COLLOQUIUM MATHEMATICUM

VOL. 148 2017 NO. 2

OPEN RETRACTIONS OF INDECOMPOSABLE CONTINUA

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Abstract. We show that for each continuum X there exist an indecomposable continuum Y which contains X and an open retraction $r : Y \to X$ such that each fiber of r is homeomorphic to the Cantor set. Furthermore, Y is homeomorphic to the closure of a countable union of topological copies of X in some continuum. This result is a strengthening of a result proved by Bellamy (1971).

1. Introduction. In this paper, unless otherwise stated, all spaces are assumed to be metrizable and maps are continuous. A compact space is called a *compactum*, and *continuum* means a connected compactum. A continuum is said to be *indecomposable* if it is not the union of two proper subcontinua. We denote the closed interval $[0, 1]$ by I.

In [3, Corollary 4], Bellamy proved that if X is a continuum, then there exists an indecomposable continuum Y which contains X as a retract. In [6], van Mill proved that for each homogeneous continuum X there exists a non-metrizable indecomposable homogeneous continuum Y such that X is an open retract of Y (see also [4] and [8]).

In this paper, we prove the following result.

THEOREM 1.1. *For each continuum X there exist an indecomposable continuum Y which contains X and an open retraction $r : Y \to X$ such that each fiber of r is homeomorphic to the Cantor set. Furthermore, Y is homeomorphic to the closure of a countable union of topological copies of X in some continuum.*

If $f : X \to Y$ is a map between compacta, then let $\mathcal{D}_f = \{f^{-1}(y) \mid y \in Y\}$. It is known that a map $f : X \to Y$ between compacta is an open map if and only if $\mathcal{D}_f$ is a continuous decomposition of X (see [7, Corollary 13.11]). Hence, Theorem 1.1 is equivalent to the following.

THEOREM 1.2. *For each continuum X there exist an indecomposable continuum Y which contains X and a retraction $r : Y \to X$ such that $\mathcal{D}_r$*

2010 *Mathematics Subject Classification*: Primary 54F15; Secondary 54C10.
Key words and phrases: indecomposable continuum, retraction, Cantor set, open map.
Received 10 March 2016; revised 20 June 2016.
Published online 24 February 2017.

DOI: 10.4064/cm6912-7-2016

is a continuous decomposition of Y *and each member of* $\mathcal{D}_r$ *is homeomorphic to the Cantor set. Furthermore,* Y *is homeomorphic to the closure of a countable union of topological copies of* X *in some continuum.*

2. Preliminaries. In this section we give some notation and terminology. Also, we introduce some auxiliary results.

If A is a subset of a space X, then $\mathrm{Cl}_X(A)$ denotes the closure of A in X. Also, if $\delta > 0$, then $\operatorname{diam} A$ denotes the diameter of A.

Let $\{X_i, f_i\}_{i\in\mathbb{N}}$ be an inverse sequence. Then, if $j > i+1$, we denote the map $f_i \circ \cdots \circ f_{j-1} : X_j \to X_i$ by f_{ij}. Also, let $f_{i,i+1} = f_i$.

Let $\{X_i, f_i\}_{i=1}^{\infty}$ be an inverse sequence such that each X_i is a continuum. Then $\{X_i, f_i\}_{i=1}^{\infty}$ is called an *indecomposable inverse sequence* if for every $i \in \mathbb{N}$, whenever A_{i+1} and B_{i+1} are subcontinua of X_{i+1} such that $X_{i+1} = A_{i+1} \cup B_{i+1}$, then $f_i(A_{i+1}) = X_i$ or $f_i(B_{i+1}) = X_i$ (see [7, Definition 2.5]).

To prove Theorem 1.1, we use the following results.

THEOREM 2.1 ([1, Theorem I]). *Let* (S, d) *be a compact metric space and let* $\{X_i, f_i\}_{i=1}^{\infty}$ *be an inverse sequence where each* X_i *is a non-empty compact subset of* S *and each* f_i *is a surjective map from* X_{i+1} *to* X_i*. Suppose that* (1) *for every* $i \in \mathbb{N}$ *and every* $\delta > 0$ *there exists* $\delta' > 0$ *such that if* $j > i$, $p, q \in X_j$ *and* $d(f_{ij}(p), f_{ij}(q)) > \delta$, *then* $d(p,q) > \delta'$, *and* (2) *for every* $\varepsilon > 0$ *there exists* $k \in \mathbb{N}$ *such that if* $p \in X_k$, *then* $\operatorname{diam} \bigcup_{j=k+1}^{\infty} f_{kj}^{-1}(p) < \varepsilon$*. Then* $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ *is homeomorphic to* $\bigcap_{i=1}^{\infty} \mathrm{Cl}_S(\bigcup_{k=i}^{\infty} X_k)$.

THEOREM 2.2 ([7, Theorem 2.7]). *Let* $\{X_i, f_i\}_{i=1}^{\infty}$ *be an indecomposable inverse sequence. Then* $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ *is an indecomposable continuum.*

3. Proof of Theorem 1.1. It is enough to show the following:

($\star$) For each continuum X there exist an indecomposable continuum Y which contains a topological copy X' of X and an open retraction $r : Y \to X'$ such that each fiber of r is homeomorphic to the Cantor set. Furthermore, Y is homeomorphic to the closure of a countable union of topological copies of X in some continuum.

Proof of ($\star$). The proof employs techniques used in [9, Example 2.3.1] (see also [2], [4] and [8]). For each $i \in \mathbb{N}$, let $I_i = I$ and $S = X \times \prod_{j=1}^{\infty} I_j$. If d is a compatible metric on X, we can define a metric ρ on S by the formula

$$\rho\big((x, x_1, x_2, \ldots), (y, y_1, y_2, \ldots)\big) = d(x,y) + \sum_{i=1}^{\infty} \frac{1}{2^i}|x_i - y_i|.$$

Let $M_1 = X$ and let $X_1 = M_1 \times \{0\} \times \{0\} \times \cdots \subset S$. Assume M_i and X_i have been defined. Take a countable dense subset $\{a_n^i\}_{n=1}^{\infty}$ in M_i. For each $n \in \mathbb{N}$, let $g_n^i : M_i \to [1/(n+1), 1/n]$ be a surjective map such that ${g_n^i}^{-1}(1/n) = \{a_n^i\}$ and ${g_n^i}^{-1}(1/(n+1)) = \{a_{n+1}^i\}$. For each $n \in \mathbb{N}$, let $G_n^i = \{(x, g_n^i(x)) \mid x \in M_i\} \subset M_i \times I_i$. Also, let $M_{i+1} = (M_i \times \{0\}) \cup \bigcup_{n=1}^{\infty} G_n^i \subset M_i \times I_i$ and $X_{i+1} = M_{i+1} \times \{0\} \times \{0\} \times \cdots \subset M_{i+1} \times \prod_{j=i+1}^{\infty} I_j$. Thus, we have defined M_{i+1} and X_{i+1}. Note that for each $i \in \mathbb{N}$, M_{i+1} is a subcontinuum of $M_i \times I_i$, M_i is homeomorphic to X_i, X_i is a countable union of topological copies of X, $X_i \subset S$ and $X_i \subset X_{i+1}$. Define $f_i : X_{i+1} \to X_i$ by $f_i(x, x_1, x_2, \ldots, x_{i-1}, x_i, 0, 0, \ldots) = (x, x_1, x_2, \ldots, x_{i-1}, 0, 0, 0, \ldots)$ and $f_i' : M_{i+1} \to M_i$ by $f_i'(x, x_1, x_2, \ldots, x_{i-1}, x_i) = (x, x_1, x_2, \ldots, x_{i-1})$.

Let $Y = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$. We show that $\{X_i, f_i\}_{i=1}^{\infty}$ is an indecomposable sequence. To see this, it is enough show the following:

($\sharp$) *If $i \in \mathbb{N}$ and A_{i+1} and B_{i+1} are subcontinua of M_{i+1} such that $M_{i+1} = A_{i+1} \cup B_{i+1}$, then $f_i'(A_{i+1}) = M_i$ or $f_i'(B_{i+1}) = M_i$.*

From now on we prove ($\sharp$). Let $i \in \mathbb{N}$ and let A_{i+1}, B_{i+1} be subcontinua of M_{i+1} such that $M_{i+1} = A_{i+1} \cup B_{i+1}$. Let $\mathrm{pr}_i : M_{i+1} \to I_i$ be the projection. Assume that $(M_i \times \{0\}) \setminus A_{i+1} \neq \emptyset \neq (M_i \times \{0\}) \setminus B_{i+1}$. We may also assume that $A_{i+1} \setminus (M_i \times \{0\}) \neq \emptyset$. Let $a = \max\{\mathrm{pr}_i(z) \mid z \in A_{i+1}\}$. Take an open subset $U \subset M_{i+1}$ such that $U \cap (M_i \times \{0\}) \neq \emptyset$ and $U \cap A_{i+1} = \emptyset$. Since $\{a_n^i\}_{n=1}^{\infty}$ is dense in M_i, there exists $n_0 \in \mathbb{N}$ such that $1/n_0 < a$ and $(a_{n_0}^i, 1/n_0) \in U$. Then, $A_{i+1} \subset \mathrm{pr}_i^{-1}([0, 1/n_0)) \cup \mathrm{pr}_i^{-1}((1/n_0, 1])$ and $A_{i+1} \cap \mathrm{pr}_i^{-1}([0, 1/n_0)) \neq \emptyset \neq A_{i+1} \cap \mathrm{pr}_i^{-1}((1/n_0, 1])$. Since A_{i+1} is a continuum, this is a contradiction. Hence, $M_i \times \{0\} \subset A_{i+1}$ or $M_i \times \{0\} \subset B_{i+1}$. Therefore, $f_i'(A_{i+1}) = M_i$ or $f_i'(B_{i+1}) = M_i$. This completes the proof of ($\sharp$) and we see that $\{X_i, f_i\}_{i=1}^{\infty}$ is an indecomposable sequence. Hence, by Theorem 2.2, Y is an indecomposable continuum.

For each $i \in \mathbb{N}$, it is not difficult to see that f_i is an open map since g_n^i is continuous for each $n \in \mathbb{N}$. Hence, by [5, Corollary 2.1.10], the projection map $\pi_1 : Y \to X_1$ is an open map.

Let $S' = S \times S \times \cdots$. Also, let $\mathbf{0} = (0, 0, \ldots) \in \prod_{j=1}^{\infty} I_j$, $X'' = (X \times \{\mathbf{0}\}) \times (X \times \{\mathbf{0}\}) \times \cdots \subset S'$ and $X' = \{((x, \mathbf{0}), (x, \mathbf{0}), \ldots) \in X'' \mid x \in X\} \subset X''$. Note that X' is homeomorphic to X and $X' \subset Y$. Define $h : X_1 \to X'$ by $h((x, \mathbf{0})) = ((x, \mathbf{0}), (x, \mathbf{0}), \ldots)$ for each $(x, \mathbf{0}) \in X_1$. Then it is easy to see that h is a homeomorphism and $r = h \circ \pi_1 : Y \to X'$ is a retraction. Since π_1 is an open map, r is also an open map.

Next, we show that each fiber of r is homeomorphic to the Cantor set. Let $x \in X'$. Since r is a continuous map from a compactum, $r^{-1}(x)$ is compact. Also, since each fiber of f_i is totally disconnected for each $i \in \mathbb{N}$, it is easy to see that $r^{-1}(x)$ is totally disconnected. Furthermore, since the cardinality of each fiber of f_i is greater than 1 for each $i \in \mathbb{N}$, $r^{-1}(x)$ is perfect. Hence, by [7, Theorem 7.14], $r^{-1}(x)$ is homeomorphic to the Cantor set.

Finally, we show that Y is homeomorphic to $\mathrm{Cl}_S(\bigcup_{i=1}^{\infty} X_i)$. Let $i \in \mathbb{N}$ and $\delta > 0$. We can see that if $j > i$, $p, q \in X_j$ and $\rho(f_{ij}(p), f_{ij}(q)) > \delta$, then $\rho(p, q) \geq \rho(f_{ij}(p), f_{ij}(q)) > \delta$. Also, let $\varepsilon > 0$. Take $k \in \mathbb{N}$ such that $\sum_{i=k+1}^{\infty} 2^{-i} < \varepsilon$. Then, for each $p \in X_k$, $\operatorname{diam} \bigcup_{j=k+1}^{\infty} f_{kj}^{-1}(p) < \varepsilon$. Hence, by Theorem 2.1 and the fact that $X_i \subset X_{i+1}$ for each $i \in \mathbb{N}$, we see that Y is homeomorphic to $\mathrm{Cl}_S(\bigcup_{i=1}^{\infty} X_i)$. ■

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