X]$, where X is any set of indeterminates.*

Proof. The case $X = \emptyset$ is clear. Suppose therefore that X is nonempty but finite. Then by Proposition 2.11(iv) and straightforward induction, and since $R[x]$ satisfies the condition in Proposition 2.6(iv), we deduce that $R[X]$ is n-absorbing. Now let $f_1 \cdots f_{n+1} = 0$, where $f_1, \ldots, f_{n+1} \in R[X]$, and let Y_i be the set of all indeterminates that appear in f_i. Then $Y := \bigcup_{1 \le i \le n+1} Y_i$ is finite, so that $R[Y]$ is n-absorbing. Since $f_1, \ldots, f_{n+1} \in R[Y]$, we infer that $f_1 \cdots f_n = 0$, say, and the proof is complete. ∎

The next result shows that a power series version of the Armendariz property holds for 2-absorbing rings. The argument we use provides, in particular, an alternative approach to proving [AB, Theorem 4.15].

Proposition 2.14. *Let S be a 2-absorbing ring and let $f=\sum_{i\geq 0} f_i x^i$, $g=\sum_{i\geq 0} g_i x^i \in S[[x]]$ be such that $fg=0$. Then $f_i g_j = 0$ for all $i, j \geq 0$ (and hence S is Armendariz). In particular, if I is a 2-absorbing ideal of R, then $I[[x]]$ is a 2-absorbing ideal of the power series ring $R[[x]]$ (and therefore $I[x]$ is a 2-absorbing ideal of the ring $R[x]$).*

Proof. We first prove that for each i, either $f_i^2 = 0$ or $f_i g = 0$. Suppose that $f_0^2 \neq 0$. We claim that $f_0 g_j = 0$ for all j: We have $f_0 g_0 = 0$, and assuming $f_0 g_j = 0$ for all $0 \leq j \leq k$, we obtain $0 = f_0(f_0 g_{k+1} + f_1 g_k + \cdots + f_{k+1} g_0) = f_0^2 g_{k+1}$, i.e. $f_0 g_{k+1} = 0$, since S is 2-absorbing. Hence, by induction, $f_0 g_j = 0$ for all j, and our claim is proved.

We therefore have $\left(\frac{f-f_0}{x}\right)^2 g = 0$, and repeating the previous argument, we conclude that either $f_1^2 = 0$ or $f_1 g = 0$. In the same way, for each $k \in \mathbb{N}$, there exists a sufficiently large $r \in \mathbb{N}$ for which

$$\left(\frac{f - \sum_{i=0}^{k-1} f_i x^i}{x^k}\right)^r g = 0.$$

This yields $f_k^{2r} = 0$ or $f_k^r g = 0$, and again using the fact that S is 2-absorbing, we obtain $f_k^2 = 0$ or $f_k g = 0$ for each $k \in \mathbb{N}$, as required. We now have $(f_i + g_j) f_i g_j = 0$ for all $i, j \geq 0$, so that $f_i g_j = 0$, as S is 2-absorbing.

For the second part of the proposition, we may assume $I = 0$, since $R[[x]]/I[[x]] \cong (R/I)[[x]]$. Let $f = \sum_{i\geq 0} f_i x^i$, $g = \sum_{i\geq 0} g_i x^i$, $h = \sum_{i\geq 0} h_i x^i$ in $R[[x]]$ be such that $fgh = 0$, let $c(f)$, $c(g)$, $c(h)$ be their contents and let $\alpha \in c(f)$, $\beta \in c(g)$, $\gamma \in c(h)$. By the first part, $\alpha gh = 0$, i.e. $(\alpha h) g = 0$, so that $(\alpha h)\beta = 0$, which gives $\alpha\beta\gamma = 0$ (cf. [AC, Proposition 1]). This implies $c(f)c(g)c(h) = 0$, and since R is a strongly 2-absorbing ring, we infer that $c(f)c(g) = 0$, say, i.e. $fg = 0$. This completes the proof. ∎

Example 2.15. Rings R for which $c(f)c(g) = 0$ whenever $f, g \in R[[x]]$ and $fg = 0$ are called *power-series-wise Armendariz* (see [KLL]). Proposition 2.14 therefore states that 2-absorbing rings are power-series-wise Armendariz. The converse is not true: Let $R = K[[x]]/(x^3)$ where K is a field; then R, a homomorphic image of a principal ideal ring, is power-series-wise Armendariz by [KLL, Proposition 3.2], but R is clearly not 2-absorbing.

We end this note with a result that shows that when attempting to prove either of (C1), (C2), or (C3), it is enough to restrict our attention to n-absorbing total rings of quotients (and hence to n-absorbing Prüfer rings). More precisely:

Proposition 2.16. *Let I be an ideal of R and let Q be the total ring of fractions of R/I. If* (C1), (C2), *or* (C3) *holds for the zero ideal of Q, then it holds for I.*

Proof. Let I be an n-absorbing ideal of R, so that the ring $S := R/I$ is n-absorbing. We claim that the ring Q is n-absorbing. For if $\frac{a_1}{s_1} \cdots \frac{a_{n+1}}{s_{n+1}} = 0$ in Q, where each a_i is in S and no s_i is a zero-divisor in S, then there is a non-zero-divisor s in S such that $sa_1 \cdots a_{n+1} = 0$, i.e. $a_1 \cdots a_{n+1} = 0$. We may clearly assume that $a_1 \cdots a_n = 0$ and obtain $\frac{a_1}{s_1} \cdots \frac{a_n}{s_n} = 0$, as claimed. Suppose first that (C1) holds for the zero ideal of Q (so Q is strongly n-absorbing) and that $A_1 \cdots A_{n+1} = 0$ for some ideals $A_1, \ldots, A_{n+1}$ of S. Then the ideals A_iQ of Q satisfy $A_1Q \cdots A_{n+1}Q = 0$, and we may assume that $A_1Q \cdots A_nQ = 0$. This implies $A_1 \cdots A_n = 0$, and so the ring S is strongly n-absorbing.

Suppose now that (C2) holds for the zero ideal of Q. Then $Q[x]$ and its subring $S[x]$ are n-absorbing. The case of (C3) is trivial. ∎

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