

## Flows with Ratner's property have discrete essential centralizer

by

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**Abstract.** We show that a free, ergodic action of  $\mathbb{R}$  with (finite) Ratner's property has countable discrete essential centralizer. Moreover, we show that such flows are mildly mixing.

**1. Introduction.** In the 1980's, when studying dynamics of horocycle flows acting on the unit tangent spaces of surfaces with constant negative curvature, M. Ratner discovered a special property, now called Ratner's property, which is a particular way of divergence of orbits of nearby points. Namely, the speed of that divergence is polynomial, and this seems to be a characteristic property for so called *parabolic* dynamics. This property was used by Ratner [18] to prove some important joining rigidity phenomena in the class of horocycle flows (see also [15], [19]). Namely, for every ergodic joining, different from product measure, of an ergodic flow  $(T_t)$  acting on a standard Borel probability space  $(X, \mathcal{B}, \mu)$  with a flow having Ratner's property, the projection map on the  $X$ -coordinate has finite fibers. Moreover, flows with Ratner's property enjoy the pairwise independence property (PID), that is, any self-joining (of arbitrary order) which is pairwise independent is just product measure. As noticed in [10], the PID property forces mixing to be mixing of all orders. During the last decade many other flows have been shown to enjoy Ratner's property (with some modifications of the original definition but keeping the aforementioned joining phenomena). Mixing of all orders of horocycle flows was known before Ratner's work (see [16]), but it is only very recently that, via Ratner's property, mixing of all orders was established for some classes of smooth flows on surfaces [1] (see

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also [14]). Ratner's property itself however does not imply mixing. Indeed, all other known classes of flows satisfying Ratner's property (see [2], [4], [5], [11], [12], [13]) are not mixing. On the other hand, all these examples are mildly mixing. As a matter of fact, it was already asked by J.-P. Thouvenot in the 1990's whether Ratner's property is compatible with the rigidity property of flows. More precisely, Thouvenot asked whether there exists a flow  $(T_t)_{t \in \mathbb{R}}$  with Ratner's property acting on a standard Borel probability space  $(X, \mathcal{B}, \mu)$  for which for some  $t_n \rightarrow \infty$ , we have  $f \circ T_{t_n} \rightarrow f$  in  $L^2$  for each  $f \in L^2(X, \mathcal{B}, \mu)$ . Recall that mild mixing can be defined as the absence of non-trivial rigid factors, so no one of known examples of flows with Ratner's property was rigid.

In this note, we will prove the following results.

**THEOREM 1.** *Assume that a measurable, measure-preserving flow  $\mathcal{T} = (T_t)$  is free, ergodic and enjoys Ratner's property. Then its essential centralizer  $EC(\mathcal{T}) := C(\mathcal{T})/\{T_t : t \in \mathbb{R}\}$  is discrete and countable.*

As the essential centralizer of a rigid flow is uncountable, this in particular answers Thouvenot's question.

**THEOREM 2.** *Assume that a measurable, measure-preserving flow  $\mathcal{T} = (T_t)$  is free, ergodic and has Ratner's property. Then, for each  $S \in C(\mathcal{T})$ , either  $S$  is mildly mixing or  $S$  is of finite order.*

We have the following immediate corollary (which also answers Thouvenot's question):

**COROLLARY 1.1.** *A flow with Ratner's property is mildly mixing.*

Let us mention that some of the ideas of our proof remind those used in [8] to show that the Chacon transformation (which has Ratner's property as a  $\mathbb{Z}$ -action) commutes only with its powers.

**2. Basic definitions.** Let  $(X, \mathcal{B}, \mu)$  be a standard Borel probability space. We denote by  $\text{Aut}(X, \mathcal{B}, \mu)$  the group of all (measure-preserving) automorphisms of that space. Each element  $S \in \text{Aut}(X, \mathcal{B}, \mu)$  can also be treated as a unitary operator on  $L^2(X, \mathcal{B}, \mu)$  via  $Sf := f \circ S$ . Endowed with the strong operator topology,  $\text{Aut}(X, \mathcal{B}, \mu)$  becomes a Polish group.

Throughout, we consider only measurable  $\mathbb{R}$ -representations in  $\text{Aut}(X, \mathcal{B}, \mu)$ , i.e. flows  $\mathcal{T} = (T_t)$  on  $(X, \mathcal{B}, \mu)$ . Measurability means that the map  $X \times \mathbb{R} \ni (x, t) \mapsto T_t x$  is measurable. A flow is called *rigid* if for a sequence  $(t_n)_{n \geq 1}$ ,  $t_n \rightarrow \infty$ , we have

$$\lim_{n \rightarrow \infty} \mu(T_{t_n} A \triangle A) = 0 \quad \text{for every } A \in \mathcal{B}.$$

This is equivalent to saying that, as operators on  $L^2(X, \mathcal{B}, \mu)$ ,  $T_{t_n} \rightarrow \text{Id}$  strongly.

REMARK 2.1. If  $S_n \in \text{Aut}(X, \mathcal{B}, \mu)$ ,  $n \geq 1$ , and  $S_n \rightarrow \text{Id}$  strongly on  $L^2(X, \mathcal{B}, \mu)$ , then  $S_n \rightarrow \text{Id}$  in measure. Indeed, there is no harm to assume that  $X$  is a compact metric space with a metric  $d$ . Fix  $\varepsilon > 0$  and cover  $X$  by finitely many balls  $B_1, \dots, B_m$  of radius  $\varepsilon$ . Now, given  $\delta > 0$ , for  $n \geq n_0$ , we have  $\mu(B_i \cap S_n^{-1}B_i) \geq (1 - \delta)\mu(B_i)$  for  $i = 1, \dots, m$ . Hence  $\mu(\{x \in X : d(x, S_n x) \geq 2\varepsilon\}) < 2m\delta$ , and the claim follows.

It follows that if  $S_n \rightarrow \text{Id}$  strongly on  $L^2(X, \mathcal{B}, \mu)$ , then for a subsequence  $(n_k)$ , we have  $S_{n_k} \rightarrow \text{Id}$   $\mu$ -a.e.

Following [6], a flow  $\mathcal{T}$  is called *mildly mixing* if it has no non-trivial rigid factors, i.e.

$$\liminf_{t \rightarrow \infty} \mu(T_{-t}B \triangle B) > 0 \quad \text{for every } B \in \mathcal{B} \text{ with } 0 < \mu(B) < 1.$$

Let  $C(\mathcal{T}) := \{S \in \text{Aut}(X, \mathcal{B}, \mu) : ST_t = T_tS \text{ for every } t \in \mathbb{R}\}$  denote the *centralizer* of  $\mathcal{T}$  and let the *essential centralizer* of  $\mathcal{T}$  be defined as

$$EC(\mathcal{T}) = C(\mathcal{T}) / \{T_t : t \in \mathbb{R}\}.$$

Then  $C(\mathcal{T})$  is closed in  $\text{Aut}(X, \mathcal{B}, \mu)$ , hence is a Polish group. Moreover,  $EC(\mathcal{T})$  is also a group, although its topological properties depend on whether the subgroup  $\{T_t : t \in \mathbb{R}\}$  is closed.

REMARK 2.2. If  $\mathcal{T}$  is rigid then  $EC(\mathcal{T})$  is uncountable. This result is folklore but we provide an argument for completeness.

First notice that in a Polish group  $G$ , if we have a dense subgroup  $H$  then either  $H = G$  or  $H$  is a set of first category. Indeed, if  $H$  is of second category then  $H = HH^{-1}$  contains a neighborhood  $U$  of 1, so  $H$  is also open. But it is also closed (the complement of  $H$  is a union of cosets of  $H$ ), so  $H$  is a clopen subgroup. Since it is dense,  $H = G$ .

If  $H$  is of first category and if  $G = \bigcup_{i \geq 1} g_i H$  then  $G$  is still of first category, which contradicts the fact that  $G$  is Polish.

In our context, we use this for  $H := \{T_t : t \in \mathbb{R}\} \subset \overline{\{T_t : t \in \mathbb{R}\}} =: G$  <sup>(1)</sup>.

For joining theory of dynamical systems, we refer the reader to [7].

### 3. The centralizer of flows with Ratner's property

**3.1. Ratner's property.** Recall now the notion of (finite) Ratner's property introduced in [2]. This is a weakening of Ratner's original property introduced in [18] in the context of horocycle flows. Assume that  $X$  is a  $\sigma$ -compact metric space with a metric  $d$ . Let  $(T_t) \subset \text{Aut}(X, \mathcal{B}, \mu)$  be an ergodic flow.

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<sup>(1)</sup> This argument has been communicated to us by A. Danilenko and replaced our first argument based on the open map theorem for Polish groups together with the fact that a solenoidal group is either  $\mathbb{R}$ , or it is compact, or else it is not locally compact.

DEFINITION 3.1. Fix a finite set  $P \subset \mathbb{R} \setminus \{0\}$  and  $t_0 \in \mathbb{R}$ . Then  $(T_t)$  is said to have the  $R(t_0, P)$  property if for every  $\varepsilon > 0$  and  $N \in \mathbb{N}$  there exist  $\kappa = \kappa(\varepsilon) > 0$ ,  $\delta = \delta(\varepsilon, N)$  and a set  $Z = Z(\varepsilon, N) \subset X$  with  $\mu(Z) > 1 - \varepsilon$  such that for any  $x, y \in Z$  with  $x$  not in the orbit of  $y$  and  $d(x, y) < \delta$  there exist  $M = M(x, y)$  and  $L = L(x, y)$ , with  $M, L \geq N$  and  $L/M \geq \kappa$ , and there exists  $p = p(x, y) \in P$  such that

$$(1) \quad \frac{1}{L} |\{n \in [M, M + L] : d(T_{nt_0}x, T_{nt_0+py}) < \varepsilon\}| > 1 - \varepsilon.$$

We say that  $(T_t)$  has *Ratner’s property* (with the set  $P$ ) if the set

$$\{s \in \mathbb{R} : (T_t) \text{ has the } R(s, P) \text{ property}\}$$

is uncountable.

REMARK 3.2. (1) If  $(X, \mathcal{B}, \mu)$  is a standard Borel probability space and there is no good metric structure on  $X$  then we say that a flow  $\mathcal{T} = (T_t) \subset \text{Aut}(X, \mathcal{B}, \mu)$  has Ratner’s property (with some set  $P$ ) if there exist a  $\sigma$ -compact metric space  $(X', d')$  and a flow  $\mathcal{T}' = (T'_t) \subset \text{Aut}(X', \mathcal{B}', \mu')$  which has Ratner’s property (with the set  $P$ ) such that the flows  $\mathcal{T}$  and  $\mathcal{T}'$  are measure-theoretically isomorphic. In such a situation we say that  $\mathcal{T}'$  is a *good metric model* of  $\mathcal{T}$ .

(2) It is shown in [2] that Ratner’s property does not depend on the choice of a good metric model. In particular, Ratner’s property does not depend on the choice of metric in Definition 3.1. It follows that Ratner’s property can be defined unambiguously for flows defined on standard Borel probability spaces.

(3) If  $(T_t) \subset \text{Aut}(X, \mathcal{B}, \mu)$  is a free ergodic  $\mathbb{R}$ -action with  $(X, d)$  a  $\sigma$ -compact metric space and all  $T_t$  being isometries then  $\mathcal{T}$  does not have Ratner’s property (e.g. consider the flow  $T_t(x, y) = (x + t, y + \alpha t)$  on  $\mathbb{T}^2$  with  $\alpha$  irrational).

REMARK 3.3. Notice that if  $R_t : \mathbb{T} \rightarrow \mathbb{T}$ ,  $t \in \mathbb{R}$ , is the *linear* flow on the additive circle, i.e.  $R_t(x) = x + t \pmod 1$ , then  $(R_t)$  satisfies Ratner’s property (with any finite set  $P$ ). Indeed, any two points are in one orbit, so Definition 3.1 holds trivially. The group of eigenvalues of this flow is  $\mathbb{Z}$ . It is not hard to see that the only ergodic flows which have discrete spectrum and (infinite) cyclic group of eigenvalues are rescalings of the linear flow. Such  $\mathbb{R}$ -actions are, up to isomorphisms, all ergodic (non-trivial) non-free  $\mathbb{R}$ -actions <sup>(2)</sup>.

In view of Remark 3.2, from now on, we assume that a flow  $\mathcal{T} = (T_t)$  acts on a standard Borel probability space  $(X, \mathcal{B}, \mu)$ , where  $(X, d)$  is a compact metric space. In view of Remark 3.3, we will assume that the  $\mathbb{R}$ -actions under consideration are free.

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<sup>(2)</sup> Note that such flows have the minimal self-joining property.

**3.2. Proof of Theorem 1.** We begin with the following lemma:

LEMMA 3.4. *If  $\mathcal{T} = (T_t)$  is ergodic and has Ratner's property (with a set  $P$ ) then  $\{T_t : t \in \mathbb{R}\}$  is open in  $C(\mathcal{T})$ .*

Notice that Theorem 1 is a straightforward consequence of Lemma 3.4:

*Proof of Theorem 1.* By Lemma 3.4, it follows that the topological group  $EC(\mathcal{T}) = C(\mathcal{T})/\{T_t : t \in \mathbb{R}\}$  is discrete (as  $\{T_t : t \in \mathbb{R}\}$  is normal and clopen). But  $EC(\mathcal{T})$  is separable (since  $C(\mathcal{T})$  is separable). Hence  $EC(\mathcal{T})$  is countable. ■

Therefore, it is enough to prove Lemma 3.4. Before we provide its proof, more definitions and observations will be needed. Let  $\mathcal{T} = (T_t)$  be an ergodic flow such that  $T_{t_0}$  is ergodic. Fix a finite set  $P \subset \mathbb{R} \setminus \{0\}$ . Set

$$(2) \quad A_k := \{x \in X : d(x, T_p x) > k^{-1} \text{ for all } p \in P\}.$$

We have the following:

$$(P1) \quad \lim_{k \rightarrow \infty} \mu(A_k) = 1.$$

(P2) There exists  $k_0 \in \mathbb{N}$  such that for every  $\kappa > 0$ , there exist a set  $W_\kappa \subset X$  with  $\mu(W_\kappa) > 0.99$  and a number  $N_0 \in \mathbb{N}$  such that for any  $M, L \geq N_0$  with  $L/M \geq \kappa$ , and every  $x \in W_\kappa$ , we have

$$(3) \quad \frac{1}{L} |\{n \in [M, M + L] : T_{nt_0} x \in A_{k_0}\}| \geq 0.99.$$

(P3) Let  $B \in \mathcal{B}$  with  $\mu(B) = 1$ . For every  $\kappa > 0$  there exist a set  $V_\kappa \subset X$  with  $\mu(V_\kappa) > 0.99$  and a number  $N_1 \in \mathbb{N}$  such that for any  $M, L \geq N_1$  with  $L/M \geq \kappa$  and every  $x \in V_\kappa$ , we have

$$(4) \quad \frac{1}{L} |\{n \in [M, M + L] : T_{nt_0} x \in B\}| \geq 0.99.$$

Indeed, notice that for every  $k \in \mathbb{N}$ , we have  $A_k \subset A_{k+1}$ . Let  $A := \bigcup_{k \geq 1} A_k$ . Then  $A^c$  is the set of periodic points for  $\mathcal{T}$  with periods belonging to  $P$ , so by ergodicity (of  $\mathcal{T}$ ), it has measure 0. Therefore  $\mu(A) = 1$ , and since  $A_k \subset A_{k+1}$  for  $k \geq 1$ , we obtain (P1). To prove (P2) we use (P1) to obtain the existence of  $k_0 \in \mathbb{N}$  such that  $\mu(A_{k_0}) \geq 1 - 10^{-4}$ . Then (P2) follows by using the pointwise ergodic theorem for  $\mathbf{1}_{A_{k_0}}$  and  $T_{t_0}$ , and then Egorov's theorem which gives a uniform convergence on a set of arbitrarily large measure (in our case this set is  $W_\kappa$ ). For (P3), just note that it is a particular case of the same argument used to show (P2) (for a constant sequence of sets).

Assume now that  $(S_n)_{n \in \mathbb{N}} \subset C(\mathcal{T})$  and define

$$(5) \quad B_k := \{x \in X : (\exists n_k) d(x, S_{n_k} x) < k^{-1} \text{ for all } n \geq n_k\}, \quad B := \bigcap_{k \geq 1} B_k.$$

We have the following:

(P4) If  $S_n \rightarrow \text{Id}$   $\mu$ -a.e., then  $\mu(B) = 1$  (note that if we assume only that  $S_n \rightarrow \text{Id}$  strongly, as operators on  $L^2(X, \mathcal{B}, \mu)$ , then by Remark 2.1, we can replace  $(S_n)$  by a  $\mu$ -a.e. convergent subsequence).

(P5) If  $\{S_n : n \in \mathbb{N}\} \cap \{T_t : t \in \mathbb{R}\} = \emptyset$ , then there exists a set  $X_0$  with  $\mu(X_0) = 1$  such that for every  $n \in \mathbb{N}$  and every  $x \in X_0$ ,

$$(6) \quad S_n x \notin \{T_t x : t \in \mathbb{R}\}.$$

Indeed, for (P4) it is enough to notice that  $\mu(B_k) = 1$  for every  $k \geq 1$ . To obtain (P5) notice that for every  $n \in \mathbb{N}$  the set

$$X_n := \{x \in X : S_n(x) \in \{T_t x : t \in \mathbb{R}\}\}$$

is measurable: it is the  $X$ -projection of the measurable set  $\{(x, t) \in X \times \mathbb{R} : S_n x = T_t x\}$ . Moreover,  $X_n$  is  $\mathcal{T}$ -invariant, hence by ergodicity we have  $\mu(X_n) \in \{0, 1\}$ . But by assumption,  $S_n \notin \{T_t : t \in \mathbb{R}\}$ , hence  $\mu(X_n) = 0$ . Finally, set  $X_0 = \bigcap_{n \geq 1} X_n^c$ .

*Proof of Lemma 3.4.* To show that  $\{T_t : t \in \mathbb{R}\}$  is open it is enough to show that for any sequence  $(S_n)_{n \in \mathbb{N}} \subset C(\mathcal{T})$  such that  $S_n \rightarrow \text{Id}$  strongly, there exists  $n_0$  such that for  $n \geq n_0$ ,  $S_n = T_{t_n}$  for some  $t_n \in \mathbb{R}$ . Assume for contradiction that there exists  $(S_n)_{n \in \mathbb{N}} \subset C(\mathcal{T})$  such that  $S_n \rightarrow \text{Id}$  strongly and

$$\{S_n : n \in \mathbb{N}\} \cap \{T_t : t \in \mathbb{R}\} = \emptyset.$$

By (P5), after restricting to  $X_0$ , we may assume that (6) holds for  $x \in X$ . Fix  $t_0 \in \mathbb{R}$  such that  $T_{t_0}$  is ergodic. We will show that for every such  $t_0$ ,  $\mathcal{T}$  fails the  $R(t_0, P)$  property. Since an ergodic flow can have at most countably many non-ergodic time automorphisms, this will finish the proof of Lemma 3.4. Since  $t_0 \in \mathbb{R}$  is fixed, we will write  $T_{t_0} =: T$ .

Fix  $k_0$  from (P2) and let  $0 < \epsilon < \min(0.01, k_0^{-2})$  with  $\kappa = \kappa(\epsilon)$  coming from the  $R(t_0, P)$  property. Let  $N_0$  come from (P2) and  $N_1$  from (P3) (for  $B \in \mathcal{B}$  defined in (5); notice that by (P4),  $\mu(B) = 1$ ). Take  $N \geq \max(N_0, N_1)$ . Let  $\delta = \delta(\epsilon, N)$  and suppose a set  $Z = Z(\epsilon, N)$  with  $\mu(Z) \geq 1 - \epsilon$  is given by the  $R(t_0, P)$  property. Let  $k > \max(\delta^{-2}, \epsilon^{-2})$  and set  $n = n_k$ , where  $n_k$  is given by (5). We will show that for every  $x \in B \cap W_\kappa \cap V_\kappa$ ,  $x$  and  $S_n x$  do not satisfy (1). This means that  $(x, S_n x) \notin Z \times Z$ . This will finish the proof since by (6),  $x$  and  $S_n x$  are not in the same orbit of  $\mathcal{T}$ , by (5) we have  $d(x, S_n x) < k^{-1} < \delta^2 < \delta$ , and

$$S_n^{-1}(Z) \cap (B \cap W_\kappa \cap V_\kappa \cap Z) = \emptyset,$$

which yields a contradiction since all the sets in the intersection above have measure at least 0.99.

Fix  $x \in B \cap W_\kappa \cap V_\kappa$ . Assume that  $(x, S_n x) \in Z \times Z$ . Then there exist  $M, L \geq N$  with  $L/M \geq \kappa$  such that (1) is satisfied for  $x, S_n x$ . Let

(see (2), (5))

$$(7) \quad W := \{i \in [M, M + L] : T_i x \in A_{k_0} \cap B\}.$$

From (3) and (4) we deduce that  $|W| \geq 3L/4$ . By (2) and (5), it follows that for all  $i \in W$  and  $p \in P$ , we have (by the choice of  $k_0$  and  $k$ )

$$\begin{aligned} d(T_{i+p}x, T_i(S_n x)) &\geq d(T_{i+p}x, T_i x) - d(T_i x, T_i(S_n x)) \\ &= d(T_{i+p}x, T_i x) - d(T_i x, S_n(T_i x)) \geq k_0^{-1} - k^{-1} \geq \epsilon^{1/2} - \epsilon^2 > \epsilon. \end{aligned}$$

Now, since  $|W| \geq 3L/4$ , (1) is not satisfied for  $x, S_n x$ . This completes the proof. ■

### 3.3. Proof of Theorem 2

PROPOSITION 3.5. *If  $\mathcal{T} = (T_t)$  is ergodic and has Ratner's property then every (non-trivial) factor  $\mathcal{S} = (S_t)$  of  $\mathcal{T}$  acts freely and has Ratner's property.*

We will use the following lemma (see [2, Remark 2 and Theorem 5.1])

LEMMA 3.6. *Let  $\mathcal{T} = (T_t)$  be an ergodic flow on a standard Borel probability space  $(X, \mathcal{B}, \mu)$ . If  $T$  has Ratner's property, then  $T$  is a finite extension of each of its non-trivial factors.*

Now, we prove Proposition 3.5.

*Proof of Proposition 3.5.* Let  $\mathcal{S} = (S_t) : (Y, \mathcal{C}, \nu) \rightarrow (Y, \mathcal{C}, \nu)$  be a factor of  $\mathcal{T}$ , where  $(Y, \mathcal{C}, \nu)$  is a standard Borel probability space, and  $(Y, d_Y)$  is a compact metric space. In view of Lemma 3.6, we can assume that  $X = Y \times \{0, \dots, k - 1\}$  and  $\mu$  is the product of  $\nu$  and the normalized counting measure on  $\{0, \dots, k - 1\}$ . Moreover,  $X$  becomes a compact metric space with the metric  $d(x, x') = d_Y(y, y') + d_k(i, i')$ , where  $x = (y, i)$ ,  $x' = (y', i')$  and  $d_k$  stands for the discrete metric on  $\{0, \dots, k - 1\}$ . Let  $\pi : X \rightarrow Y$  be given by  $\pi(y, i) = y$  so that  $\pi \circ T_t = S_t \circ \pi$ .

Let us first show that the action of  $(S_t)$  is free. Suppose not. Then for  $\nu$ -a.e.  $y \in Y$  there exists  $r \in \mathbb{R}$  such that  $S_r y = y$ . Let  $x \in X$  be such that  $\pi(x) = y$ . Then, for every  $n \geq 1$ ,

$$\pi(T_{nr}x) = S_{nr}(\pi(x)) = S_{nr}(y) = y,$$

so for every  $n \geq 1$ , we have  $T_{nr}x \in \pi^{-1}(y)$ . But since the action of  $(T_t)$  is free, for  $\mu$ -a.e.  $x \in X$ , we have  $T_{n_1 s}x \neq T_{n_2 s}x$  whenever  $n_1 \neq n_2$ . Hence  $\pi^{-1}y$  is not finite, which is a contradiction.

It remains to show that  $\mathcal{S}$  has Ratner's property. Fix  $t_0$  such that  $(T_t)$  has the  $R(t_0, P)$  property. We will show that  $(S_t)$  also has the  $R(t_0, P)$  property (with the metric  $d_Y$ ). Fix  $\epsilon, N$  and let  $\kappa, \delta, Z$  be as in the definition of  $R(t_0, P)$  property for  $(T_t)$ . If  $\epsilon > 0$  is small enough compared to  $1/k$ , then  $Z \supset Z_Y \times \{0, \dots, k - 1\}$ , where  $Z_Y \in \mathcal{C}$  and  $\nu(Z_Y) > 1 - \epsilon$ . Notice that any

$x, x' \in Z$  such that  $d(x, x') < \delta$  are of the form  $x = (y, i), x' = (y', i)$  for some  $y, y' \in Z_Y$  with  $d_Y(y, y') < \delta$ . Take any  $y, y' \in Z_Y$  such that  $y$  is not in the orbit of  $y'$  and  $d_Y(y, y') < \delta$ . Next, let  $M, L, p$  be as in the definition of  $R(t_0, P)$  for  $x = (y, i)$  and  $x' = (y', i)$ . Then

$$\begin{aligned} & \frac{1}{L} |\{n \in [M, M + L] : d_Y(S_{nt_0}y, S_{nt_0+p}y') < \varepsilon\}| \\ & \geq \frac{1}{L} |\{n \in [M, M + L] : d(T_{nt_0}x, T_{nt_0+p}x') < \varepsilon\}| \geq 1 - \varepsilon. \end{aligned}$$

It follows that  $\mathcal{S}$  has the  $R(t_0, P)$  property. This completes the proof of Proposition 3.5. ■

REMARK 3.7. It follows from [2] that Lemma 3.6 holds without the freeness assumption. By Remark 3.3, also the assertion of Proposition 3.5 holds without the assumption that the action of  $\mathcal{T}$  is free.

Now we give the proof of Theorem 2:

*Proof of Theorem 2.* Let  $ST_t = T_tS$  for all  $t \in \mathbb{R}$ . Assume first that  $S$  is not ergodic. Notice that  $\mathcal{A} := \{A \in \mathcal{B} : SA = A\}$  is a factor of  $(T_t)$ . By Lemma 3.6 it follows that  $(T_t)$  can be represented as a skew product over  $Y$  (where  $Y = X/\mathcal{A}$ ) with finite fibers. Let us denote this space by  $Y \times \{0, \dots, k - 1\}$  for some  $k \in \mathbb{N}$ . Since  $S \in C(\mathcal{T})$ , it follows that  $S$  acts on  $Y \times \{0, \dots, k - 1\}$  as the identity on the first coordinate. It is hence of the form  $S(x, i) = (x, \tau_x(i))$ , with  $\tau_x$  being a bijection of  $\{0, \dots, k - 1\}$ . It easily follows that  $S^{k!} = \text{Id}$ .

Assume now that  $S$  is ergodic. We will show that  $S$  is mildly mixing. Since  $\mathcal{T}$  has Ratner’s property, the Kronecker factor for  $\mathcal{T}$  is trivial. Indeed, if not then by Proposition 3.5 the action on the Kronecker factor is free, so by Remarks 3.2 and 3.3, it does not have Ratner’s property, contrary to Proposition 3.5. Thus,  $\mathcal{T}$  is weakly mixing. Furthermore, since  $S \in C(\mathcal{T})$ ,  $S$  is also weakly mixing. Suppose that  $S$  is not mildly mixing and let  $\mathcal{F}$  be a non-trivial rigid factor of it. Then  $S|_{\mathcal{F}}$  is rigid and  $\mathcal{F}$  is invariant under all  $T_t$ , whence  $\mathcal{F}$  is a non-trivial factor for  $\mathcal{T}$ . Moreover,  $S|_{\mathcal{F}}$  is still weakly mixing, whence all its powers  $(S|_{\mathcal{F}})^n, n \in \mathbb{Z}$ , are distinct (and in particular  $(S|_{\mathcal{F}})^n \neq \text{Id}$  for  $n \neq 0$ ). What is more, if for some  $0 \neq n \in \mathbb{Z}$ , we have  $(S|_{\mathcal{F}})^n = T_{r_n}|_{\mathcal{F}}$ , then  $\mathcal{T}|_{\mathcal{F}}$  is rigid (since  $(S|_{\mathcal{F}})^n$  is rigid), and hence its essential centralizer would be uncountable, in contradiction with Theorem 1 applied to  $\mathcal{T}|_{\mathcal{F}}$ , since the latter action has Ratner’s property by Proposition 3.5. It follows that

$$(8) \quad (S|_{\mathcal{F}})^k \notin \{T_t|_{\mathcal{F}} : t \in \mathbb{R}\} \quad \text{for } 0 \neq k \in \mathbb{Z}.$$

Since  $S|_{\mathcal{F}}$  is rigid, the group  $\{T_t|_{\mathcal{F}} : t \in \mathbb{R}\}$  is not open in  $C(\mathcal{T}|_{\mathcal{F}})$ . This yields a contradiction with Theorem 1. Hence  $S$  has no non-trivial rigid factors, and so it is mildly mixing. This completes the proof of Theorem 2. ■



**3.4. Final remarks.** We have proved that for flows satisfying Ratner's property the essential centralizer is countable (and discrete). We emphasize that this fact is not a consequence of the finite fiber property of ergodic joinings of flows with Ratner's property, mentioned in the Introduction. Indeed, it seems to be possible to adapt the construction of a rigid and simple automorphism from [9] to obtain a simple and rigid flow. We have been unable to decide whether the essential centralizer of a flow with Ratner's property can indeed be infinite (in particular, whether it can contain an element of infinite order). If  $\mathcal{H} = (h_t)$  is a horocycle flow then  $EC(\mathcal{H})$  is finite (see [17, Corollary 4]). For another class of flows enjoying Ratner's property, so called von Neumann special flows over irrational rotations (with the rotation of bounded type), the same phenomenon has been proved [2, 3].

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