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SYMMETRIC AND ANTIPERSYMMETRIC EXTREMAL RANK SOLUTIONS FOR LINEAR MATRIX EQUATIONS AND THEIR APPROXIMATION

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Abstract. This study establishes solvability conditions and explicit expressions of symmetric and antipersymmetric solutions of a matrix equation $AX = B$. The maximal and minimal ranks of the solutions are then derived. Finally, the matrix closest to a given matrix in the Frobenius norm is given explicitly in the minimal rank solution set of the matrix equation $AX = B$.

1. Introduction. Let $\mathbb{C}^{m\times n}$, $\mathbb{R}^{m\times n}$, $\mathbb{OR}^{n\times n}$, $\mathbb{SR}^{n\times n}$, $\mathbb{ASR}^{n\times n}$, and $\mathbb{SOR}^{m\times n}$ be the sets of all $m\times n$ complex matrices, all $m\times n$ real matrices, all $n\times n$ real orthogonal matrices, all $n\times n$ real symmetric matrices, all $n\times n$ real antisymmetric matrices, and all $n\times n$ real symmetric orthogonal matrices, respectively.

The symbols A^T, A^*, $r(A)$, and $\operatorname{tr}(A)$ denote the transpose, conjugate transpose, rank, and trace of the matrix $A\in\mathbb{R}^{m\times n}$, respectively. I_n represents the identity matrix of order n. $[A,B]$ denotes a row block matrix.

Given two matrices $A,B\in\mathbb{R}^{m\times n}$, their inner product is defined by $\langle A,B\rangle=\operatorname{tr}(B^TA)$. In this way $\mathbb{R}^{m\times n}$ is a complete inner product space. Typically, the matrix norm considered in this study is the *Frobenius norm*, i.e., $\|A\|=\sqrt{\operatorname{tr}(A^TA))}$.

A space $V\subseteq\mathbb{R}^{m\times n}$ is the orthogonal direct sum of subspaces $V_1,V_2\subseteq\mathbb{R}^{m\times n}$, written $V=V_1\oplus V_2$, if (1) for each $A\in V$ there are $A_1\in V_1$ and $A_2\in V_2$ such that $A=A_1+A_2$, (2) $V_1\cap V_2=0$, and (3) for all $A\in V_1$ and $B\in V_2$, $\langle A,B\rangle=0$.

The *Moore–Penrose generalized inverse* $A^\dagger$ of the matrix $A\in\mathbb{R}^{m\times n}$ is defined as the unique solution $X\in\mathbb{R}^{m\times n}$ of the following system of four matrix equations:

$$AXA=A,\quad XAX=X,\quad (AX)^*=AX,\quad (XA)^*=XA.$$

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Furthermore, Q_A and P_A are the orthogonal projectors $Q_A = I_m - AA^\dagger$ and $P_A = I_n - A^\dagger A$. Let

$$S_n = (e_n\, e_{n-1}\, \cdots\, e_1),$$

where e_i is the ith column of the identity matrix I_n.

DEFINITION 1.1. An $n \times n$ real matrix A is *symmetric* if $A = A^T$, and *antipersymmetric* if $A = -S_n A S_n$.

DEFINITION 1.2. An $n \times n$ real matrix A is *bisymmetric* if $A = A^T$ and $A = S_n A S_n$.

The set of all $n \times n$ symmetric and antipersymmetric matrices and the set of all $n \times n$ bisymmetric matrices are denoted by $\mathbb{SASR}^{n\times n}$ and $\mathbb{BSR}^{n\times n}$, respectively. Bisymmetric, symmetric and antipersymmetric matrices are useful for addressing engineering problems and have been thoroughly studied (see [9, 13, 20–22, 27–30]).

Linear matrix equations have been extensively studied. For instance, Deng, Bai and Gao [3] investigated consistency conditions and general expressions for Hermitian solutions of the equations

$$AXB = C \tag{1.1}$$

and

$$AX = C, \qquad XB = D. \tag{1.2}$$

Necessary and sufficient conditions for the existence of solutions of (1.2) have been established (see e.g. [12]).

By applying the generalized singular value decomposition (GSVD), Liao and Bai [10, 11] derived general analytic formulas for the least square solution of (1.1) in terms of the symmetric semidefinite solution of the matrix equation

$$A^T X A = B. \tag{1.3}$$

As a special case of (1.1)–(1.3), the well-known linear matrix equation

$$AX = B \tag{1.4}$$

has also been examined (see [4–8, 18, 19, 24–26, 31]). For example, Wang [21] investigated bisymmetric solutions of the quaternion matrix equation in the form (1.4). Zhao et al. [32] obtained (1.4) while searching for bisymmetric least square solutions under a central principal submatrix constraint.

Solutions with extremal ranks, i.e., with maximal and minimal ranks of certain matrix expressions, can be applied in statistics, control theory, and economics (see [1, 2, 14, 16]).

Motivated by the aforementioned work, we aim to derive symmetric and antipersymmetric maximal and minimal rank solutions to (1.4).

Moreover, we consider the matrix nearness problem

$$\min_{X\in\mathcal{S}_{\min}} \|X-\tilde{X}\|, \tag{1.5}$$

where $\tilde{X}$ is a given matrix in $\mathbb{R}^{m\times n}$ and $\mathcal{S}_{\min}$ is the minimal rank solution set of (1.4).

The matrix nearness problem (1.5) is the so-called optimal approximation problem, which has significant practical applicability. This problem has been widely discussed (see e.g. [4, 5, 8, 24, 26, 30, 31] and references therein).

The paper is organized as follows. In Section 2, we determine the maximal and minimal ranks of general solutions to (1.2) through the matrix rank method. In Section 3, we investigate the maximal and minimal ranks of symmetric and antipersymmetric solutions to (1.4) via SVD and special decompositions of symmetric and antipersymmetric matrices. In Section 4, we prove that $\mathcal{S}_{\min}$ is not just a closed convex cone, but in fact an affine space, and we give an expression for the solution of the matrix nearness problem (1.5).

2. Extremal rank solutions to (1.2). To derive the maximal and minimal ranks of general solutions to (1.2), we will use the following lemmas.

LEMMA 2.1 ([15]). *Given $A\in\mathbb{R}^{p\times m}$, $B\in\mathbb{R}^{n\times q}$, $C\in\mathbb{R}^{p\times n}$, $D\in\mathbb{R}^{m\times q}$, with $r(A)=k$ and $r(B)=t$, we can express the SVDs of A and B as follows:*

$$A=U\begin{bmatrix}\Sigma & 0\\ 0 & 0\end{bmatrix}V^T=U_1\Sigma V_1^T, \tag{2.1}$$

where $U=[U_1,U_2]\in\mathbb{OR}^{p\times p}$, $U_1\in\mathbb{R}^{p\times k}$, $V=[V_1,V_2]\in\mathbb{OR}^{m\times m}$, $V_1\in\mathbb{R}^{m\times k}$, $\Sigma=\operatorname{diag}(\sigma_1,\dots,\sigma_k)$, $\sigma_1\ge\dots\ge\sigma_k>0$; and

$$B=P\begin{bmatrix}\Gamma & 0\\ 0 & 0\end{bmatrix}Q^T=P_1\Gamma Q_1^T, \tag{2.2}$$

where $P=[P_1,P_2]\in\mathbb{OR}^{n\times n}$, $P_1\in\mathbb{R}^{n\times t}$, $Q=[Q_1,Q_2]\in\mathbb{OR}^{q\times q}$, $Q_1\in\mathbb{R}^{q\times t}$, and $\Gamma=\operatorname{diag}(\gamma_1,\dots,\gamma_t)$, $\gamma_1\ge\dots\ge\gamma_t>0$. Then a solution X of the matrix equation (1.2) exists if and only if

$$CB=AD,\qquad AA^\dagger C=C,\qquad DB^\dagger B=D. \tag{2.3}$$

In this case, the general solution can be expressed as

$$X=DB^\dagger+A^\dagger C-A^\dagger ADB^\dagger+P_AZQ_B \tag{2.4}$$

$$=V\begin{bmatrix}\Sigma^{-1}U_1^TCP_1 & \Sigma^{-1}U_1^TCP_2\\ V_2^TDQ_1\Gamma^{-1} & X_{22}\end{bmatrix}P^T, \tag{2.5}$$

where $Z\in\mathbb{R}^{m\times n}$ and $X_{22}\in\mathbb{R}^{(m-k)\times(n-t)}$ are arbitrary.

Lemma 2.2 ([23]). *Assume that* $K \in \mathbb{R}^{m\times n}$ *and* $Y \in \mathbb{R}^{p\times m}$ *are of full column rank. In addition, assume that* $Z \in \mathbb{R}^{n\times q}$ *is of full row rank. Then*

$$r(K) = r(YK) = r(KZ) = r(YKZ).$$

Lemma 2.3 ([23]). *Suppose that* $L \in \mathbb{R}^{n\times n}$ *is a projection matrix, i.e.,* $L^2 = L$. *Then*

$$r(L) = \mathrm{tr}(L).$$

Lemma 2.4 ([23]). *Let* $M \in \mathbb{R}^{m\times n}$. *Then*

$$r(MM^\dagger) = r(M).$$

Lemma 2.5 ([17]). *Let* $E \in \mathbb{R}^{m\times n}$, $F \in \mathbb{R}^{m\times k}$, $I \in \mathbb{R}^{l\times n}$ *and* $J \in \mathbb{R}^{l\times k}$. *Then*

$$r[E, F] = r(E) + r(Q_E F),$$

$$r\begin{bmatrix} E \\ I \end{bmatrix} = r(E) + r(IP_E),$$

$$r\begin{bmatrix} E & F \\ I & J \end{bmatrix} = r\begin{bmatrix} E \\ I \end{bmatrix} + r[E, F] - r(E) + r[Q_{I_1}(J - IE^\dagger F)P_{F_1}],$$

where $I_1 = IP_E$ *and* $F_1 = Q_E F$.

We suppose that a solution of (1.2) exists and that the general solution can be written as

$$X = V\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} P^T, \tag{2.6}$$

where

$$X_{11} = \varSigma^{-1} U_1^T C P_1, \qquad X_{12} = \varSigma^{-1} U_1^T C P_2, \qquad X_{21} = V_2^T D Q_1 \varGamma^{-1},$$

and $X_{22} \in \mathbb{R}^{(m-k)\times(n-t)}$ is arbitrary.

Let $G_1 = X_{21} P_{X_{11}}$ and $H_1 = Q_{X_{11}} X_{12}$. We assume that the SVDs of G_1 and $H_1^\dagger$ are

$$G_1 = U_{G_1}\begin{bmatrix} \varSigma_1 & 0 \\ 0 & 0 \end{bmatrix} V_{G_1}^T = U_{11}\varSigma_1 V_{11}^T, \tag{2.7}$$

where $U_{G_1} = [U_{11}, U_{12}] \in \mathbb{OR}^{(m-k)\times(m-k)}$, $U_{11} \in \mathbb{R}^{(m-k)\times k_1}$, $V_{G_1} = [V_{11}, V_{12}] \in \mathbb{OR}^{k\times k}$, $V_{11} \in \mathbb{R}^{k\times k_1}$, $k_1 = r(G_1)$, $\varSigma_1 = \mathrm{diag}(\alpha_1, \ldots, \alpha_{k_1})$ and $\alpha_1 \geq \cdots \geq \alpha_{k_1} > 0$; and

$$H_1^\dagger = U_{H_1^\dagger}\begin{bmatrix} \varGamma_1 & 0 \\ 0 & 0 \end{bmatrix} V_{H_1^\dagger}^T = P_{11}\varGamma_1 Q_{11}^T, \tag{2.8}$$

where $U_{H_1^\dagger} = [P_{11}, P_{12}] \in \mathbb{OR}^{(n-t)\times(n-t)}$, $P_{11} \in \mathbb{R}^{(n-t)\times t_1}$, $V_{H_1^\dagger} = [Q_{11}, Q_{12}] \in \mathbb{OR}^{k\times k}$, $Q_{11} \in \mathbb{R}^{k\times t_1}$, $t_1 = r(H_1^\dagger)$, $\varGamma_1 = \operatorname{diag}(\beta_1, \dots, \beta_{t_1})$ and $\beta_1 \geq \cdots \geq \beta_{t_1} > 0$.

Subsequently, we apply Lemmas 2.1–2.5 to derive extremal rank solutions to (1.2).

THEOREM 2.6. *Let* $A \in \mathbb{R}^{p\times m}$, $B \in \mathbb{R}^{n\times q}$, $C \in \mathbb{R}^{p\times n}$ *and* $D \in \mathbb{R}^{m\times q}$. *Moreover, assume that* (1.2) *is consistent. Suppose the SVDs of the matrices* A, B, G_1, *and* $H_1^\dagger$ *are given by* (2.1), (2.2), (2.7), *and* (2.8), *respectively. Let* $\varOmega$ *be the set of all solutions of* (1.2). *Then the extreme ranks of* X *are determined as follows:*

(a) *The minimal rank of* X *is*

$$\min_{X\in\varOmega} r(X) = r(C) + r(D) - r(CB). \tag{2.9}$$

The general expression of X *that satisfies* (2.9) *is*

$$\begin{aligned} X = {} & DB^\dagger + A^\dagger C - A^\dagger ADB^\dagger + P_A DB^\dagger (A^\dagger CBB^\dagger)^\dagger A^\dagger CQ_B \\ & + V_2 U_{11} U_{11}^T \tilde{Y} P_{11} P_{11}^T P_2^T, \end{aligned} \tag{2.10}$$

where $\tilde{Y} \in \mathbb{R}^{(m-k)\times(n-t)}$ *is arbitrary.*

(b) *The maximal rank of* X *is determined by*

$$\max_{X\in\varOmega} r(X) = \min\bigl(m + r(C) - r(A), n + r(D) - r(B)\bigr). \tag{2.11}$$

The general expression of X *that satisfies* (2.11) *is*

$$X = DB^\dagger + A^\dagger C - A^\dagger ADB^\dagger + P_A DB^\dagger (A^\dagger CBB^\dagger)^\dagger A^+ CQ_B + V_2 Y P_2^T,$$

where $Y \in \mathbb{R}^{(m-k)\times(n-t)}$ *satisfies* $r(Q_{G_1} Y P_{H_1}) = r(CB) + \min(m - r(A) - r(D), n - r(C) - r(B))$.

Proof. Suppose (1.2) has a solution X. By applying Lemma 2.5 to (2.6), we obtain

$$\begin{aligned} r(X) &= r\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \\ &= r\begin{bmatrix} X_{11} \\ X_{21} \end{bmatrix} + r[X_{11}, X_{12}] - r(X_{11}) + r[Q_{G_1}(X_{22} - X_{21} X_{11}^\dagger X_{12}) P_{H_1}], \end{aligned}$$

where $G_1 = X_{21} P_{X_{11}}$ and $H_1 = Q_{X_{11}} X_{12}$.

Let

$$r\begin{bmatrix} X_{11} \\ X_{21} \end{bmatrix} + r\,[X_{11}, X_{12}] - r(X_{11}) = s. \tag{2.12}$$

Then

$$\min_X r(X) = s + \min_{X_{22}} r[Q_{G_1}(X_{22} - X_{21}X_{11}^{\dagger}X_{12})P_{H_1}] = s, \tag{2.13}$$

$$\begin{aligned}\max_X r(X) &= s + \max_{X_{22}} r[Q_{G_1}(X_{22} - X_{21}X_{11}^{\dagger}X_{12})P_{H_1}] \\ &= s + \min(r(Q_{G_1}), r(P_{H_1})).\end{aligned} \tag{2.14}$$

Applying Lemmas 2.2–2.4, we get

$$\begin{aligned}r\begin{bmatrix}X_{11}\\X_{21}\end{bmatrix} &= r\begin{bmatrix}V_1^T DQ_1\Gamma^{-1}\\V_2^T DQ_1\Gamma^{-1}\end{bmatrix} = r(V^T DQ_1\Gamma^{-1}) \\ &= r(DQ_1) = r(DQ_1Q_1^T) = r(DB^{\dagger}B) = r(D), \\ r[X_{11}, X_{12}] &= r[\Sigma^{-1}U_1^T CP_1, \Sigma^{-1}U_1^T CP_2] = r(\Sigma^{-1}U_1^T C[P_1, P_2]) \\ &= r(\Sigma^{-1}U_1^T C) = r(U_1U_1^T C) = r(AA^{\dagger}C) = r(C), \\ r(X_{11}) &= r(\Sigma^{-1}U_1^T CP_1) = r(U_1U_1^T CP_1\Gamma Q_1^T) = r(AA^{\dagger}CB) = r(CB),\end{aligned}$$

and

$$\begin{aligned}r(Q_{G_1}) &= \operatorname{tr}(Q_{G_1}) = m - k - r(G_1G_1^{\dagger}) = m - k - r(G_1) \\ &= m - k - r(X_{21}P_{X_{11}}) = m - k - r\begin{bmatrix}X_{11}\\X_{21}\end{bmatrix} + r(X_{11}) \\ &= m - k - r(D) + r(CB) = m + r(CB) - r(A) - r(D), \\ r(P_{H_1}) &= \operatorname{tr}(P_{H_1}) = n - t - r(H_1^{\dagger}H_1) = n - t - r(H_1) \\ &= n - t - r(Q_{X_{11}}X_{12}) = n - t - r\,[X_{11}, X_{12}] + r(X_{11}) \\ &= n - t - r(C) + r(CB) = n + r(CB) - r(C) - r(B).\end{aligned}$$

The substitution of these formulas into (2.12)–(2.14) yields (2.9) and (2.11).

(a) The matrix X_{22}, which satisfies (2.9), is the precise general solution of the consistent linear matrix equation $Q_{G_1}(X_{22} - X_{21}X_{11}^{\dagger}X_{12})P_{H_1} = 0$, which can be written as

$$X_{22} = X_{21}X_{11}^{\dagger}X_{12} + Y,$$

where $Y \in \mathbb{R}^{(m-k)\times(n-t)}$ satisfies $Q_{G_1}YP_{H_1} = 0$.

By Lemma 2.1, the general expression of X that satisfies (2.9) can be written as

$$X = DB^{\dagger} + A^{\dagger}C - A^{\dagger}ADB^{\dagger} + V_2X_{21}X_{11}^{\dagger}X_{12}P_2^T + V_2YP_2^T, \tag{2.15}$$

where $Y \in \mathbb{R}^{(m-k)\times(n-t)}$ satisfies $Q_{G_1}YP_{H_1} = 0$.

From (2.1), (2.2) and (2.6), we get $A^\dagger = V_1\Sigma^{-1}U_1^T$, $BB^\dagger = P_1P_1^T$, $X_{11} = \Sigma^{-1}U_1^TCP_1$, $X_{12} = \Sigma^{-1}U_1^TCP_2$, $X_{21} = V_2^TDQ_1\Gamma^{-1}$. Then

$$\begin{aligned} V_1X_{11}P_1^T = A^\dagger CBB^\dagger \ &\Leftrightarrow\ P_1X_{11}^\dagger V_1^T = (A^\dagger CBB^\dagger)^\dagger \\ &\Leftrightarrow\ X_{11}^\dagger = P_1^T(A^\dagger CBB^\dagger)^\dagger V_1. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} V_2X_{21}X_{11}^\dagger X_{12}P_2^T &= V_2V_2^TDQ_1\Gamma^{-1}P_1^T(A^\dagger CBB^\dagger)^\dagger V_1\Sigma^{-1}U_1^TCP_2P_2^T \\ &= (I - AA^\dagger)DB^\dagger(A^\dagger CBB^\dagger)^\dagger A^\dagger C(I - BB^\dagger) \\ &= P_ADB^+(A^\dagger CBB^\dagger)^\dagger A^\dagger CQ_B. \end{aligned} \tag{2.16}$$

According to (2.7) and (2.8),

$$\begin{aligned} G_1G_1^\dagger &= U_{11}U_{11}^T, &\quad Q_{G_1} &= I - U_{11}U_{11}^T = U_{12}U_{12}^T, \\ H_1^\dagger H_1 &= P_{11}P_{11}^T, &\quad P_{H_1} &= I - P_{11}P_{11}^T = P_{12}P_{12}^T. \end{aligned}$$

Thus $Q_{G_1}YP_{H_1} = 0$, i.e. $U_{12}U_{12}^TYP_{12}P_{12}^T = 0$, and we obtain

$$Y = U_{11}U_{11}^T\tilde{Y}P_{11}P_{11}^T, \tag{2.17}$$

where $\tilde{Y} \in \mathbb{R}^{(m-k)\times(n-t)}$ is arbitrary.

The substitution of (2.16) and (2.17) into (2.15) yields (2.10).

(b) As in (a), based on Lemma 2.1, the general expression of X that satisfies (2.11) can be written as follows:

$$X = DB^\dagger + A^\dagger C - A^\dagger ADB^\dagger + P_ADB^\dagger(A^\dagger CBB^\dagger)^\dagger A^+CQ_B + V_2YP_2^T,$$

where $Y \in \mathbb{R}^{(m-k)\times(n-t)}$ satisfies $r(Q_{G_1}YP_{H_1}) = r(CB) + \min(m - r(A) - r(D), n - r(C) - r(B))$. The proof is therefore complete. ■

3. Symmetric and antipersymmetric extremal rank solutions to (1.4). In this section, we first derive formulas for the maximal and minimal ranks of symmetric and antipersymmetric solutions of (1.4). Subsequently, we give expressions for symmetric and antipersymmetric maximal and minimal rank solutions to (1.4).

Let $k = [n/2]$, where $[x]$ is the maximum integer not greater than x. Define

$$\begin{aligned} D_n &= \frac{1}{\sqrt{2}}\begin{bmatrix} I_k & I_k \\ S_k & -S_k \end{bmatrix} &\quad &\text{if } n = 2k, \\ D_n &= \frac{1}{\sqrt{2}}\begin{bmatrix} I_k & 0 & I_k \\ 0 & \sqrt{2} & 0 \\ S_k & 0 & -S_k \end{bmatrix} &\quad &\text{if } n = 2k+1. \end{aligned} \tag{3.1}$$

It is easy to verify that the matrices D_n are orthogonal.

LEMMA 3.1 ([28]). *Let* $X \in \mathbb{R}^{n\times n}$ *and let* $D = D_n$ *be given by* (3.1). *Then* $X \in \mathbb{SASR}^{n\times n}$ *if and only if*

$$X = D\begin{bmatrix} 0 & F \\ F^T & 0 \end{bmatrix} D^T \quad \textit{for some } F \in \mathbb{R}^{(n-k)\times k}. \tag{3.2}$$

Given $A, B \in \mathbb{R}^{m\times n}$, let

$$AD = [A_1, A_2], \qquad BD = [B_1, B_2], \tag{3.3}$$

where $A_1 \in \mathbb{R}^{m\times(n-k)}$, $A_2 \in \mathbb{R}^{m\times k}$, $B_1 \in \mathbb{R}^{m\times(n-k)}$, and $B_2 \in \mathbb{R}^{m\times k}$.

Let the SVDs of A_1 and A_2^T be

$$A_1 = U_{A_1}\begin{bmatrix} \Sigma_2 & 0 \\ 0 & 0 \end{bmatrix} V_{A_1}^T = \tilde{U}_1\Sigma_2\tilde{V}_1^T, \tag{3.4}$$

$$A_2^T = U_{A_2^T}\begin{bmatrix} \Gamma_2 & 0 \\ 0 & 0 \end{bmatrix} V_{A_2^T}^T = \tilde{P}_1\Gamma_2\tilde{Q}_1^T, \tag{3.5}$$

where the matrices

$$U_{A_1} = [\tilde{U}_1, \tilde{U}_2], \qquad V_{A_1} = [\tilde{V}_1, \tilde{V}_2], \qquad U_{A_2^T} = [\tilde{P}_1, \tilde{P}_2], \qquad V_{A_2^T} = [\tilde{Q}_1, \tilde{Q}_2]$$

are all orthogonal. Furthermore, the partitions are compatible with the sizes of

$$\Sigma_2 = \mathrm{diag}(\xi_1, \ldots, \xi_l) > 0, \quad \Gamma_2 = \mathrm{diag}(\omega_1, \ldots, \omega_p) > 0, \quad l = r(A_1),\ p = r(A_2).$$

Let

$$F_{11} = \Sigma_2^{-1}\tilde{U}_1^T B_2\tilde{P}_1, \qquad F_{12} = \Sigma_2^{-1}\tilde{U}_1^T B_2\tilde{P}_2, \qquad F_{21} = \tilde{V}_2^T B_1^T\tilde{Q}_1\Gamma_2^{-1},$$
$$G_2 = F_{21}P_{F_{11}}, \qquad H_2 = Q_{F_{11}}F_{12}.$$

Let the SVDs of G_2 and $H_2^\dagger$ be

$$G_2 = U_{G_2}\begin{bmatrix} \Sigma_3 & 0 \\ 0 & 0 \end{bmatrix} V_{G_2}^T = \tilde{U}_{11}\Sigma_3\tilde{V}_{11}^T, \tag{3.6}$$

$$H_2^\dagger = U_{H_2^\dagger}\begin{bmatrix} \Gamma_3 & 0 \\ 0 & 0 \end{bmatrix} V_{H_2^\dagger}^T = \tilde{P}_{11}\Gamma_3\tilde{Q}_{11}^T, \tag{3.7}$$

where the matrices

$$U_{G_2} = [\tilde{U}_{11}, \tilde{U}_{12}], \quad V_{G_2} = [\tilde{V}_{11}, \tilde{V}_{12}], \quad U_{H_2^\dagger} = [\tilde{P}_{11}, \tilde{P}_{12}], \quad V_{H_2^\dagger} = [\tilde{Q}_{11}, \tilde{Q}_{12}]$$

are all orthogonal. Moreover, the partitions are compatible with the sizes of

$$\Sigma_3 = \mathrm{diag}(\rho_1, \ldots, \rho_{l_1}) > 0, \qquad \Gamma_3 = \mathrm{diag}(\varphi_1, \ldots, \varphi_{p_1}) > 0,$$
$$l_1 = r(G_2), \qquad p_1 = r(H_2^\dagger).$$

We can then establish the existence theorem:

THEOREM 3.2. *Let $A, B \in \mathbb{R}^{m\times n}$, and let $D = D_n$ be given by* (3.1). *Let AD, BD, and the SVDs of A_1, A_2^T, G_2, and $H_2^\dagger$ follow the partitions of* (3.3)–(3.7). *Then* (1.4) *has a symmetric and antipersymmetric solution X if and only if*

$$A_1A_1^\dagger B_2 = B_2, \qquad A_2A_2^\dagger B_1 = B_1, \qquad A_1B_1^T = B_2A_2^T. \tag{3.8}$$

When these conditions are satisfied, let Ω_1 be the set of all symmetric and antipersymmetric solutions to (1.4). *The extreme ranks of X are then as follows:*

(a) *The minimal rank of X is*

$$\min_{X\in\Omega_1} r(X) = 2r(B_2) + 2r(B_1) - 2r(B_2A_2^T). \tag{3.9}$$

The general expression of X that satisfies (3.9) *is*

$$X = D\begin{bmatrix} 0 & F_0 + \tilde{V}_2\tilde{U}_{11}\tilde{U}_{11}^T\tilde{Z}\tilde{P}_{11}\tilde{P}_{11}^T\tilde{P}_2^T \\ F_0^T + \tilde{P}_2\tilde{P}_{11}\tilde{P}_{11}^T\tilde{Z}^T\tilde{U}_{11}\tilde{U}_{11}^T\tilde{V}_2^T & 0 \end{bmatrix} D^T, \tag{3.10}$$

where

$$\begin{aligned} F_0 = {} & (A_2^\dagger B_1)^T + A_1^\dagger B_2 - A_1^\dagger A_1(A_2^\dagger B_1)^T \\ & + P_{A_1}(A_2^\dagger B_1)^T(A_1^\dagger B_2(A_2^\dagger A_2)^T)^\dagger A_1^\dagger B_2Q_{A_2^T}, \end{aligned}$$

and $\tilde{Z} \in \mathbb{R}^{(n-k-l)\times(k-p)}$ is arbitrary.

(b) *The maximal rank of X is*

$$\max_{X\in\Omega_1} r(X) = \min\bigl(2n - 2k + 2r(B_2) - 2r(A_1), 2k + 2r(B_1) - 2r(A_2)\bigr). \tag{3.11}$$

The general expression of X that satisfies (3.11) *is*

$$X = D\begin{bmatrix} 0 & F_0 + \tilde{V}_2Z\tilde{P}_2^T \\ F_0^T + \tilde{P}_2Z^T\tilde{V}_2^T & 0 \end{bmatrix} D^T, \tag{3.12}$$

where

$$\begin{aligned} F_0 = {} & (A_2^\dagger B_1)^T + A_1^\dagger B_2 - A_1^\dagger A_1(A_2^\dagger B_1)^T \\ & + P_{A_1}(A_2^\dagger B_1)^T(A_1^\dagger B_2(A_2^\dagger A_2)^T)^\dagger A_1^\dagger B_2Q_{A_2^T}, \end{aligned}$$

and $Z \in \mathbb{R}^{(n-k-l)\times(k-p)}$ satisfies

$$r(Q_{G_2}ZP_{H_2}) = r(B_2A_2^T) + \min\bigl(n - k - r(A_1) - r(B_1), k - r(B_2) - r(A_2)\bigr).$$

Proof. If (1.4) has a solution X that is both symmetric and antipersymmetric, then by Lemma 3.1 there is $F \in \mathbb{R}^{(n-k)\times k}$ that satisfies

$$X = D\begin{bmatrix} 0 & F \\ F^T & 0 \end{bmatrix} D^T \quad \text{and} \quad AX = B. \tag{3.13}$$

Based on the partitions in (3.3), that is,

$$[A_1, A_2]\begin{bmatrix} 0 & F \\ F^T & 0 \end{bmatrix} = [B_1, B_2], \tag{3.14}$$

we get

$$A_1F = B_2, \qquad FA_2^T = B_1^T. \tag{3.15}$$

Therefore, (3.8) holds in view of Lemma 2.1.

According to (3.13), $r(X) = 2r(F)$. On the basis of (3.15) and Theorem 2.6, and the partitions in (3.4)–(3.7), the maximal and minimal ranks of the solution $X \in \mathbb{SASR}^{n\times n}$ to (1.4) can be written as (3.9) and (3.11).

According to the general expression of the solution in Theorem 2.6, the remaining parts of (a) and (b) are easy to verify. ∎

4. Expression of the optimal approximation solution to the set of minimal rank solutions. Let $\Omega_1 = \{X : AX = B, X = X^T, X = -S_nXS_n\}$ be the solution set and $\mathcal{S}_{\min} = \{X \in \Omega_1 : r(X) = \min\{r(X') : X' \in \Omega_1\}\}$ be the set of minimal rank solutions.

Remark 4.1. It follows from Theorems 2.6 and 3.2 that $\mathcal{S}_{\min}$ is an affine space since there are no conditions on the free matrices $\tilde{Y}$ in Theorem 2.6 and $\tilde{Z}$ in Theorem 3.2.

Lemma 4.2. *Let $G \in \mathbb{R}^{n\times m}$, and let $Q_1 \in \mathbb{R}^{n\times n}$ and $Q_2 \in \mathbb{R}^{m\times m}$ be projection matrices, i.e., $Q_1^2 = Q_1$, $Q_2^2 = Q_2$. Then the minimization problem*

$$\|G - Q_1\hat{F}Q_2\| = \min_{F\in\mathbb{R}^{n\times m}} \|G - Q_1FQ_2\| \tag{4.1}$$

has a solution $\hat{F} \in \mathbb{R}^{n\times m}$, and $\hat{F}$ can be expressed as

$$\hat{F} = G + (I_n - Q_1)E(I_m - Q_2), \tag{4.2}$$

where $E \in \mathbb{R}^{n\times m}$ is an arbitrary matrix.

Proof. Note that Q_1, $I_n - Q_1$, Q_2, and $I_m - Q_2$ are projection matrices, and $Q_1(I_n - Q_1) = 0$, $Q_2(I_m - Q_2) = 0$. When the inner product as defined in Section 1 is applied, we obtain

$$\begin{aligned}\|G - Q_1FQ_2\|^2 &= \|(I_n - Q_1)G\|^2 + \|Q_1G - Q_1FQ_2\|^2 \\ &= \|(I_n - Q_1)G\|^2 + \|Q_1G(I_m - Q_2)\|^2 + \|Q_1(G - F)Q_2\|^2.\end{aligned}$$

Hence,

$$\min_{F\in\mathbb{R}^{n\times m}} \|G - Q_1FQ_2\|$$

is equivalent to

$$\min_{F\in\mathbb{R}^{n\times m}} \|Q_1(G - F)Q_2\|.$$

The matrix $\hat{F} = G + (I_n - Q_1)E(I_m - Q_2)$ with $E \in \mathbb{R}^{n\times m}$ arbitrary is the solution to the minimization problem described above. Therefore, the solution of (4.1) can be expressed as in (4.2). ∎

LEMMA 4.3 ([4]).

(1) $\mathbb{R}^{n\times n} = \mathbb{ASR}^{n\times n} \oplus \mathbb{BSR}^{n\times n} \oplus \mathbb{SASR}^{n\times n}$.

(2) *For any* $X \in \mathbb{R}^{n\times n}$, *there exist unique* $X_0 \in \mathbb{ASR}^{n\times n}$, $X_1 \in \mathbb{BSR}^{n\times n}$ *and* $X_2 \in \mathbb{SASR}^{n\times n}$ *such that*

$$X = X_0 + X_1 + X_2$$

and $\langle X_i, X_j\rangle = 0$, $i \neq j$, $i, j = 0, 1, 2$; *here*

$$X_0 = \frac{X - X^T}{2}, \qquad X_1 = \frac{X + X^T + S_n(X + X^T)S_n}{4}, \qquad X_2 = \frac{X + X^T - S_n(X + X^T)S_n}{4}. \tag{4.3}$$

We can then determine a unique solution to the optimal approximation problem (1.5) in $\mathcal{S}_{\min}$. For a given matrix $\tilde{X} \in \mathbb{R}^{n\times n}$, unique $\tilde{X}_0 \in \mathbb{ASR}^{n\times n}$, $\tilde{X}_1 \in \mathbb{BSR}^{n\times n}$ and $\tilde{X}_2 \in \mathbb{SASR}^{n\times n}$ are obtained based on Lemma 4.3, such that $\tilde{X} = \tilde{X}_0 + \tilde{X}_1 + \tilde{X}_2$. In view of Lemma 3.1, $\tilde{X}_2 \in \mathbb{SASR}^{n\times n}$ can be expressed as

$$\tilde{X}_2 = D\begin{bmatrix} 0 & \tilde{F} \\ \tilde{F}^T & 0 \end{bmatrix} D^T. \tag{4.4}$$

THEOREM 4.4. *Assume* $\tilde{X} \in \mathbb{R}^{n\times n}$, $A, B \in \mathbb{R}^{m\times n}$, *and* A, B *satisfy the conditions of Theorem* 3.2. *Let* $D = D_n$ *be given by* (3.1). *Let* AD, BD, *and the SVDs of* A_1, A_2^T, G_2, *and* $H_2^\dagger$ *follow the partitions of* (3.3)–(3.7). *Suppose that*

$$V_{A_1}^T(\tilde{F} - F_0)U_{A_2^T} = \begin{bmatrix} \tilde{F}_{11} & \tilde{F}_{12} \\ \tilde{F}_{21} & \tilde{F}_{22} \end{bmatrix}. \tag{4.5}$$

Then the matrix nearness problem (1.5) *has a unique optimal approximate solution which can be represented as*

$$\hat{X} = D\begin{bmatrix} 0 & \hat{F} \\ \hat{F}^T & 0 \end{bmatrix} D^T, \tag{4.6}$$

where

$$\hat{F} = F_0 + \tilde{V}_2\tilde{U}_{11}\tilde{U}_{11}^T\big(\tilde{V}_2^T(\tilde{F} - F_0)\tilde{P}_2\big)\tilde{P}_{11}\tilde{P}_{11}^T\tilde{P}_2^T.$$

Proof. Because A and B satisfy the conditions of Theorem 3.2, the solution set $\mathcal{S}_{\min}$ is nonempty. From Remark 4.1 we know $\mathcal{S}_{\min}$ is an affine space. Hence the corresponding problem (1.5) has a unique optimal approximate solution (see [29]). Let X be an arbitrary solution in $\mathcal{S}_{\min}$. Then by (3.10), X can be expressed as

(4.7)
$$X = D\begin{bmatrix} 0 & F_0 + \tilde{V}_2\tilde{U}_{11}\tilde{U}_{11}^T\tilde{Z}\tilde{P}_{11}\tilde{P}_{11}^T\tilde{P}_2^T \\ F_0^T + \tilde{P}_2\tilde{P}_{11}\tilde{P}_{11}^T\tilde{Z}^T\tilde{U}_{11}\tilde{U}_{11}^T\tilde{V}_2^T & 0 \end{bmatrix} D^T,$$

where $\tilde{Z} \in \mathbb{R}^{(n-k-l)\times(k-p)}$ is arbitrary.

Applying Lemma 4.3, from (4.4), (4.5), and (4.7) as well as the orthogonal invariance of the Frobenius norm we obtain

$$\begin{aligned} \|\tilde{X} - X\|^2 &= \|\tilde{X}_0 + \tilde{X}_1 + \tilde{X}_2 - X\|^2 \\ &= \|\tilde{X}_0\|^2 + \|\tilde{X}_1\|^2 + \|\tilde{X}_2 - X\|^2 \\ &= \|\tilde{X}_0\|^2 + \|\tilde{X}_1\|^2 + 2\|\tilde{F} - F_0 - \tilde{V}_2\tilde{U}_{11}\tilde{U}_{11}^T\tilde{Z}\tilde{P}_{11}\tilde{P}_{11}^T\tilde{P}_2^T\|^2 \\ &= \|\tilde{X}_0\|^2 + \|\tilde{X}_1\|^2 + 2\left\|\tilde{F} - F_0 - V_{A_1}\begin{bmatrix} 0 & 0 \\ 0 & \tilde{U}_{11}\tilde{U}_{11}^T\tilde{Z}\tilde{P}_{11}\tilde{P}_{11}^T \end{bmatrix} U_{A_2^T}^T\right\|^2 \\ &= \|\tilde{X}_0\|^2 + \|\tilde{X}_1\|^2 + 2\|\tilde{F}_{11}\|^2 + 2\|\tilde{F}_{12}\|^2 + 2\|\tilde{F}_{21}\|^2 \\ &\quad + 2\|\tilde{F}_{22} - \tilde{U}_{11}\tilde{U}_{11}^T\tilde{Z}\tilde{P}_{11}\tilde{P}_{11}^T\|^2. \end{aligned}$$

Here $\tilde{U}_{11}\tilde{U}_{11}^T$ and $\tilde{P}_{11}\tilde{P}_{11}^T$ are projection matrices. By the discussion above, the problem $\min_{X\in\mathcal{S}_{\min}} \|\tilde{X} - X\|$ is equivalent to

$$\min_{\tilde{Z}\in\mathbb{R}^{(n-k-l)\times(k-p)}} \|\tilde{F}_{22} - \tilde{U}_{11}\tilde{U}_{11}^T\tilde{Z}\tilde{P}_{11}\tilde{P}_{11}^T\|.$$

Based on Lemma 4.2, when

(4.8)
$$\tilde{Z} = \tilde{F}_{22} + (I - \tilde{U}_{11}\tilde{U}_{11}^T)Z(I - \tilde{P}_{11}\tilde{P}_{11}^T), \qquad \forall Z \in \mathbb{R}^{(n-k-l)\times(k-p)},$$

$\|\tilde{F}_{22} - \tilde{U}_{11}\tilde{U}_{11}^T\tilde{Z}\tilde{P}_{11}\tilde{P}_{11}^T\|$ is minimized.

Substituting (4.8) into (4.7) yields (4.6), completing the proof. ■

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