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On separation of points from additive subgroups of l_p^n by linear functionals and positive definite functions

by

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Abstract. Let X be a finite-dimensional real normed space, and K a closed additive subgroup of X. Let $a \in X \setminus K$ and let $d_X(a, K)$ be the distance from a to K. We say that a linear functional $f \in X^*$ separates a from K if $d_{\mathbb{R}}(f(a), f(K)) > 0$. We say that a continuous positive definite function $\varphi : X \to \mathbb{C}$ separates a from K if φ is constant on K and $\varphi(a) \neq \varphi(0)$. We consider the following question: how well can a be separated from K by linear functionals and positive definite functions? We introduce certain quantities, denoted by $wd_X(a, K)$ and $pd_X(a, K)$, which measure the 'distance' from a to K with respect to linear functionals and positive definite functions, respectively. Then we define

$$\mathrm{wp}(X) := \sup \frac{pd_X(a, K)}{wd_X(a, K)}, \qquad \mathrm{ps}(X) := \sup \frac{d_X(a, K)}{pd_X(a, K)},$$

the suprema taken over all closed subgroups $K \subset X$ and all $a \in X \setminus K$. We give some estimates of $\mathrm{wp}(X)$ and $\mathrm{ps}(X)$, mainly for $X = l_p^n$. In particular we prove that $\mathrm{wp}(l_p^n) \asymp_n n^{\max\{1/2, 1/p\}}$ if $1 \leq p \leq \infty$, and $\mathrm{ps}(l_p^n) \asymp_n n^{1/2}$ if $2 \leq p < \infty$. The results may be treated as finite-dimensional analogs of those obtained in Banaszczyk and Stegliński (2008, Sec. 5) for diagonal operators in l_p spaces.

1. Introduction and notation.

Let X be a real normed space. Let K be an additive subgroup of X and let $a \in X \setminus K$. We say that a is *strongly separated* from K if it is separated from K in norm, i.e. $d_X(a, K) > 0$, where d_X is the metric in X.

By a *character* of X we mean a homomorphism of the additive group of X into the multiplicative group of complex numbers with modulus 1. We say that a character χ of X *separates a from K* if $\chi_{|K} \equiv 1$ and $\chi(a) \neq 1$. We say that a linear functional $f \in X^*$ *separates a from K* if $f(K)$ is a discrete subgroup of $\mathbb{R}$ and $f(a) \notin f(K)$. We say that a is *weakly separated*

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from K if it is separated from K by some continuous linear functional. This holds if and only if a is separated from K by some continuous character (there is a one-to-one correspondence between linear functionals $f \in X^*$ with $f(K) \subset \mathbb{Z}$ and continuous characters χ of X with $\chi_{|K} \equiv 1$, given by $e^{2\pi i f} = \chi$ (see e.g. [HR, (23.32)]).

A complex-valued function φ on X is called *positive definite* (p.d. for short) if $\sum_{i,j=1}^{n} \lambda_i \overline{\lambda}_j \varphi(x_i - x_j) \geq 0$ for all $n \geq 1$, all $x_1, \ldots, x_n \in X$ and all $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. We say that a p.d. function φ *separates* a *from* K if $\varphi(x) = \varphi(0)$ for each $x \in K$, and $\varphi(a) \neq \varphi(0)$. We say that a is $\mathcal{P}$*-separated* from K if it can be separated from K by some continuous p.d. function φ; we may of course assume that $\varphi(0) = 1$.

If a is weakly separated from K, then it is $\mathcal{P}$-separated, and consequently strongly separated, from K. If $\dim X = \infty$, the three separation conditions are not equivalent, as was proved in [B1] and [S]. The differences between these conditions can be described in terms of linear operators between Banach spaces; the corresponding classes of operators were investigated in [BS].

If $\dim X < \infty$ and a is strongly separated from K, then clearly it is also weakly separated, so that the three separation conditions are equivalent. Nevertheless, also in the finite-dimensional case it is 'easier' to separate a from K in norm than to separate it by a continuous p.d. function. Similarly, it is 'easier' to separate a from K by a continuous p.d. function than by a continuous character (or equivalently by a linear functional). The present paper is an attempt to give to the above vague statements some more precise meaning. To this end we need to define the distances from a to K with respect to linear functionals and p.d. functions, respectively.

From now on all normed spaces occurring are assumed to be real and finite-dimensional.

Let X be a normed space. We denote by $\mathcal{A}(X)$ the family of all closed additive subgroups of X. If $K \in \mathcal{A}(X)$, then K^* denotes the dual subgroup: $K^* := \{f \in X^* : f(K) \subset \mathbb{Z}\}$. We use the same symbol to denote the dual space X^* and the dual subgroup K^*; this should not cause confusion. We denote by $\mathcal{P}(X)$ the family of all continuous p.d. functions φ on X such that $\varphi(0) = 1$.

By an *open plank* in X we mean a set of the form $\{x : c_1 < f(x) < c_2\}$, where $0 \neq f \in X^*$ and $-\infty < c_1 < c_2 < \infty$. The *width* of such a plank is defined as the distance between its boundary hyperplanes; it is equal to $(c_2 - c_1)/\|f\|$.

Let $K \in \mathcal{A}(X)$ and $a \in X \setminus K$. The *weak distance* from a to K which we denote by $wd_X(a, K)$, is defined as half the maximum of the widths of all those open planks symmetric about a which are disjoint from K. It is easy to see that

$$wd_X(a, K) = \max_f \frac{d(f(a), \mathbb{Z})}{\|f\|}, \tag{1.1}$$

where the maximum is taken over all non-zero $f \in K^*$. It is clear that

$$wd_X(ta, tK) = t \cdot wd_X(a, K), \qquad t > 0. \tag{1.2}$$

There is apparently no natural way to define the distance from a to K with respect to p.d. functions. Loosely speaking, a can be well separated from K by $\varphi \in \mathcal{P}(X)$ if $\varphi_{|K} \equiv 1$, φ is close to 1 in some not too small neighbourhood of 0, and far from 1 at a. For the purpose of the present paper we adopt the following definition.

For $\varphi \in \mathcal{P}(X)$, we denote

$$r(\varphi) := \sup\{r > 0 : \operatorname{Re}\varphi(x) \geq 5/6 \text{ if } \|x\| \leq r\}.$$

In other words, $r(\varphi)$ is the inradius of the set $\{x : \operatorname{Re}\varphi(x) \geq 5/6\}$. The $\mathcal{P}$-*distance* from a to K is the number

$$pd_X(a, K) := 3 \sup_\varphi r(\varphi),$$

where the supremum is taken over all $\varphi \in \mathcal{P}(X)$ with $\varphi_{|K} \equiv 1$ and $\operatorname{Re}\varphi(a) \leq 1/6$. It is clear that

$$pd_X(ta, tK) = t \cdot pd_X(a, K), \qquad t > 0. \tag{1.3}$$

Remark. If we replace in the above definition p.d. functions by characters, then we obtain a number which might be called the *character distance* from a to K. More precisely, let us write

$$cd_X(a, K) := 3 \sup_\chi r(\chi),$$

where the supremum is taken over all continuous characters χ of X with $\chi_{|K} \equiv 1$ and $\operatorname{Re}\chi(a) \leq 1/6$. A standard argument shows that the character distance is equivalent to the weak distance. Direct computations show that

$$c_1 \leq \frac{cd_X(a, K)}{wd_X(a, K)} \leq c_2,$$

where $c_1 = (3/\pi)\arccos(5/6) = 0.559\ldots$ and $c_2 = 3\arccos(5/6)/\arccos(1/6) = 1.252\ldots$.

Let X be a normed space. We define

$$\operatorname{ws}(X) := \sup_{K,a} \frac{d_X(a, K)}{wd_X(a, K)}, \qquad \operatorname{wp}(X) := \sup_{K,a} \frac{pd_X(a, K)}{wd_X(a, K)},$$

$$\operatorname{ps}(X) := \sup_{K,a} \frac{d_X(a, K)}{pd_X(a, K)},$$

where each supremum is taken over all $K \in \mathcal{A}(X)$ and $a \in X \setminus K$.

As usual, l_p^n ($1 \le p \le \infty$) is the space $\mathbb{R}^n$ endowed with the norm

$$\|x\|_p = \Big(\sum_{k=1}^n |x_k|^p\Big)^{1/p}, \qquad x = (x_1, \dots, x_n).$$

The unit ball in l_p^n is denoted by B_p^n. The euclidean inner product of vectors $x, y \in l_2^n$ is denoted by xy. If $X = l_p^n$, then instead of $d_X(a, K)$, $wd_X(a, K)$ and $pd_X(a, K)$ we write $d_p(a, K)$, $wd_p(a, K)$ and $pd_p(a, K)$, respectively.

Estimates of $\mathrm{ws}(X)$ are closely related to the so-called transference theorems in the geometry of numbers (see e.g. [C, Ch. XI, §3.3]). By a *lattice* in $\mathbb{R}^n$ we mean an additive subgroup generated by n linearly independent vectors. Let $\mathcal{L}_n$ be the family of all lattices in $\mathbb{R}^n$. If $L \in \mathcal{L}_n$, then

$$L^* := \{x \in \mathbb{R}^n : xy \in \mathbb{Z} \text{ for each } y \in L\}$$

is the *dual lattice*.

Let $\mathcal{C}_n$ be the family of all symmetric convex bodies in $\mathbb{R}^n$. If $U \in \mathcal{C}_n$, then

$$U^0 := \{x \in \mathbb{R}^n : |xy| \le 1 \text{ for each } y \in U\}$$

is the *polar body*. We denote by $\|\cdot\|_U$ the norm on $\mathbb{R}^n$ induced by U, and d_U is the corresponding metric.

Let $U \in \mathcal{C}_n$. We define

$$\mathrm{kh}(U) := \sup_{L \in \mathcal{L}_n} \sup_{\substack{x \in \mathbb{R}^n \\ x \notin L}} \sup_{\substack{y \in L^* \\ xy \notin \mathbb{Z}}} \frac{d_U(x, L) \cdot \|y\|_{U^0}}{d(xy, \mathbb{Z})}.$$

It is clear that $\mathrm{kh}(U)$ is an affine invariant of U. A standard approximation argument shows that in the above definition the supremum over all lattices $L \in \mathcal{L}_n$ may be replaced by the supremum over all subgroups $L \in \mathcal{A}(\mathbb{R}^n)$ (loosely speaking, every closed additive subgroup of $\mathbb{R}^n$ is the limit of a sequence of lattices).

If $U \in \mathcal{C}_n$ and if X is the normed space $(\mathbb{R}^n, \|\cdot\|_U)$, then the dual space X^* may be identified with $(\mathbb{R}^n, \|\cdot\|_{U^0})$, and from the above definitions it follows directly that $\mathrm{ws}(X) = \mathrm{kh}(U)$.

It was proved in [B2, Cor. 3.4] that

$$\mathrm{kh}(B_p^n) \le C_p n, \qquad 1 < p < \infty,$$

where C_p depends only on p. Next, it was proved in [B3] that

$$(2\pi e)^{-1} n < \mathrm{kh}(U) \le C n (1 + \log n)$$

for any $U \in \mathcal{C}_n$, and $\mathrm{kh}(U) \le C' n (1 + \log n)^{1/2}$ if U is symmetric with respect to the coordinate hyperplanes; here C and C' are some universal constants.

Translated to the language of normed spaces, this means that

$$\mathrm{ws}(l_p^n) \le C_p n, \qquad 1 < p < \infty,$$

$$(2\pi e)^{-1} n < \mathrm{ws}(X) \le C n (1 + \log n) \tag{1.4}$$

for any n-dimensional normed space X, and $\mathrm{ws}(X) \le C' n (1 + \log n)^{1/2}$ if X has a 1-unconditional basis.

The main idea in the proofs of these inequalities is the following. Suppose L is a lattice in X. First we show that if $d_X(a, L)$ is large enough, then a can be well separated from L by a certain p.d. function $\varphi \in \mathcal{P}(X)$. Then we represent φ as an integral of characters (with respect to the corresponding purely atomic measure with support in L^*) and show that a can be well separated from L by one of these characters. This raises the following questions: 1° If a is well separated from K in norm, how well can it be separated from K by p.d. functions? 2° If a is well separated from K by p.d. functions, how well can it be separated from K by linear functionals?

The aim of the present paper is to give some estimates of $\mathrm{wp}(X)$ and $\mathrm{ps}(X)$, mainly for $X = l_p^n$. In particular we prove that

$$\mathrm{wp}(l_p^n) \asymp_n n^{\max\{1/2, 1/p\}}, \qquad 1 \le p \le \infty,$$
$$\mathrm{ps}(l_p^n) \asymp_n n^{1/2}, \qquad 2 \le p < \infty.$$

The results of the paper may be treated as finite-dimensional analogs of the results obtained in [BS, Sec. 5] for diagonal operators in l_p spaces.

We begin with several lemmas.

LEMMA 1.1. *Let X be a normed space. Then*

$$\mathrm{ws}(X) \le \mathrm{wp}(X) \cdot \mathrm{ps}(X).$$

This is a direct consequence of definitions.

LEMMA 1.2. *Let Y be a subspace of a normed space X. Then*

(i) $\mathrm{ws}(X) \ge \mathrm{ws}(Y)$, $\quad \mathrm{ws}(X) \ge \mathrm{ws}(X/Y)$,
(ii) $\mathrm{wp}(X) \ge \mathrm{wp}(X/Y)$,
(iii) $\mathrm{ps}(X) \ge \mathrm{ps}(Y)$, $\quad \mathrm{ps}(X) \ge \mathrm{ps}(X/Y)$.

Proof. Let $K \in \mathcal{A}(Y)$ and $a \in Y \setminus K$. Obviously, $d_Y(a, K) = d_X(a, K)$.

If $f \in X^*$ separates a from K, then so does $g := f_{|Y} \in Y^*$. Obviously, $\|g\| \le \|f\|$. Hence $wd_X(a, K) \le wd_Y(a, K)$ (in fact, $wd_X(a, K) = wd_Y(a, K)$ due to the Hahn–Banach theorem). Thus $\mathrm{ws}(X) \ge \mathrm{ws}(Y)$.

Next, if $\varphi \in \mathcal{P}(X)$ separates a from K, then so does $\psi := \varphi_{|Y} \in \mathcal{P}(Y)$. Obviously, $r(\psi) \ge r(\varphi)$. Hence $pd_X(a, K) \le pd_Y(a, K)$. Thus $\mathrm{ps}(X) \ge \mathrm{ps}(Y)$.

Now, let $K \in \mathcal{A}(X/Y)$ and $a \in (X/Y) \setminus K$. Let $T : X \to X/Y$ be the quotient mapping. Choose some $\widetilde{a} \in X$ with $T\widetilde{a} = a$ and let $\widetilde{K} := T^{-1}(K)$.

Then it is not hard to see that

$$d_X(\widetilde{a},\widetilde{K}) = d_{X/Y}(a,K), \qquad wd_X(\widetilde{a},\widetilde{K}) = wd_{X/Y}(a,K)$$

and $pd_X(\widetilde{a},\widetilde{K}) = pd_{X/Y}(a,K)$. This implies that

$$\mathrm{ws}(X) \geq \mathrm{ws}(X/Y), \quad \mathrm{wp}(X) \geq \mathrm{wp}(X/Y) \quad \text{and} \quad \mathrm{ps}(X) \geq \mathrm{ps}(X/Y). \ \blacksquare$$

Lemma 1.3. *Let $T : X \to Y$ be a linear operator between normed spaces, and let $K \in \mathcal{A}(X)$ and $a \in X \setminus K$. Then*

(i) $d_Y(T(a),T(K)) \leq \|T\| \cdot d_X(a,K)$,
(ii) $wd_Y(T(a),T(K)) \leq \|T\| \cdot wd_X(a,K)$,
(iii) $pd_Y(T(a),T(K)) \leq \|T\| \cdot pd_X(a,K)$.

Inequality (i) is obvious, while (ii) and (iii) are direct consequences of the corresponding definitions.

The Banach–Mazur distance between normed spaces X and Y is denoted by $d(X,Y)$.

Lemma 1.4. *Let X, Y be normed spaces. Then*

(i) $\mathrm{ws}(X) \leq \mathrm{ws}(Y) \cdot d(X,Y)$,
(ii) $\mathrm{wp}(X) \leq \mathrm{wp}(Y) \cdot d(X,Y)$,
(iii) $\mathrm{ps}(X) \leq \mathrm{ps}(Y) \cdot d(X,Y)$.

Proof. Choose a linear isomorphism $T : X \to Y$ with $\|T\| \cdot \|T^{-1}\| = d(X,Y)$. Let $K \in \mathcal{A}(X)$ and $a \in X \setminus K$. By Lemma 1.3(i) applied to the operator $T^{-1} : Y \to X$, we have

$$d_X(a,K) \leq \|T^{-1}\| \cdot d_Y(T(a),T(K)), \tag{1.5}$$

and Lemma 1.3(ii) says that

$$wd_X(a,K) \geq \|T\|^{-1} \cdot wd_Y(T(a),T(K)). \tag{1.6}$$

Similarly, by Lemma 1.3(iii) we have

$$pd_X(a,K) \leq \|T^{-1}\| \cdot pd_Y(T(a),T(K)), \tag{1.7}$$

$$pd_X(a,K) \geq \|T\|^{-1} \cdot pd_Y(T(a),T(K)). \tag{1.8}$$

Thus

$$\begin{aligned}\mathrm{ws}(X) &\stackrel{\text{def}}{=} \sup_{\substack{K\in\mathcal{A}(X)\\ a\in X\setminus K}} \frac{d_X(a,K)}{wd_X(a,K)} \stackrel{(1.5),(1.6)}{\leq} \sup_{\substack{K\in\mathcal{A}(X)\\ a\in X\setminus K}} \frac{\|T^{-1}\| \cdot d_Y(T(a),T(K))}{\|T\|^{-1} \cdot wd_Y(T(a),T(K))} \\ &= \|T\| \cdot \|T^{-1}\| \cdot \sup_{\substack{L\in\mathcal{A}(Y)\\ b\in Y\setminus L}} \frac{d_Y(b,L)}{wd_Y(b,L)} = d(X,Y) \cdot \mathrm{ws}(Y),\end{aligned}$$

which proves (i). Next,

$$\mathrm{wp}(X) \stackrel{\text{def}}{=} \sup_{\substack{K\in\mathcal{A}(X)\\ a\in X\setminus K}} \frac{pd_X(a,K)}{wd_X(a,K)} \stackrel{(1.7),(1.6)}{\le} \sup_{\substack{K\in\mathcal{A}(X)\\ a\in X\setminus K}} \frac{\|T^{-1}\|\cdot pd_Y(T(a),T(K))}{\|T\|^{-1}\cdot wd_Y(T(a),T(K))}$$

$$= \|T\|\cdot\|T^{-1}\|\cdot \sup_{\substack{L\in\mathcal{A}(Y)\\ b\in Y\setminus L}} \frac{pd_Y(b,L)}{wd_Y(b,L)} = d(X,Y)\cdot \mathrm{wp}(Y),$$

which proves (ii). Finally,

$$\mathrm{ps}(X) \stackrel{\text{def}}{=} \sup_{\substack{K\in\mathcal{A}(X)\\ a\in X\setminus K}} \frac{d_X(a,K)}{pd_X(a,K)} \stackrel{(1.5),(1.8)}{\le} \sup_{\substack{K\in\mathcal{A}(X)\\ a\in X\setminus K}} \frac{\|T^{-1}\|\cdot d_Y(T(a),T(K))}{\|T\|^{-1}\cdot pd_Y(T(a),T(K))}$$

$$= \|T\|\cdot\|T^{-1}\|\cdot \sup_{\substack{L\in\mathcal{A}(Y)\\ b\in Y\setminus L}} \frac{d_Y(b,L)}{pd_Y(b,L)} = d(X,Y)\cdot \mathrm{ps}(Y),$$

which proves (iii). ∎

We will need the following standard facts about Banach–Mazur distances between l_p^n spaces:

$$d(l_p^n, l_2^n) = n^{1/p-1/2}, \quad 1\le p\le 2, \tag{1.9}$$

$$d(l_p^n, l_1^n) = n^{1-1/p}, \quad 1\le p\le 2, \tag{1.10}$$

$$d(l_p^n, l_1^n) \le 3n^{1/2}, \quad 2< p\le \infty \tag{1.11}$$

(see e.g. [JL, pp. 43–44]).

2. Estimates of $\mathrm{wp}(l_p^n)$**.** Let μ be a Radon probability measure on $\mathbb{R}^n$. The *Fourier transform* of μ is defined by

$$\widehat{\mu}(x) = \int_{\mathbb{R}^n} e^{2\pi i xy}\, d\mu(y), \quad x\in\mathbb{R}^n.$$

LEMMA 2.1. *Let μ be a Radon probability measure on $\mathbb{R}^n$. Suppose that for some $0<\varepsilon<1/2$,*

$$\mathrm{Re}\,\widehat{\mu}(x) \ge 1-\varepsilon \quad \textit{for all } x\in B_\infty^n. \tag{2.1}$$

Then $\mu(\{y\in\mathbb{R}^n : \|y\|_2 > 1/3\}) < 2\varepsilon$.

Proof. Let $V = \{y\in\mathbb{R}^n : \|y\|_2 > 1/3\}$ and $U = \mathbb{R}^n\setminus V$. Let ν be the probability measure uniformly distributed on B_∞^n. Naturally, $\widehat{\nu}$ is real-valued and we may write

$$\vartheta := \int_{B_\infty^n} \mathrm{Re}\,\widehat{\mu}(x)\, d\nu(x) = \mathrm{Re}\int_{\mathbb{R}^n} \widehat{\mu}(x)\, d\nu(x)$$

$$= \mathrm{Re}\int_{\mathbb{R}^n} \widehat{\nu}(y)\, d\mu(y) = \int_{\mathbb{R}^n} \widehat{\nu}(y)\, d\mu(y) = \Big(\int_U + \int_V\Big)\widehat{\nu}(y)\, d\mu(y).$$

Standard calculations show that if $y = (y_1, \ldots, y_n) \in V$, then

$$\widehat{\nu}(y) = \prod_{k=1}^{n} \frac{\sin(2\pi y_k)}{2\pi y_k} < \frac{1}{2}.$$

Consequently,

$$\vartheta \le \int_U d\mu(y) + \frac{1}{2}\int_V d\mu(y) = \mu(U) + \mu(V)/2 = 1 - \mu(V)/2.$$

On the other hand, (2.1) implies that $\vartheta > 1 - \varepsilon$. Hence $\mu(V) < 2\varepsilon$. ∎

Lemma 2.2. *Let $K \in \mathcal{A}(\mathbb{R}^n)$ and $a \in \mathbb{R}^n \setminus K$. If $pd_\infty(a, K) \ge 1$, then $wd_2(a, K) \ge 1/9$.*

Proof. In view of (1.2) and (1.3) it is enough to show that $pd_\infty(a, K) > 3$ implies $wd_2(a, K) \ge 1/3$. So, assume that $pd_\infty(a, K) > 3$. Then there is some $\varphi \in \mathcal{P}(\mathbb{R}^n)$ with $\varphi_{|K} \equiv 1$ such that $\operatorname{Re}\varphi(a) \le 1/6$ and $\operatorname{Re}\varphi(x) \ge 5/6$ for $x \in B_\infty^n$. According to the Bochner theorem, there is a (unique) Radon probability measure μ on $\mathbb{R}^n$ with $\widehat{\mu} = \varphi$. Since $\varphi_{|K} \equiv 1$, the measure μ is concentrated on the dual subgroup

$$K^* := \{x \in \mathbb{R}^n : xy \in \mathbb{Z} \text{ for each } y \in K\},$$

i.e. $\mu(\mathbb{R}^n \setminus K^*) = 0$. Let $U = \{y \in \mathbb{R}^n : \|y\|_2 \le 1/3\}$ and $V = \mathbb{R}^n \setminus U$. Setting $\varepsilon = 1/6$ in Lemma 2.1, we get $\mu(V) < 1/3$, so that $\mu(K^* \cap U) = \mu(U) > 2/3$.

If $\cos(2\pi ay) > 3/4$ for all $y \in K^* \cap U$, then

$$\begin{aligned}\frac{1}{6} \ge \operatorname{Re}\varphi(a) &= \int_{K^*} \cos(2\pi ay)\, d\mu(y) = \Big(\int_{K^*\cap U} + \int_{K^*\cap V}\Big) \cos(2\pi ay)\, d\mu(y) \\ &> \frac{3}{4}\mu(K^* \cap U) - \mu(K^* \cap V) > \frac{3}{4}\cdot\frac{2}{3} - \frac{1}{3} = \frac{1}{6},\end{aligned}$$

which is impossible. So, there must be some $y_0 \in K^* \cap U$ with $\cos(2\pi a y_0) \le 3/4$. Then $d(ay_0, \mathbb{Z}) \ge (2\pi)^{-1}\arccos(3/4) > 1/9$ and

$$wd_2(a, K) = \max_{\substack{y \in K^* \\ y \ne 0}} \frac{d(ay, \mathbb{Z})}{\|y\|_2} \ge \frac{d(ay_0, \mathbb{Z})}{\|y_0\|_2} > \frac{1/9}{1/3} = \frac{1}{3}.$$

The first equality above follows from (1.1) (we identify here $(l_2^n)^*$ with l_2^n in the usual way). ∎

Lemma 2.3. *One has $\operatorname{wp}(l_1^n) \ge \frac{3}{5}n$ for all $n \ge 1$.*

Proof. Fix n and consider the function $\psi : \mathbb{R} \to \mathbb{R}$ given by

$$\psi(x) = 1 - \tfrac{5}{3} n\, d(x, n^{-1}\mathbb{Z}), \qquad -\infty < x < \infty.$$

Clearly, ψ is a piecewise-linear, continuous and positive definite periodic function on $\mathbb{R}$, with period n^{-1}; direct calculations show that

$$\psi(x) = \frac{7}{12} + \frac{10}{3\pi^2} \sum_{k=0}^{\infty} \frac{\cos[(2k+1)2\pi nx]}{(2k+1)^2}, \qquad -\infty < x < \infty.$$

Thus $\psi \in \mathcal{P}(\mathbb{R})$. We have $\psi\big(\frac{1}{2}n^{-1}\big) = 1/6$ and

$$\psi(x) \geq 1 - \tfrac{5}{3}n|x|, \qquad -\infty < x < \infty. \tag{2.2}$$

Let $K = n^{-1}\mathbb{Z}^n \subset \mathbb{R}^n$ and $a = (a_1, \ldots, a_n)$, where $a_k = \frac{1}{2}n^{-1}$ for $k = 1, \ldots, n$. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be given by

$$\varphi(x) = \frac{1}{n}\sum_{k=1}^{n} \psi(x_k), \qquad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Since $\psi \in \mathcal{P}(\mathbb{R})$, it follows that $\varphi \in \mathcal{P}(\mathbb{R}^n)$. It is clear that $\varphi_{|K} \equiv 1$ and $\varphi(a) = 1/6$. By (2.2), for each $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we have

$$\varphi(x) \geq \frac{1}{n}\sum_{k=1}^{n}\big(1 - \tfrac{5}{3}n|x_k|\big) = 1 - \tfrac{5}{3}\|x\|_1.$$

So, if $\|x\|_1 \leq r := 1/10$, then $\varphi(x) \geq 5/6$. This means that $pd_1(a, K) \geq 3r = 3/10$.

Set $K^* = \{f \in (l_1^n)^* : f(K) \subset \mathbb{Z}\}$. We may identify $(l_1^n)^*$ with l_∞^n, and K^* with $n\mathbb{Z}^n$. If $f \in K^*$ and $f \neq 0$, then $\|f\| \geq n$. Thus

$$wd_1(a, K) \overset{(1.1)}{=} \max_{\substack{f \in K^* \\ f \neq 0}} \frac{d(f(a), \mathbb{Z})}{\|f\|} \leq \frac{1}{2n}.$$

Consequently,

$$\mathrm{wp}(l_1^n) \geq \frac{pd_1(a, K)}{wd_1(a, K)} \geq \frac{3/10}{1/(2n)} = \frac{3}{5}n. \ \blacksquare$$

Theorem 2.4. *For all $n \geq 1$ one has*

(i) $\frac{3}{5}n^{1/p} \leq \mathrm{wp}(l_p^n) \leq 9n^{1/p}, \quad 1 \leq p \leq 2,$
(ii) $\frac{1}{5}n^{1/2} \leq \mathrm{wp}(l_p^n) \leq 9n^{1/2}, \quad 2 < p \leq \infty.$

Proof. By Lemmas 2.3 and 1.4(ii) we have

$$\tfrac{3}{5}n \leq \mathrm{wp}(l_1^n) \leq \mathrm{wp}(l_p^n) \cdot d(l_1^n, l_p^n), \qquad 1 \leq p \leq \infty.$$

Hence

$$\mathrm{wp}(l_p^n) \geq \tfrac{3}{5}n\, d(l_1^n, l_p^n)^{-1} \overset{(1.10)}{\geq} \tfrac{3}{5}n \cdot n^{1/p-1} = \tfrac{3}{5}n^{1/p}, \qquad 1 \leq p \leq 2,$$

$$\mathrm{wp}(l_p^n) \geq \tfrac{3}{5}n\, d(l_1^n, l_p^n)^{-1} \overset{(1.11)}{\geq} \tfrac{3}{5}n \cdot \tfrac{1}{3}n^{-1/2} = \tfrac{1}{5}n^{1/2}, \qquad 2 < p \leq \infty.$$

This yields the lower estimates in (i) and (ii).

To obtain the upper estimates, suppose first that $p \geq 2$. Let $K \in \mathcal{A}(\mathbb{R}^n)$ and $a \in \mathbb{R}^n \setminus K$. We want to show that

$$\frac{pd_p(a,K)}{wd_p(a,K)} \leq 9n^{1/2}. \tag{2.3}$$

In view of (1.2) and (1.3) we may assume that $pd_p(a,K) = n^{1/p}$. Then $pd_\infty(a,K) \geq 1$, because $\|\cdot\|_\infty/\|\cdot\|_p \geq n^{-1/p}$. Hence $wd_2(a,K) \geq 1/9$ due to Lemma 2.2. Since $\|\cdot\|_p/\|\cdot\|_2 \geq n^{1/p-1/2}$, it follows that

$$wd_p(a,K) \geq n^{1/p-1/2} wd_2(a,K) \geq \tfrac{1}{9} n^{1/p-1/2},$$

which proves (2.3).

We have thus proved that $\mathrm{wp}(l_p^n) \leq 9n^{1/2}$ for $p \geq 2$, in particular for $p = 2$. Hence, by Lemma 1.4(ii), for $p < 2$ we obtain

$$\mathrm{wp}(l_p^n) \leq \mathrm{wp}(l_2^n) \cdot d(l_p^n, l_2^n) \overset{(1.9)}{\leq} 9n^{1/2} \cdot n^{1/p-1/2} = 9n^{1/p}. \ \blacksquare$$

Remark. Let X be an n-dimensional normed space. By Dvoretzky's theorem, there exists a subspace Y of X such that $m := \dim(X/Y) \geq c_0 \log n$ and $d(X/Y, l_2^m) \leq 2$, where $c_0 > 0$ is some universal constant. Hence, by Lemmas 1.2(ii), 1.4(ii) and Theorem 2.4(i), we get

$$\mathrm{wp}(X) \geq \mathrm{wp}(X/Y) \geq \tfrac{1}{2}\,\mathrm{wp}(l_2^m) \geq \tfrac{3}{10} m^{1/2} \geq \tfrac{3}{10} c_0^{1/2} (\log n)^{1/2}.$$

For some special classes of spaces the estimate $m \geq c_0 \log n$ can be much improved, which leads to the corresponding improvement of the lower estimate of $\mathrm{wp}(X)$. Conjecture: there is a universal constant $c > 0$ such that $\mathrm{wp}(X) \geq cn^{1/2}$ for any n-dimensional normed space X.

3. Estimates of $\mathrm{ps}(l_p^n)$

Theorem 3.1. *There is a universal constant $c > 0$ such that $\mathrm{ps}(X) \geq cn^{1/2}$ for any n-dimensional normed space X.*

Proof. According to the quotient of subspace theorem of Milman (see e.g. [LM, Th. 3.1.1] or [GM, Th. 5.3.1]), there are subspaces $Z \subset Y \subset X$ such that $m := \dim(Y/Z) \geq n/2$ and $d(Y/Z, l_2^m) \leq C$, where C is some universal constant. From Lemmas 1.2(iii) and 1.4(iii) we obtain

$$\mathrm{ps}(X) \geq \mathrm{ps}(Y) \geq \mathrm{ps}(Y/Z) \geq d(Y/Z, l_2^m)^{-1} \cdot \mathrm{ps}(l_2^m) \geq C^{-1}\,\mathrm{ps}(l_2^m). \tag{3.1}$$

From Theorem 2.4(i) and Lemma 1.1 it follows that

$$9m^{1/2} \cdot \mathrm{ps}(l_2^m) \geq \mathrm{wp}(l_2^m) \cdot \mathrm{ps}(l_2^m) \geq \mathrm{ws}(l_2^m).$$

Next, we have $\mathrm{ws}(l_2^m) > (2\pi e)^{-1} m$ (see (1.4)). Hence $\mathrm{ps}(l_2^m) \geq c_1 m^{1/2}$, where $c_1 = (18\pi e)^{-1}$. As $m \geq n/2$, it follows that $\mathrm{ps}(l_2^m) \geq 2^{-1/2} c_1 n^{1/2}$. From this and (3.1) we obtain $\mathrm{ps}(X) \geq 2^{-1/2} c_1 C^{-1} n^{1/2}$. $\blacksquare$

Let $K \in \mathcal{A}(\mathbb{R}^n)$. Let m be the dimension of the components of K and let λ_K be the m-dimensional Lebesgue measure on K. We define

$$\varphi_K(x) = \int\limits_K e^{-\pi\|x-y\|_2^2}\, d\lambda_K(y) \Big/ \int\limits_K e^{-\pi\|y\|_2^2}\, d\lambda_K(y), \qquad x \in \mathbb{R}^n.$$

It is not hard to see that $\varphi_K \in \mathcal{P}(\mathbb{R}^n)$ (see [B4, Lemma 4.4]). For a convex body $U \in \mathcal{C}_n$ we define

$$\beta(U) = \sup_{L\in\mathcal{L}_n} \sup_{x\in\mathbb{R}^n} \Big(\sum_{y\in(L+x)\setminus U} e^{-\pi\|y\|_2^2} \Big/ \sum_{y\in L} e^{-\pi\|y\|_2^2} \Big).$$

LEMMA 3.2. *Let $K \in \mathcal{A}(\mathbb{R}^n)$, $a \in \mathbb{R}^n$ and $U \in \mathcal{C}_n$. If $(U + a) \cap K = \emptyset$, then $\varphi_K(a) \le \beta(U)$.*

This is a direct consequence of [B2, Lemma 1.3].

LEMMA 3.3. *For all $n \ge 1$ one has*

(i) $\beta(rB_p^n) < pn\pi^{-p/2}\Gamma\big(\frac{1}{2}p\big)r^{-p}, \quad p \ge 1,\ r > 0,$

(ii) $\beta(rB_2^n) < 2(2\pi e)^{n/2}n^{-n/2}r^n e^{-\pi r^2}, \quad r \ge (n/2\pi)^{1/2},$

(iii) $\beta(rB_\infty^n) < 2ne^{-\pi r^2}, \quad r > 0.$

This was proved in [B2, Lemmas 2.8–2.10].

THEOREM 3.4. *For all $n \ge 1$ one has*

(i) $\mathrm{ps}(l_p^n) < \frac{7}{5}n^{1/p}, \quad 1 \le p \le 2,$

(ii) $\mathrm{ps}(l_p^n) \le 2p^{1/2}\, n^{1/2}, \quad 2 < p < \infty,$

(iii) $\mathrm{ps}(l_\infty^n) \le n^{1/2}(3 + \log n)^{1/2}.$

Proof. (i) Suppose first that $p = 2$. Fix $n \ge 1$. Next, take $K \in \mathcal{A}(\mathbb{R}^n)$ and $a \in \mathbb{R}^n$ such that

$$d_2(a, K) > n^{1/2}. \tag{3.2}$$

We will prove that

$$\varphi_K(a) \le 1/6. \tag{3.3}$$

Suppose first that $n = 1$. If $K = \{0\}$, then $\varphi_K(x) = e^{-\pi x^2}$ for $x \in \mathbb{R}$, so that $\varphi_K(a) = e^{-\pi a^2} \le e^{-\pi} < 1/6$. If $K \ne \{0\}$, then $K = \vartheta\mathbb{Z}$ for some $\vartheta > 0$, and

$$\varphi_K(x) = \sum_{k\in\mathbb{Z}} e^{-\pi(x+k\vartheta)^2} \Big/ \sum_{k\in\mathbb{Z}} e^{-\pi k^2\vartheta^2}, \qquad x \in \mathbb{R}.$$

As $d(a, \vartheta\mathbb{Z}) \ge 1$, we have $\vartheta \ge 2$ and it is not hard to see that

$$\varphi_K(a) < \sum_{k\in\mathbb{Z}} e^{-\pi(a+k\vartheta)^2} \le 2(e^{-\pi} + e^{-9\pi} + e^{-25\pi} + \cdots) < 1/6.$$

If $n \geq 2$, then, by Lemma 3.3(ii),

$$\beta(n^{1/2}B_2^n) < 2(2\pi e)^{n/2}\, e^{-\pi n} = 2(2\pi e^{1-2\pi})^{n/2} < 1/6,$$

and (3.3) follows from Lemma 3.2.

By [B4, Lemma 4.4(ii)] we have $\varphi_K(x) \geq e^{-\pi\|x\|_2^2}$ for all $x \in \mathbb{R}^n$. This means that $\varphi_K(x) \geq 5/6$ whenever $\|x\|_2 \leq c_0 := \big(\frac{1}{\pi}\log\frac{6}{5}\big)^{1/2}$. Hence, by (3.3),

$$pd_2(a,K) \geq 3c_0. \tag{3.4}$$

We have thus shown that (3.2) implies (3.4), for all K and a. In view of (1.3), this means that

$$\mathrm{ps}(l_2^n) \stackrel{\mathrm{def}}{=} \sup_{K,a} \frac{d_2(a,K)}{pd_2(a,K)} \leq \frac{n^{1/2}}{3c_0} < \tfrac{7}{5}n^{1/2}. \tag{3.5}$$

If $p < 2$, then from Lemma 1.4(iii) we obtain

$$\mathrm{ps}(l_p^n) \leq \mathrm{ps}(l_2^n)\cdot d(l_p^n, l_2^n) \stackrel{(3.5),(1.9)}{<} \tfrac{7}{5}n^{1/2}\cdot n^{1/p-1/2} = \tfrac{7}{5}n^{1/p}.$$

(ii) Fix $n \geq 1$ and $p > 2$. Let $r := \frac{7}{5}p^{1/2}n^{1/2}$. From Lemma 3.3(i), after easy computations based on Stirling's formula, we get

$$\beta(rB_p^n) < 1/6. \tag{3.6}$$

Take $K \in \mathcal{A}(\mathbb{R}^n)$ and $a \in \mathbb{R}^n$ such that

$$d_p(a,K) \geq r. \tag{3.7}$$

Then from (3.6) and Lemma 3.2 we get (3.3). Hence, as before, we obtain (3.4). As $\|\cdot\|_p/\|\cdot\|_2 \geq n^{1/p-1/2}$, it follows that

$$pd_p(a,K) \geq 3c_0 n^{1/p-1/2}. \tag{3.8}$$

We have thus shown that (3.7) implies (3.8), for all K and a. In view of (1.3), this yields

$$\mathrm{ps}(l_p^n) \stackrel{\mathrm{def}}{=} \sup_{K,a} \frac{d_p(a,K)}{pd_p(a,K)} \leq \frac{r}{3c_0 n^{1/p-1/2}} = \frac{7}{15c_0}\,p^{1/2}n^{1/2} < 2p^{1/2}n^{1/2}.$$

The proof of (iii) is analogous, only Lemma 3.3(i) should be replaced by Lemma 3.3(iii). ∎

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