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Asymptotic structure and coarse Lipschitz geometry of Banach spaces

by

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Abstract. We study the coarse Lipschitz geometry of Banach spaces with several asymptotic properties. Specifically, we look at asymptotic uniform smoothness and convexity, and several distinct Banach–Saks-like properties. We characterize the Banach spaces which are either coarsely or uniformly homeomorphic to $T^{p_1} \oplus \cdots \oplus T^{p_n}$, where each T^{p_j} denotes the p_j-convexification of the Tsirelson space, for $p_1, \ldots, p_n \in (1, \ldots, \infty)$ and $2 \notin \{p_1, \ldots, p_n\}$. We obtain applications to the coarse Lipschitz geometry of the p-convexifications of the Schlumprecht space, and some hereditarily indecomposable Banach spaces. We also obtain some new results in the linear theory of Banach spaces.

1. Introduction. In this paper, we study nonlinear embeddings and nonlinear equivalences between Banach spaces. For that, we look at a Banach space $(X, \|\cdot\|)$ as a metric space endowed with the metric $\|\cdot - \cdot\|$. Let (M, d) and (N, ∂) be metric spaces, and $f : M \to N$ be a map. For each $t \in [0, \infty)$, we define the *expansion modulus* of f as

$$\omega_f(t) = \sup\{\partial(f(x), f(y)) \mid d(x, y) \leq t\}$$

and the *compression modulus* of f as

$$\rho_f(t) = \inf\{\partial(f(x), f(y)) \mid d(x, y) \geq t\}.$$

We say that f is a *coarse map* if $\omega_f(t) < \infty$ for all $t \in [0, \infty)$. If, in addition, $\lim_{t\to\infty} \rho_f(t) = \infty$, then f is a *coarse embedding.* We say that f is a *coarse equivalence* if f is both a coarse embedding and *cobounded*, i.e., $\sup_{y\in N} \partial(y, f(M)) < \infty$. The map f is a *uniform embedding* if $\lim_{t\to 0_+} \omega_f(t) = 0$ and $\rho_f(t) > 0$ for all $t \in (0, \infty)$. A surjective uniform embedding is called a *uniform homeomorphism.* If there exists $L > 0$ such that $\omega_f(t) \leq Lt + L$ for all $t \in [0, \infty)$, then we call f a *coarse Lipschitz map.* If, in addition, $\rho_f(t) \geq L^{-1}t - L$ for all $t \in [0, \infty)$, then f is a *coarse Lipschitz embedding.*

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A uniformly continuous map $f : X \to N$ from a Banach space X to a metric space N is automatically a coarse map (again see [K, Lemma 1.4]). Similarly, $f : X \to M$ is a coarse map if and only if it is a coarse Lipschitz map (see [K, Lemma 1.4]). Also, if two Banach spaces X and Y are coarsely equivalent (resp. uniformly homeomorphic) then X coarse Lipschitz embeds into Y (see [K, Proposition 1.5]).

In these notes, we are mainly interested in what kind of stability properties those notions of nonlinear embeddings and nonlinear equivalences may have, and we will mainly work with Banach spaces having some kind of asymptotic property. More specifically, we are concerned with asymptotically uniformly smooth Banach spaces, asymptotically uniformly convex Banach spaces, and Banach spaces having several different Banach–Saks-like properties (we refer to Section 2 for precise definitions).

The following general question is a central problem when dealing with nonlinear embeddings between Banach spaces.

PROBLEM 1.1. *Let $\mathcal{P}$ and $\mathcal{P}'$ be classes of Banach spaces and $\mathcal{E}$ be a kind of nonlinear embedding between Banach spaces. If a Banach space X $\mathcal{E}$-embeds into a Banach space Y in $\mathcal{P}$, does it follow that X is in $\mathcal{P}'$?*

For example, if a separable Banach space X coarse Lipschitz embeds into a super-reflexive Banach space, then X is also super-reflexive (this follows from [K, Proposition 1.6 and Theorem 2.4] but was first proved for uniform equivalences in [Ri, Theorem 1A]). Another example was given by M. Mendel and A. Naor [MN, Theorem 1.11], where they showed that if a Banach space X coarsely embeds into a Banach space Y with cotype q and nontrivial type, then X has cotype $q + \varepsilon$ for all $\varepsilon > 0$.

If we look at nonlinear equivalences between Banach spaces, the following is a central problem in the theory.

PROBLEM 1.2. *Let X be a Banach space and $\mathcal{E}$ be a kind of nonlinear equivalence between Banach spaces. If a Banach space Y is $\mathcal{E}$-equivalent to X, what can we say about the isomorphism type of Y? More precisely:*

(i) *Is the linear structure of X determined by its $\mathcal{E}$-structure, i.e., if a Banach space Y is $\mathcal{E}$-equivalent to X, does it follow that Y is linearly isomorphic to X?*
(ii) *Let $\mathcal{P}$ be a class of Banach spaces. If Y is $\mathcal{E}$-equivalent to X, does is follow that Y is linearly isomorphic to $X \oplus Z$ for some Banach space Z in $\mathcal{P}$?*

Along those lines, it was shown in [JLS, Theorem 2.1] that the coarse (resp. uniform) structure of ℓ_p completely determines its linear structure for any $p \in (1, \infty)$. For $p = 1$, we do not even know if the Lipschitz structure of ℓ_1 determines its linear structure. N. Kalton and N. Randrianarivony [KR, Theorem 5.4] proved that, for any $p_1, \dots, p_n \in (1,\infty)$ with $2 \notin \{p_1, \dots, p_n\}$, the linear structure of $\ell_{p_1} \oplus \cdots \oplus \ell_{p_n}$ is determined by its coarse (resp. uniform) structure (see also [JLS, Theorem 2.2]). This problem is still open if $2 \in \{p_1, \dots, p_n\}$.

Let T denote the Tsirelson space introduced by T. Figiel and W. Johnson [FJ]. For each $p \in [1,\infty)$, let T^p be the p-convexification of T (see Subsection 2.6 for definitions). W. Johnson, J. Lindenstrauss and G. Schechtman addressed Problem 1.2(ii) above by proving the following (see [JLS, Theorem 5.8]): Suppose that either $1 < p_1 < \cdots < p_n < 2$ or $2 < p_1 < \cdots < p_n$ and set $X = T^{p_1} \oplus \cdots \oplus T^{p_n}$. Then a Banach space Y is coarsely equivalent (resp. uniformly homeomorphic) to X if and only if Y is linearly isomorphic to $X \oplus \bigoplus_{j\in F} \ell_{p_j}$ for some $F \subset \{1, \dots, n\}$.

We now describe the organization and some of the results of this paper. Firstly, in order not to make this introduction too extensive, we will postpone some technical definitions as well as our more technical results. The reader will find all the background and notation necessary for this paper in Section 2.

In relation to Problem 1.1, we prove the following in Section 3.

Theorem 1.3. *Let Y be a reflexive asymptotically uniformly smooth Banach space, and assume that a Banach space X coarse Lipschitz embeds into Y. Then X has the Banach–Saks property.*

As the Banach–Saks property implies reflexivity, Theorem 1.3 above is a strengthening of [BKL, Theorem 4.1] where the authors showed that if a separable Banach space X coarse Lipschitz embeds into a reflexive asymptotically uniformly smooth Banach space, then X must be reflexive. As T is a reflexive Banach space without the Banach–Saks property, Theorem 1.3 gives us the following new corollary.

Corollary 1.4. *The Tsirelson space does not coarse Lipschitz embed into any reflexive asymptotically uniformly smooth Banach space.*

In Section 3, we also prove some results in the linear theory of Banach spaces. Precisely, we show that an asymptotically uniformly smooth Banach space X must have the alternating Banach–Saks property (see Corollary 3.2). Using descriptive-set-theoretical arguments, we also show that the converse does not hold, i.e., that there are Banach spaces with the alternating Banach–Saks property which do not admit an asymptotically uniformly smooth renorming (see Proposition 3.8).

In Section 4, we study coarse embeddings $f : X \to Y$ between Banach spaces X and Y with specific asymptotic properties, and obtain a general result on how close to an affine map the compression modulus ρ_f can be (see Theorem 4.1). More precisely, E. Guentner and J. Kaminker [GK] introduced the following quantity: for Banach spaces X and Y, define $\alpha_Y(X)$ as the supremum of all $\alpha > 0$ for which there exists a coarse embedding $f : X \to Y$ and $L > 0$ such that $\rho_f(t) \geq L^{-1}t^\alpha - L$ for all $t \geq 0$. We call $\alpha_Y(X)$ the *compression exponent of X in Y*. As a simple consequence of Theorem 4.1, we obtain Theorem 1.5 below.

We denote by S the Schlumprecht space introduced in [Sc], and for each $p \in [1, \infty)$, we let S^p be the p-convexification of S, and T^p be the p-convexification of the Tsirelson space T (see Subsection 2.6 for definitions).

THEOREM 1.5. *Let $1 \leq p < q$. Then*

(i) $\alpha_{T^q}(T^p) \leq p/q$, *and*
(ii) $\alpha_{S^q}(S^p) \leq p/q$.

In particular, T^p (resp. S^p) does not coarse Lipschitz embed into T^q (resp. S^q).

The proof of Theorem 1.5 is asymptotical in nature, hence we obtain equivalent estimates for the compression exponent $\alpha_Y(X)$, where X and Y are Banach spaces satisfying some special asymptotic properties. In particular, the spaces T^q and S^q can be replaced in Theorem 1.5 by $(\bigoplus_n E_n)_{T^q}$ and $(\bigoplus_n E_n)_{S^q}$, where $(E_n)_{n=1}^\infty$ is any sequence of finite-dimensional Banach spaces. See Theorems 4.3 and 4.5 and Corollary 4.7 for precise statements.

We also apply our results to the hereditarily indecomposable Banach spaces $\mathfrak{X}^p$ defined by N. Dew [D], and deduce that $\alpha_{\mathfrak{X}^q}(\mathfrak{X}^p) \leq p/q$ for $1 < p < q$ (see Corollary 4.8).

In Section 5, we prove a general theorem regarding the nonexistence of coarse Lipschitz embeddings $X \to Y_1 \oplus Y_2$ for Banach spaces X, Y_1, Y_2 with specific asymptotic properties (see Theorem 5.6). With that result in hand, we prove the following.

THEOREM 1.6. *Let $1 \leq p_1 < \cdots < p_n < \infty$ and $p \in [1, \infty) \setminus \{p_1, \ldots, p_n\}$. Then neither T^p nor ℓ_p coarse Lipschitz embeds into $T^{p_1} \oplus \cdots \oplus T^{p_n}$. In particular, T^p coarse Lipschitz embeds into T^q for no $p, q \in [1, \infty)$ with $p \neq q$.*

Finally, we use Theorem 1.6 to obtain the following characterization.

THEOREM 1.7. *Let $1 < p_1 < \cdots < p_n < \infty$ with $2 \notin \{p_1, \ldots, p_n\}$. A Banach space Y is coarsely equivalent (resp. uniformly homeomorphic) to $X = T^{p_1} \oplus \cdots \oplus T^{p_n}$ if and only if Y is linearly isomorphic to $X \oplus \bigoplus_{j \in F} \ell_{p_j}$ for some $F \subset \{1, \ldots, n\}$.*

Clearly, Theorem 1.7 is a strengthening of [JLS, Theorem 5.8]. However, just as for $\ell_{p_1} \oplus \cdots \oplus \ell_{p_n}$, we do not know whether the theorem above holds if $2 \in \{p_1, \ldots, p_n\}$.

2. Notation and background

2.1. Basic definitions. All the Banach spaces in these notes are assumed to be infinite-dimensional unless otherwise stated. Let X be a Banach space. We denote the closed unit ball of X by B_X, and its unit sphere by ∂B_X. If Y is also a Banach space, we write $X \cong Y$ if X is linearly isomorphic to Y. Given a Banach space X with norm $\|\cdot\|_X$, we simply write $\|\cdot\|$ as long as it is clear from the context which space the elements inside the norm belong to. A sequence $(x_n)_{n=1}^\infty$ in a Banach space X is called *seminormalized* if it is bounded and $\inf_n \|x_n\| > 0$.

Say $(e_n)_{n=1}^\infty$ is a basis for the Banach space X. For $x = \sum_{n=1}^\infty x_n e_n \in X$, we write $\mathrm{supp}(x) = \{n \in \mathbb{N} \mid x_n \neq 0\}$. For all finite subsets $E, F \subset \mathbb{N}$, we write $E < F$ (resp. $E \leq F$) if $\max E < \min F$ (resp. $\max E \leq \min F$). We call a sequence $(y_n)_{n=1}^\infty$ in X a *block sequence of* $(e_n)_{n=1}^\infty$ if $\mathrm{supp}(y_n) < \mathrm{supp}(y_{n+1})$ for all $n \in \mathbb{N}$.

Let $(X_n)_{n=1}^\infty$ be a sequence of Banach spaces. Let $\mathcal{E} = (e_n)_{n=1}^\infty$ be a 1-unconditional basic sequence in a Banach space E with norm $\|\cdot\|_E$. We define the sum $(\bigoplus_n X_n)_{\mathcal{E}}$ to be the space of sequences $(x_n)_{n=1}^\infty$, where $x_n \in X_n$ for all $n \in \mathbb{N}$, such that

$$\|(x_n)_{n=1}^\infty\| := \Big\|\sum_{n\in\mathbb{N}} \|x_n\| e_n\Big\|_E < \infty.$$

One can check that $(\bigoplus_n X_n)_{\mathcal{E}}$ endowed with the norm $\|\cdot\|$ defined above is a Banach space. If the X_n's are all the same, say $X_n = X$ for all $n \in \mathbb{N}$, we write $(\bigoplus X)_{\mathcal{E}}$. Also, if it is implicit what is the basis $\mathcal{E}$ of the Banach space E that we are working with, we write $(\bigoplus_n X_n)_E$.

2.2. p-convex and p-concave Banach spaces. Let X be a Banach space with 1-unconditional basis $(e_n)_{n=1}^\infty$, and let $p \in (1,\infty)$. We say that the basis $(e_n)_{n=1}^\infty$ is *p-convex with convexity constant C* (resp. *p-concave with concavity constant C*) if

$$\Big\|\sum_{j\in\mathbb{N}} (|x_j^1|^p + \cdots + |x_j^k|^p)^{1/p} e_j\Big\|^p \leq C^p \sum_{n=1}^k \|x^n\|^p$$

$$\Big(\text{resp. } C^p \Big\|\sum_{j\in\mathbb{N}} (|x_j^1|^p + \cdots + |x_j^k|^p)^{1/p} e_j\Big\|^p \geq \sum_{n=1}^k \|x^n\|^p\Big)$$

for all $x^1 = \sum_{j=1}^\infty x_j^1 e_j, \ldots, x^k = \sum_{j=1}^\infty x_j^k e_j \in X$. We say that the basis $(e_n)_{n=1}^\infty$ satisfies an *upper ℓ_p-estimate with constant C* (resp. *lower ℓ_p-estimate with constant C*) if

$$\|x_1 + \cdots + x_k\|^p \leq C^p \sum_{n=1}^k \|x_n\|^p \quad \Big(\text{resp. } C^p \|x_1 + \cdots + x_k\|^p \geq \sum_{n=1}^k \|x_n\|^p\Big)$$

for all $x_1, \dots, x_k \in X$ with disjoint supports. Clearly, a p-convex (resp. p-concave) basis with constant C satisfies an upper (resp. lower) ℓ_p-estimate with constant C.

2.3. p-convexification. Let X be a Banach space with a 1-unconditional basis $(e_n)_{n=1}^\infty$. For any $p \in [1,\infty)$, we define the *p-convexification* of X as follows. Let

$$X^p = \Big\{(x_n)_{n=1}^\infty \in \mathbb{R}^{\mathbb{N}} \;\Big|\; x^p := \sum_{n\in\mathbb{N}} |x_n|^p e_n \in X\Big\},$$

and endow X^p with the norm $\|x\|_p = \|x^p\|^{1/p}$ for all $x \in X^p$. By abuse of notation, we denote by $(e_n)_{n=1}^\infty$ the sequence of coordinate vectors in X^p. It is clear that $(e_n)_{n=1}^\infty$ is a 1-unconditional basis for X^p and that $X^1 = X$. Also, the triangle inequality implies that X^p is p-convex with constant 1.

2.4. Asymptotically p-uniformly smooth and convex spaces. Let X be a Banach space. We define the *modulus of asymptotic uniform smoothness* of X as

$$\overline{\rho}_X(t) = \sup_{x\in\partial B_X} \inf_{\dim(X/E)<\infty} \sup_{h\in\partial B_E} \|x+th\| - 1.$$

We say that X is *asymptotically uniformly smooth* if $\lim_{t\to 0_+} \overline{\rho}_X(t)/t = 0$. If there exist $p \in (1,\infty)$ and $C > 0$ such that $\overline{\rho}_X(t) \le Ct^p$ for all $t \in [0,1]$, we say that X is *asymptotically p-uniformly smooth*. Every asymptotically uniformly smooth Banach space is asymptotically p-uniformly smooth for some $p \in (1,\infty)$ (this was first proved in [KOS] for separable Banach spaces, and later generalized to any Banach space in [Ra, Theorem 1.2]).

Let X be a Banach space. We define the *modulus of asymptotic uniform convexity* of X as

$$\overline{\delta}_X(t) = \inf_{x\in\partial B_X} \sup_{\dim(X/E)<\infty} \inf_{h\in\partial B_E} \|x+th\| - 1.$$

We say that X is *asymptotically uniformly convex* if $\overline{\delta}_X(t) > 0$ for all $t > 0$. If there exist $p \in (1,\infty)$ and $C > 0$ such that $\overline{\delta}_X(t) \ge Ct^p$ for all $t \in [0,1]$, we say that X is *asymptotically p-uniformly convex*.

The following proposition is straightforward.

PROPOSITION 2.1. *Let $p \in (1,\infty)$ and let X be a Banach space with a 1-unconditional basis satisfying an upper ℓ_p-estimate (resp. lower ℓ_p-estimate) with constant 1. Then X is asymptotically p-uniformly smooth (resp. asymptotically p-uniformly convex).*

2.5. Banach–Saks properties. A Banach space X is said to have the *Banach–Saks property* if every bounded sequence $(x_n)_{n=1}^\infty$ in X has a subsequence $(x_{n_j})_{j=1}^\infty$ such that $(k^{-1}\sum_{j=1}^k x_{n_j})_{k=1}^\infty$ converges. A Banach space X is said to have the *alternating Banach–Saks property* if every bounded sequence $(x_n)_{n=1}^\infty$ in X has a subsequence $(x_{n_j})_{j=1}^\infty$ such that $(k^{-1}\sum_{j=1}^k \varepsilon_j x_{n_j})_{k=1}^\infty$ converges for some $(\varepsilon_j)_{j=1}^\infty \in \{-1,1\}^{\mathbb{N}}$. For a detailed study of these properties, we refer to [Be].

Let $p \in (1,\infty)$. A Banach space X is said to have the *p-Banach–Saks property* (resp. *p-co-Banach–Saks property*) if for every seminormalized weakly null sequence $(x_n)_{n=1}^\infty$ in X there exists a subsequence $(x_{n_j})_{j=1}^\infty$ and $c > 0$ such that

$$\|x_{n_1} + \cdots + x_{n_k}\| \le ck^{1/p} \quad (\text{resp. } \|x_{n_1} + \cdots + x_{n_k}\| \ge ck^{1/p})$$

for all $k \in \mathbb{N}$ and all $k \le n_1 < \cdots < n_k$.

The following is a combination of [DGJ, Propositions 1.2, 1.3 and 1.6] ([DGJ, Proposition 1.6] only mentions the p-Banach–Saks property, but a straightforward modification of the proof gives the result for the p-co-Banach–Saks property).

PROPOSITION 2.2. *Let $p \in (1,\infty)$ and let X be a Banach space. If X is asymptotically p-uniformly smooth* (*resp. asymptotically p-uniformly convex*), *then X has the p-Banach–Saks property* (*resp. p-co-Banach–Saks property*).

2.6. Tsirelson and Schlumprecht spaces. Let c_{00} denote the set of sequences of real numbers which are eventually zero, and let $\|\cdot\|_0$ be the max norm on c_{00}. We denote by T the *Tsirelson space* defined in [FJ], i.e., T is the completion of c_{00} under the unique norm $\|\cdot\|$ satisfying

$$\|x\| = \max\bigg\{\|x\|_0, \frac{1}{2}\sup\Big(\sum_{j=1}^k \|E_j x\|\Big)\bigg\},$$

where the supremum is taken over all finite sequences $(E_j)_{j=1}^k$ of finite subsets of $\mathbb{N}$ such that $k \le E_1 < \cdots < E_k$. Therefore, for each $p \in (1,\infty)$, the norm $\|\cdot\|_p$ of the p-convexified Tsirelson space T^p satisfies

$$\|x\|_p = \max\bigg\{\|x\|_0, \frac{1}{2^{1/p}}\sup\Big(\sum_{j=1}^k \|E_j x\|_p^p\Big)^{1/p}\bigg\},$$

where the supremum is taken over all finite sequences $(E_j)_{j=1}^k$ of finite subsets of $\mathbb{N}$ such that $k \le E_1 < \cdots < E_k$ (see [CS, Chapter X, Section E]).

As T^p satisfies an upper ℓ_p-estimate with constant 1, it follows that T^p is asymptotically p-uniformly smooth and it has the p-Banach–Saks property. Also, T^p has the p-co-Banach–Saks property. Indeed, let $(e_n)_{n=1}^\infty$ be the standard basis for T^p. If $(x_n)_{n=1}^\infty$ is a normalized block subsequence of $(e_n)_{n=1}^\infty$, then

$$2^{-1/p}k^{1/p} = 2^{-1/p}\Big(\sum_{n=k}^{2k-1} \|x_n\|_p^p\Big)^{1/p} \le \Big\|\sum_{n=k}^{2k-1} x_n\Big\|_p$$

for all $k \in \mathbb{N}$. Therefore, as for any normalized weakly null sequence $(x_n)_{n=1}^\infty$ in T^p one can find a block sequence $(y_n)_{n=1}^\infty$ which is equivalent to a subsequence of $(x_n)_{n=1}^\infty$, we conclude that T^p has the p-co-Banach–Saks property.

Remark 2.3. Let $p \in (1,\infty)$. Then T^p contains ℓ_r for no $r \in [1,\infty)$ (this is shown in [J2] for T, and the result for T^p follows analogously). Similarly, by duality arguments, T^{p*} contains ℓ_r for no $r \in [1,\infty)$ (the reader can find more on T^p and similar duality arguments in [CS]).

The *Schlumprecht space* S (see [Sc]) is defined as the completion of c_{00} under the unique norm $\|\cdot\|$ satisfying

$$\|x\| = \max\Big\{\|x\|_0, \sup \frac{1}{\log_2(k+1)} \sum_{j=1}^k \|E_j x\|\Big\}$$

where the supremum is taken over all finite sequences $(E_j)_{j=1}^k$ of finite subsets of $\mathbb{N}$ such that $E_1 < \cdots < E_k$. Similarly to the p-convexified Tsirelson space, the norm $\|\cdot\|_p$ of the p-convexified Schlumprecht space S^p is given by

$$\|x\|_p = \max\Big\{\|x\|_0, \sup\Big(\frac{1}{\log_2(k+1)} \sum_{j=1}^k \|E_j x\|_p^p\Big)^{1/p}\Big\},$$

where the supremum is taken over all finite sequences $(E_j)_{j=1}^k$ of finite subsets of $\mathbb{N}$ such that $E_1 < \cdots < E_k$ (see [D, p. 59]).

Similarly to T^p, the space S^p is asymptotically p-uniformly smooth and has the p-Banach–Saks property, for $p \in (1,\infty)$.

2.7. Almost p-co-Banach–Saks property. Although T^p has the p-co-Banach–Saks property, S^p does not. However, S^p satisfies a weaker property that will be enough for our goals. Let $p \in (1,\infty)$. We say that a Banach space X has the *almost p-co-Banach–Saks property* if for every seminormalized weakly null sequence $(x_n)_{n=1}^\infty$ in X there exists a subsequence $(x_{n_j})_{j=1}^\infty$ and a sequence $(\theta_j)_{j=1}^\infty$ of positive numbers in $[1,\infty)$ such that $\lim_{j\to\infty} j^\alpha \theta_j^{-1} = \infty$ for all $\alpha > 0$, and

$$\|x_{n_1} + \cdots + x_{n_k}\| \ge k^{1/p}\theta_k^{-1}$$

for all $k \in \mathbb{N}$ and all $k \le n_1 < \cdots < n_k$. Clearly, S^p has the almost p-co-Banach–Saks property with $\theta_k = \log_2(k+1)^{1/p}$ for all $k \in \mathbb{N}$.

3. Asymptotic uniform smoothness and the alternating Banach–Saks property. In this section, we are going to show that asymptotically uniformly smooth Banach spaces must have the alternating Banach–Saks property (Corollary 3.2), but the converse does not hold (see Proposition 3.8). Also, we show that if a Banach space X coarse Lipschitz embeds into a reflexive space Y which is also asymptotically uniformly smooth, then X must have the Banach–Saks property (Theorem 1.3). As any space with the Banach–Saks property is reflexive, this is a strengthening of [BKL, Theorem 4.1], which says that, under the same hypothesis, X must be reflexive.

PROPOSITION 3.1. *Let X be a Banach space with the p-Banach–Saks property for some $p \in (1,\infty)$, and assume that X does not contain ℓ_1. Then X has the alternating Banach–Saks property. In particular, if X is also reflexive, then X has the Banach–Saks property.*

Proof. Assume X does not have the alternating Banach–Saks property. Then there exist $\delta > 0$ and a bounded sequence $(x_n)_{n=1}^\infty$ in X such that for all $k \in \mathbb{N}$, all $\varepsilon_1, \ldots, \varepsilon_k \in \{-1,1\}$, and all $n_1 < \cdots < n_k \in \mathbb{N}$, we have

$$\Big\| \frac{1}{k} \sum_{j=1}^{k} \varepsilon_j x_{n_j} \Big\| > \delta \tag{3.1}$$

(see [Be, Theorem 1, p. 369]). As X does not contain ℓ_1, by Rosenthal's ℓ_1-theorem (see [Ro]) we can assume that $(x_n)_{n=1}^\infty$ is weakly Cauchy. Hence, the sequence $(x_{2n-1} - x_{2n})_{n=1}^\infty$ is weakly null. By (3.1), it is also seminormalized. Therefore, as X has the p-Banach–Saks property, by taking a subsequence if necessary, we see that

$$\Big\| \sum_{j=1}^{k} (x_{n_{2j-1}} - x_{n_{2j}}) \Big\| \le c k^{1/p}$$

for all $k \in \mathbb{N}$ and some constant $c > 0$ independent of k. Again by (3.1), we get

$$\delta < \Big\| \frac{1}{2k} \sum_{j=1}^{2k} (-1)^{j+1} x_{n_j} \Big\| \le \frac{c}{2} k^{1/p-1}.$$

As this holds for all $k \in \mathbb{N}$, and $p > 1$, if we let $k \to \infty$, we get $\delta = 0$, which is a contradiction.

For reflexive spaces, the alternating Banach–Saks property and the Banach–Saks property are equivalent (see [Be, Proposition 2]), so the last statement of the proposition follows. ∎

COROLLARY 3.2. *Let X be an asymptotically uniformly smooth Banach space. Then X has the alternating Banach–Saks property. In particular, if X is also reflexive, then X has the Banach–Saks property.*

Proof. As X is asymptotically uniformly smooth, X cannot contain ℓ_1. Therefore, we only need to notice that X has the p-Banach–Saks property for some $p \in (1,\infty)$, and apply Proposition 3.1. By [Ra, Theorem 1.2],

X is asymptotically p-uniformly smooth for some $p \in (1,\infty)$. Therefore, by Proposition 2.2 above, X has the p-Banach–Saks property. ∎

For each $k \in \mathbb{N}$ and each infinite subset $\mathbb{M} \subset \mathbb{N}$, we define $G_k(\mathbb{M})$ as the set of all subsets of $\mathbb{M}$ with k elements. We write $\bar{n} = (n_1, \dots, n_k) \in G_k(\mathbb{M})$ always in increasing order, i.e., $n_1 < \cdots < n_k$. We define a metric $d = d_k$ on $G_k(\mathbb{M})$ by letting

$$d(\overline{n}, \overline{m}) = |\{j \mid n_j \neq m_j\}|$$

for all $\overline{n} = (n_1, \dots, n_k), \overline{m} = (m_1, \dots, m_k) \in G_k(\mathbb{M})$.

The following result will play an important role in this paper. It was proved in [KR, Theorem 4.2] (see also [KR, Theorem 6.1]).

THEOREM 3.3. *Let $p \in (1,\infty)$, and let Y be a reflexive asymptotically p-uniformly smooth Banach space. There exists $K > 0$ such that for all infinite subsets $\mathbb{M} \subset \mathbb{N}$, all $k \in \mathbb{N}$, and all bounded maps $f : G_k(\mathbb{M}) \to Y$, there exists an infinite subset $\mathbb{M}' \subset \mathbb{M}$ such that*

$$\operatorname{diam}(f(G_k(\mathbb{M}'))) \leq K \operatorname{Lip}(f) k^{1/p}.$$

Proof of Theorem 1.3. Let $f : X \to Y$ be a coarse Lipschitz embedding. Pick $C > 0$ so that $\omega_f(t) \leq Ct + C$ and $\rho_f(t) \geq C^{-1}t - C$ for all $t \geq 0$. Assume that X does not have the Banach–Saks property. By [Be, p. 373], there exists $\delta > 0$ and a sequence $(x_n)_{n=1}^{\infty}$ in B_X such that for all $k \in \mathbb{N}$ and all $n_1 < \cdots < n_{2k} \in \mathbb{N}$, we have

$$\left\| \frac{1}{2k} \sum_{j=1}^{k} (x_{n_j} - x_{n_{k+j}}) \right\| \geq \delta.$$

For each $k \in \mathbb{N}$, define $\varphi_k : G_k(\mathbb{N}) \to X$ by $\varphi_k(n_1, \dots, n_k) = x_{n_1} + \cdots + x_{n_k}$ for all $(n_1, \dots, n_k) \in G_k(\mathbb{N})$. Therefore, $\operatorname{diam}(\varphi_k(G_k(\mathbb{M}))) \geq 2k\delta$, and we have $\operatorname{diam}(f \circ \varphi_k(G_k(\mathbb{M}))) \geq 2k\delta C^{-1} - C$ for all $k \in \mathbb{N}$ and all infinite $\mathbb{M} \subset \mathbb{N}$.

As $\operatorname{Lip}(\varphi_k) \leq 2$, we have $\operatorname{Lip}(f \circ \varphi_k) \leq 3C$. As Y is asymptotically uniformly smooth, there exists $p \in (1,\infty)$ for which Y is asymptotically p-uniformly smooth (see [Ra, Theorem 1.2]). By Theorem 3.3, there exist $K = K(Y) > 0$ and $\mathbb{M} \subset \mathbb{N}$ such that $\operatorname{diam}(f \circ \varphi_k(G_k(\mathbb{M}))) \leq 3KCk^{1/p}$ for all $k \in \mathbb{N}$. We conclude that

$$2k\delta C^{-1} - C \leq 3KCk^{1/p}$$

for all $k \in \mathbb{N}$. As $p > 1$, this leads to a contradiction if we let $k \to \infty$. ∎

The following was asked in [GLZ, Problem 2], and remains open.

PROBLEM 3.4. *If a Banach space X coarse Lipschitz embeds into a reflexive asymptotically uniformly smooth Banach space Y, does it follow that X has an asymptotically uniformly smooth renorming?*

PROBLEM 3.5. *Let N be a metric space. We say that a family $(M_k)_{k=1}^{\infty}$ of metric spaces* uniformly Lipschitz embeds into N *if there exists $C > 0$ and Lipschitz embeddings $f_k : M_k \to N$ such that* $\mathrm{Lip}(f) \cdot \mathrm{Lip}(f^{-1}) < C$ *for all $k \in \mathbb{N}$. Does the family $(G_k(\mathbb{N}), d)_{k=1}^{\infty}$ uniformly Lipschitz embed into any Banach space without an asymptotically uniformly smooth renorming?*

As noticed in [GLZ, Problem 6], a positive answer to Problem 3.5 together with Theorem 3.3 would give us a positive answer to Problem 3.4.

It is worth noticing that the Banach–Saks property is not stable under uniform equivalences, hence, it is not stable under coarse Lipschitz isomorphisms either. Indeed, if $(p_n)_{n=1}^{\infty}$ is a sequence in $(1,\infty)$ converging to 1, then $(\bigoplus_n \ell_{p_n})_{\ell_2}$ is uniformly equivalent to $(\bigoplus_n \ell_{p_n})_{\ell_2} \oplus \ell_1$ (see [BL, p. 244]). The space $(\bigoplus_n \ell_{p_n})_{\ell_2}$ has the Banach–Saks property, while $(\bigoplus_n \ell_{p_n})_{\ell_2} \oplus \ell_1$ does not.

Let $\mathcal{G}(\mathbb{N})$ denote the set of finite subsets of $\mathbb{N}$. We endow $\mathcal{G}(\mathbb{N})$ with the metric D given by

$$D(\overline{n},\overline{m}) = |\overline{n} \bigtriangleup \overline{m}|,$$

for all $\overline{n} = (n_1,\dots,n_k), \overline{m} = (m_1,\dots,m_l) \in \mathcal{G}(\mathbb{N})$, where $\overline{n} \bigtriangleup \overline{m}$ denotes the symmetric difference between the sets $\overline{n}$ and $\overline{m}$.

PROPOSITION 3.6. *$\mathcal{G}(\mathbb{N})$ Lipschitz embeds into any Banach space X without the alternating Banach–Saks property. Moreover, for any $\varepsilon > 0$, the Lipschitz embedding $f : \mathcal{G}(\mathbb{N}) \to X$ can be chosen so that* $\mathrm{Lip}(f) \cdot \mathrm{Lip}(f^{-1}) < 1 + \varepsilon$.

Proof. By [Be, Theorem 1, p. 369], for all $\eta > 0$, there exists a bounded sequence $(x_n)_{n=1}^{\infty}$ in X such that for all $k \in \mathbb{N}$, all $\varepsilon_1,\dots,\varepsilon_k \in \{-1,1\}$, and all $n_1 < \dots < n_k$, we have

$$1-\eta \le \left\| \frac{1}{k} \sum_{j=1}^{k} \varepsilon_j x_{n_j} \right\| \le 1+\eta.$$

Define $\varphi : \mathcal{G}(\mathbb{N}) \to X$ by setting $\varphi(n_1,\dots,n_k) = x_{n_1} + \dots + x_{n_k}$ for all $(n_1,\dots,n_k) \in \mathcal{G}(\mathbb{N}) \setminus \{\emptyset\}$, and $\varphi(\emptyset) = 0$. Then

$$(1-\eta)\cdot D(\overline{n},\overline{m}) \le \|\varphi_k(\overline{n}) - \varphi_k(\overline{m})\| \le (1+\eta)\cdot D(\overline{n},\overline{m})$$

for all $\overline{n},\overline{m} \in \mathcal{G}(\mathbb{N})$. ∎

PROBLEM 3.7. *If X has the Banach–Saks property, does it follow that $\mathcal{G}(\mathbb{N})$ does not Lipschitz embed into X? In other words, if X is a reflexive Banach space, does $\mathcal{G}(\mathbb{N})$ Lipschitz embed into X if and only if X does not have the Banach–Saks property?*

By Corollary 3.2 above, any Banach space with an asymptotically uniformly smooth renorming has the alternating Banach–Saks property. To the best of our knowledge, there is no known example of a Banach space which has the alternating Banach–Saks property but does not admit an asymptotically uniformly smooth renorming. However, using descriptive-set-theoretical arguments, one can show the existence of such spaces. Recall that (X, Ω) is called a *standard Borel space* if X is a set and Ω is a σ-algebra on X which is the Borel σ-algebra associated to a Polish topology on X (i.e., a topology generated by a complete separable metric). A subset $A \subset X$ is called *analytic* if it is the image of a standard Borel space under a Borel map. We refer to [Do] and [Br, Section 2] for more details on the descriptive set theory of separable Banach spaces.

Let $C[0,1]$ be the space of continuous real-valued functions on $[0,1]$ endowed with the supremum norm. Let

$$\mathrm{SB} = \{X \in C[0,1] \mid X \text{ is a closed linear subspace}\},$$

and endow SB with the Effros–Borel structure, i.e., the σ-algebra generated by

$$\{X \in \mathrm{SB} \mid X \cap U \neq \emptyset\} \quad \text{for } U \subset C[0,1] \text{ open.}$$

This makes SB into a standard Borel space and, as $C[0,1]$ contains isometric copies of every separable Banach space, SB can be seen as a coding set for the class of all separable Banach spaces. Therefore, we can talk about Borel and analytic classes of separable Banach spaces.

By [Br, Theorem 17], the subset $\mathrm{ABS} \subset \mathrm{SB}$ of Banach spaces with the alternating Banach–Saks property is not analytic. On the other hand, letting $\mathrm{AUS} = \{X \in \mathrm{SB} \mid X \text{ is asymptotically uniformly smooth}\}$, we have

$$X \in \mathrm{AUS} \;\Leftrightarrow\; \forall \varepsilon \in \mathbb{Q}_+ \;\exists \delta \in \mathbb{Q}_+ \;\forall t \in \mathbb{Q}_+ \;(t < \delta \Rightarrow \overline{\rho}_X(t) < \varepsilon t).$$

As $\{X \in \mathrm{SB} \mid \dim(C[0,1]/X) < \infty\}$ is Borel, it is easy to check that the condition $A(t,\varepsilon) \subset \mathrm{SB}$ given by

$$X \in A(t,\varepsilon) \;\Leftrightarrow\; \overline{\rho}_X(t) < \varepsilon t$$

defines an analytic subset of SB (for similar arguments, we refer to [Do, Section 2.1]). So, AUS must be analytic. Hence, letting $\mathrm{AUSable} \subset \mathrm{SB}$ be the subset of Banach spaces with an asymptotically uniformly smooth renorming, we have

$$X \in \mathrm{AUSable} \;\Leftrightarrow\; \exists Y \in \mathrm{AUS} \;(X \cong Y).$$

As the isomorphism relation in $\mathrm{SB} \times \mathrm{SB}$ forms an analytic set (see [Do, p. 11]), it follows that AUSable is analytic. This discussion together with Corollary 3.2 gives us the following.

PROPOSITION 3.8. $\mathrm{AUSable} \subsetneq \mathrm{ABS}$. *In particular, there exist separable Banach spaces with the alternating Banach–Saks property which do not admit an asymptotically uniformly smooth renorming.*

4. Asymptotically p-uniformly convex/smooth spaces. In this section, we will use results from [KR] in order to obtain some restrictions on coarse embeddings $X \to Y$, where the spaces X and Y are assumed to have some asymptotic properties (see Theorem 4.1). We obtain restrictions on the existence of coarse embeddings between the convexified Tsirelson spaces (Theorem 1.5(i)), convexified Schlumprecht spaces (Theorem 1.5(ii)), and some specific hereditarily indecomposable spaces introduced in [D] (Corollary 4.8) beelow.

THEOREM 4.1. *Let $p, q \in (1, \infty)$. Let X be an infinite-dimensional Banach space with the p-co-Banach–Saks property and not containing ℓ_1. Let Y be a reflexive asymptotically q-uniformly smooth Banach space. Then there exists no coarse embedding $f : X \to Y$ such that*

$$\limsup_{k\to\infty} \frac{\rho_f(k^{1/p})}{k^{1/q}} = \infty.$$

Proof. Let $f : X \to Y$ be a coarse embedding. So, there exists $C > 0$ such that $\omega_f(t) \le Ct + C$ for all $t > 0$. As X does not contain ℓ_1, by Rosenthal's ℓ_1-theorem we can pick a normalized weakly null sequence $(x_n)_{n=1}^\infty$ in X with $\inf_{n\neq m} \|x_n - x_m\| > 0$. For each $k \in \mathbb{N}$, define a map $\varphi_k : G_k(\mathbb{N}) \to X$ by letting

$$\varphi_k(n_1, \dots, n_k) = x_{n_1} + \dots + x_{n_k}$$

for all $(n_1, \dots, n_k) \in G_k(\mathbb{N})$. Then φ_k is a bounded map.

If $d((n_1, \dots, n_k), (m_1, \dots, m_k)) \le 1$, then $\|\sum_{j=1}^k x_{n_j} - \sum_{j=1}^k x_{m_j}\| \le 2$. So, $\mathrm{Lip}(f \circ \varphi_k) \le 3C$. By Theorem 3.3, there exists $K = K(Y) > 0$ and an infinite subset $\mathbb{M}_k \subset \mathbb{N}$ such that

$$\mathrm{diam}(f \circ \varphi_k(G_k(\mathbb{M}_k))) \le 3KCk^{1/q}.$$

Without loss of generality, we may assume that $\mathbb{M}_{k+1} \subset \mathbb{M}_k$ for all $k \in \mathbb{N}$. Let $\mathbb{M} \subset \mathbb{N}$ diagonalize the sequence $(\mathbb{M}_k)_{k=1}^\infty$, say $\mathbb{M} = (n_j)_{j=1}^\infty$. If a sequence $(y_n)_{n=1}^\infty$ is weakly null, so is $(y_{2n-1} - y_{2n})_{n=1}^\infty$. Therefore, applying the fact that X has the p-co-Banach–Saks property to the weakly null sequence $(x_{n_{2j-1}} - x_{n_{2j}})_{j=1}^\infty$, we find that there exists $c > 0$ such that for all $k \in \mathbb{N}$, there exists $m_1 < \dots < m_{2k} \in \mathbb{M}_k$ such that

$$\Big\|\sum_{j=1}^k (x_{m_{2j-1}} - x_{m_{2j}})\Big\| \ge ck^{1/p}.$$

Hence, $\mathrm{diam}(\varphi_k(G_k(\mathbb{M}_k))) \ge ck^{1/p}$, which implies $\mathrm{diam}(f \circ \varphi_k(G_k(\mathbb{M}_k))) \ge \rho_f(ck^{1/p})$ for all $k \in \mathbb{N}$. So,

$$\rho_f(ck^{1/p}) \le 3KCk^{1/q}$$

for all $k \in \mathbb{N}$. Therefore, if $\limsup_{k\to\infty} \rho_f(k^{1/p})k^{-1/q} = \infty$, we get a contradiction. ∎

REMARK 4.2. Let X be any Banach space containing a sequence $(x_n)_{n=1}^\infty$ which is asymptotically ℓ_1, i.e., there exists $L > 0$ such that for all $m \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $(x_{n_j})_{j=1}^m$ is L-equivalent to $(e_j)_{j=1}^m$ for all $k \le n_1 < \cdots < n_m \in \mathbb{N}$, where $(e_j)_{j=1}^\infty$ is the standard ℓ_1-basis. Then, proceeding exactly as above, we can show that there exists no coarse embedding $f : X \to Y$ such that

$$\limsup_{k\to\infty} \frac{\rho_f(k)}{k^{1/q}} = \infty,$$

where $q \in (1,\infty)$ and Y is a reflexive asymptotically q-uniformly smooth Banach space.

Let X and Y be Banach spaces. We define $\alpha_Y(X)$ as the supremum of all $\alpha > 0$ for which there exists a coarse embedding $f : X \to Y$ and $L > 0$ such that

$$L^{-1}\|x-y\|^\alpha - L \le \|f(x) - f(y)\|$$

for all $x, y \in X$. We call $\alpha_Y(X)$ the *compression exponent of X in Y*, or the *Y-compression of X*. If for all $\alpha > 0$, no such f and L exist, we set $\alpha_Y(X) = 0$. As ω_f is always bounded by an affine map (because X is a Banach space), it follows that $\alpha_Y(X) \in [0,1]$. Also, $\alpha_Y(X) = 0$ if X does not coarsely embed into Y.

The quantity $\alpha_Y(X)$ was introduced by E. Guentner and J. Kaminker [GK]. For a detailed study of $\alpha_{\ell_q}(\ell_p)$, $\alpha_{L_q}(\ell_p)$, $\alpha_{\ell_q}(L_p)$, and $\alpha_{L_q}(L_p)$, where $p, q \in (0,\infty)$, we refer to [B].

Using this terminology, let us reinterpret Theorem 4.1.

THEOREM 4.3. *Let $1 < p < q$. Let Y be a reflexive asymptotically q-uniformly smooth Banach space.*

(i) *If X contains a sequence which is asymptotically ℓ_1, then $\alpha_Y(X) \le 1/q$.*
(ii) *If X is an infinite-dimensional Banach space with the p-co-Banach–Saks property and not containing ℓ_1, then $\alpha_Y(X) \le p/q$.*

In particular, X does not coarse Lipschitz embed into Y.

Proof. (ii) Let $L > 0$ and $f : X \to Y$ be a coarse embedding such that $\rho_f(t) \ge L^{-1}t^\alpha - L$ for all $t > 0$. By Theorem 4.1, we must have

$$\limsup_{k\to\infty}(k^{\alpha/p-1/q}L^{-1} - Lk^{-1/q}) < \infty.$$

Therefore, $\alpha/p - 1/q \le 0$, and the result follows.

(i) This follows from Remark 4.2 and the same reasoning as (ii) above. ∎

Notice that Y being reflexive in Theorem 4.3 cannot be removed. Indeed, c_0 contains a Lipschitz copy of any separable metric space (see [A]), and it is also asymptotically q-uniformly smooth for any $q \in (1, \infty)$.

COROLLARY 4.4. *Let $1 < p < q$. Let X be asymptotically p-uniformly convex, and Y be reflexive and asymptotically q-uniformly smooth. Then $\alpha_Y(X) \leq p/q$.*

Requiring the Banach space X to have the p-co-Banach–Saks property in Theorem 4.3 is actually too much, and we can weaken this condition by only requiring X to have the almost p-co-Banach–Saks property. More precisely, we have the following.

THEOREM 4.5. *Let $1 < p < q$. Let X be an infinite-dimensional Banach space with the almost p-co-Banach–Saks property. Let Y be a reflexive asymptotically q-uniformly smooth Banach space. Then $\alpha_Y(X) \leq p/q$. In particular, X does not coarse Lipschitz embed into Y.*

Proof. Let $f : X \to Y$ be a coarse embedding and pick $C > 0$ such that $\omega_f(t) \leq Ct + C$ for all $t \geq 0$. If X contains ℓ_1, the result follows from Theorem 4.3(i). If X does not contain ℓ_1, we can pick a normalized weakly null sequence $(x_n)_{n=1}^\infty$ in X with $\inf_{n \neq m} \|x_n - x_m\| > 0$. By taking a subsequence of $(x_n)_{n=1}^\infty$ if necessary, pick $(\theta_k)_{k=1}^\infty$ as in the definition of the almost p-co-Banach–Saks property. Define $\varphi_k : G_k(\mathbb{N}) \to X$ by letting $\varphi_k(n_1, \ldots, n_k) = x_{n_1} + \cdots + x_{n_k}$ for all $(n_1, \ldots, n_k) \in G_k(\mathbb{N})$.

Following the proof of Theorem 4.1, we get

$$\rho_f(k^{1/p}\theta_k^{-1}) \leq 3KCk^{1/q}$$

for all $k \in \mathbb{N}$. Let $L > 0$ and $\alpha > 0$ be such that $\rho_f(t) \geq L^{-1}t^\alpha - L$ for all $t > 0$. Then

$$k^{\alpha/p - 1/q}\theta_k^{-\alpha}L^{-1} \leq 4KC$$

for large enough $k \in \mathbb{N}$. As $\lim_{k\to\infty} k^\beta \theta_k^{-\alpha} = \infty$ for all $\beta > 0$, we must have $\alpha/p - 1/q \leq 0$. ∎

REMARK 4.6. Let $(x_n)_{n=1}^\infty$ be a bounded sequence in a Banach space X with the following property: there exists a sequence of positive reals $(\theta_j)_{j=1}^\infty$ in $[1, \infty)$ such that $\lim_{j\to\infty} j^\alpha \theta_j^{-1} = \infty$ for all $\alpha > 0$, and

$$(*) \qquad k\theta_k^{-1} \leq \|\pm x_{n_1} \pm \cdots \pm x_{n_k}\|$$

for all $n_1 < \cdots < n_k \in \mathbb{N}$. The proof of Theorem 4.5 gives $\alpha_Y(X) \leq 1/q$ for any reflexive asymptotically q-uniformly smooth Banach space Y with $q > 1$.

Let $q > 1$, and let $(E_n)_{n=1}^\infty$ be a sequence of finite-dimensional Banach spaces. Let $\mathcal{E}$ be a 1-unconditional basic sequence. Notice that if $\mathcal{E}$ generates a reflexive asymptotically q-uniformly smooth Banach space, then $(\bigoplus_n E_n)_{\mathcal{E}}$ is also reflexive and asymptotically q-uniformly smooth. Hence, Theorems 4.3 and 4.5 have the following

COROLLARY 4.7. *Let $1 < p < q$, and let $(E_n)_{n=1}^{\infty}$ be a sequence of finite-dimensional Banach spaces. Let $\mathcal{E}$ be a 1-unconditional basic sequence generating a reflexive asymptotically q-uniformly smooth Banach space.*

(i) *If X contains a sequence with property $(*)$, then $\alpha_{(\bigoplus_n E_n)_{\mathcal{E}}}(X) \leq 1/q$.*
(ii) *If X is an infinite-dimensional Banach space with the almost p-co-Banach–Saks property, then $\alpha_{(\bigoplus_n E_n)_{\mathcal{E}}}(X) \leq p/q$.*

In particular, X does not coarse Lipschitz embed into $(\bigoplus_n E_n)_{\mathcal{E}}$.

Proof of Theorem 1.5. (i) As noticed in Subsection 2.6, T^p has the p-co-Banach–Saks property, and is asymptotically p-uniformly smooth for all $p \in (1,\infty)$. Therefore, as T^p is reflexive (see [OSZ, Proposition 5.3(b)]) for all $p \in [1,\infty)$, the result follows from Theorem 4.3 (or Corollary 4.7).

(ii) For any $p \in (1,\infty)$, S^p has the almost p-co-Banach–Saks property and is asymptotically p-uniformly smooth. By [CK, Theorem 8 and Proposition 2(2)], S^p is reflexive for all $p \in [1,\infty)$. So, the result follows from Corollary 4.7. ∎

A Banach space X is called *hereditarily indecomposable* if none of its subspaces can be decomposed as a sum of two infinite-dimensional Banach spaces. In [D, Chapter 5], for each $p \in (1,\infty)$, Dew constructed a hereditarily indecomposable space $\mathfrak{X}_p$ with a basis $(e_n)_{n=1}^{\infty}$ satisfying the following properties: (i) $\mathfrak{X}_p$ is reflexive, (ii) the base $(e_n)_{n=1}^{\infty}$ satisfies an upper ℓ_p-estimate with constant 1, and (iii) if $(x_n)_{n=1}^{\infty}$ is a block sequence of $(e_n)_{n=1}^{\infty}$, then for all $n \in \mathbb{N}$,

$$\Big\|\sum_{j=1}^{n} x_j\Big\| \geq f(n)^{-1/p}\Big(\sum_{j=1}^{n} \|x_j\|^p\Big)^{1/p},$$

where $f : \mathbb{N} \to [0,\infty)$ is a function such that, among other properties, $\lim_{n\to\infty} n^{\alpha} f(n)^{-1} = \infty$ for all $\alpha > 0$. In particular, $\mathfrak{X}_p$ has the almost p-co-Banach–Saks property, and it is asymptotically p-uniformly smooth. This, together with Theorem 4.5, gives us the following.

COROLLARY 4.8. *Let $1 < p < q$. Then $\alpha_{\mathfrak{X}^q}(\mathfrak{X}^p) \leq p/q$. In particular, $\mathfrak{X}_p$ does not coarse Lipschitz embeds into $\mathfrak{X}_q$.*

PROBLEM 4.9. *Let $1 \leq p < q$. Is it true that $\alpha_{T^q}(T^p) = \alpha_{S^q}(S^p) = p/q$? If $p > 1$, does $\alpha_{\mathfrak{X}^q}(\mathfrak{X}^p) = p/q$ hold?*

REMARK 4.10. It is worth noticing that, if $p > \max\{q, 2\}$, then $\alpha_{T^q}(T^p) = 0$. Indeed, for all $r \geq 2$, T^r has cotype $r + \varepsilon$ for all $\varepsilon > 0$ (see

[DJT, p. 305]). On the other hand, if $r < 2$, then T^r has cotype 2. This follows from the fact that, for any $\varepsilon > 0$, T^r has an equivalent norm satisfying a lower $\ell_{(r+\varepsilon)}$-estimate (we explain this in the proof of Corollary 1.6 below), therefore, by [LT, Theorem 1.f.7 and Proposition 1.f.3(i)], T^r has cotype 2. Similarly, by [LT, Theorem 1.f.7 and Proposition 1.f.3(ii)], T^r has nontrivial type for all $r \in (1, \infty)$. By [MN, Theorem 1.11], if a Banach space X coarsely embeds into a Banach space Y with nontrivial type, then

$$\inf\{q \in [2, \infty) \mid X \text{ has cotype } q\} \leq \inf\{q \in [2, \infty) \mid Y \text{ has cotype } q\}.$$

Therefore, we conclude that T^p does not coarsely embed into T^q if $p > \max\{q, 2\}$. So, $\alpha_{T^q}(T^p) = 0$.

PROBLEM 4.11. *Let $1 \leq q < p \leq 2$. What can we say about $\alpha_{T^q}(T^p)$?*

We finish this section with an application of Theorems 4.3 and 4.5 and [AB, Theorem 3.4]. By looking at the proof of the latter, one can easily see that the authors proved a stronger result than the one stated in their paper. More precisely, they proved the following.

THEOREM 4.12. *Let $0 < p < q$. There exist maps $(\psi_j : \mathbb{R} \to \mathbb{R})_{j=1}^{\infty}$ such that for all $x, y \in \mathbb{R}$,*

$$A_{p,q}|x - y|^p \leq \max\{|\psi_j(x) - \psi_j(y)|^q \mid j \in \mathbb{N}\}$$

and

$$\sum_{j \in \mathbb{N}} |\psi_j(x) - \psi_j(y)|^q \leq B_{p,q}|x - y|^p,$$

where $A_{p,q}, B_{p,q}$ are positive constants.

PROPOSITION 4.13. *Let $1 \leq p < q$. There exists a map $f : T^p \to (\oplus T^q)_{T^q}$ which is simultaneously a coarse and a uniform embedding such that $\rho_f(t) \geq Ct^{p/q}$ for some $C > 0$. In particular, $\alpha_{(\oplus T^q)_{T^q}}(T^p) = p/q$.*

Proof. Let $(\psi_j)_{j=1}^{\infty}$, $A_{p,q}$, and $B_{p,q}$ be given by Theorem 4.12. Define $f : T^p \to (\bigoplus T_q)_{T^q}$ by letting

$$f(x) = \big((\psi_j(x_n) - \psi_j(0))_{j=1}^{\infty}\big)_{n=1}^{\infty}$$

for all $x = (x_n)_{n=1}^{\infty} \in T^p$. One can easily check that f satisfies

$$A_{p,q}^{1/q}\|x - y\|^{p/q} \leq \|f(x) - f(y)\| \leq B_{p,q}^{1/q}\|x - y\|^{p/q}$$

for all $x, y \in T^p$. As T^q is q-convex, it is easy to see that $(\bigoplus T^q)_{T^q}$ is asymptotically q-uniformly smooth. Since $(\bigoplus T^q)_{T^q}$ is reflexive, we conclude that $\alpha_{(\bigoplus T^q)_{T^q}}(T^p) = p/q$. ■

Corollary 4.14. *T strongly embeds into a super-reflexive Banach space.*

Proof. It is easy to check that $(\bigoplus T^2)_{T^2}$ is super-reflexive. Indeed, super-reflexivity is equivalent to a uniformly convex renorming. Hence, if $\mathcal{E}$ is a 1-unconditional basis generating a super-reflexive space, and X is a super-reflexive space, then so is $(\bigoplus X)_{\mathcal{E}}$ (see [LT, p. 100]). ∎

Similarly, we get the following.

Proposition 4.15. *Let $1 \leq p < q$. There exists a map $f : S^p \to (\bigoplus S^q)_{S^q}$ which is simultaneously a coarse and a uniform embedding such that $\rho_f(t) \geq Ct^{p/q}$ for some $C > 0$. In particular, $\alpha_{(\bigoplus S^q)_{S^q}}(S^p) = p/q$.*

5. Coarse Lipschitz embeddings into sums. In this last section, we will be specially interested in the nonlinear geometry of the Tsirelson space and its convexifications. In order to obtain Theorem 1.6, we will prove a technical result on the coarse Lipschitz nonembeddability of certain Banach spaces into the direct sum of Banach spaces with certain p-properties (Theorem 5.6). The main goal of this section is to characterize the Banach spaces which are coarsely (resp. uniformly) equivalent to $T^{p_1} \oplus \cdots \oplus T^{p_n}$ for $p_1, \ldots, p_n \in (1, \ldots, \infty)$ and $2 \notin \{p_1, \ldots, p_n\}$.

Given $x, y \in X$ and $\delta > 0$, the *approximate midpoint between x and y with error δ* is given by

$$\mathrm{Mid}(x, y, \delta) = \{z \in X \mid \max\{\|x - z\|, \|y - z\|\} \leq 2^{-1}(1 + \delta)\|x - y\|\}.$$

The following lemma is an asymptotic version of [JLS, Lemma 1.6(i)] and [KR, Lemma 3.2].

Lemma 5.1. *Let X be an asymptotically p-uniformly smooth Banach space for some $p \in (1, \infty)$. There exists $c > 0$ such that for all $x, y \in X$, all $\delta > 0$, and all weakly null sequences $(x_n)_{n=1}^\infty$ in B_X, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, we have*

$$u + \delta^{1/p}\|v\|x_n \in \mathrm{Mid}(x, y, c\delta),$$

where $u = \frac{1}{2}(x + y)$ and $v = \frac{1}{2}(x - y)$.

Proof. By [DGJ, Proposition 1.3], there exists $c > 0$ such that for all weakly null sequences $(x_n)_{n=1}^\infty$ in B_X, we have

$$\limsup_n \|x + x_n\|^p \leq \|x\|^p + c \cdot \limsup_n \|x_n\|^p.$$

Fix such sequence. As $\big\|x - (u + \delta^{1/p}\|v\|x_n)\big\| = \big\|v - \delta^{1/p}\|v\|x_n\big\|$, we get

$$\limsup_n \big\|x - (u + \delta^{1/p}\|v\|x_n)\big\|^p \leq (1 + c\delta)\|v\|^p.$$

Therefore, as $(1 + c\delta)^{1/p} < 1 + c\delta$, we see that there exists $n_0 \in \mathbb{N}$ such that $\|x - (u + \delta^{1/p}\|v\|x_n)\| \leq (1 + c\delta)\|v\|$ for all $n > n_0$. Similarly we can prove that $\|y - (u + \delta^{1/p}\|v\|x_n)\| \leq (1 + c\delta)\|v\|$ for all $n > n_0$. ∎

The following lemma is a simple modification of [KR, Lemma 3.3], or [JLS, Lemma 1.6(ii)], so we omit its proof.

LEMMA 5.2. *Suppose $1 \le p < \infty$, and let X be Banach space with a 1-unconditional basis $(e_n)_{n=1}^\infty$ satisfying a lower ℓ_p-estimate with constant 1. For all $x, y \in X$ and $\delta > 0$, there exists a compact subset $K \subset X$ that*

$$\mathrm{Mid}(x, y, \delta) \subset K + 2\delta^{1/p}\|v\|B_X,$$

where $u = \frac{1}{2}(x+y)$ and $v = \frac{1}{2}(x-y)$.

For each $s > 0$, let

$$\mathrm{Lip}_s(f) = \sup_{t \ge s} \frac{\omega_f(t)}{t} \quad \text{and} \quad \mathrm{Lip}_\infty(f) = \inf_{s>0} \mathrm{Lip}_s(f).$$

We will need the following proposition (see [KR, Proposition 3.1]).

PROPOSITION 5.3. *Let X be a Banach space and M be a metric space. Let $f : X \to M$ be a coarse map with $\mathrm{Lip}_\infty(f) > 0$. Then for all $\varepsilon, t > 0$ and all $\delta \in (0,1)$, there exists $x, y \in X$ with $\|x - y\| > t$ such that*

$$f(\mathrm{Mid}(x, y, \delta)) \subset \mathrm{Mid}(f(x), f(y), (1+\varepsilon)\delta).$$

The following lemma will play the same role in our settings as Proposition 3.5 of [KR] did in that paper.

LEMMA 5.4. *Let $1 \le q < p$. Let X be an asymptotically p-uniformly smooth Banach space, and Y be a Banach space with a 1-unconditional basis satisfying a lower ℓ_q-estimate with constant 1. Let $f : X \to Y$ be a coarse map. Then, for any $t > 0$ and any $\delta \in (0,1)$, there exist $x \in X$, $\tau > t$, and a compact subset $K \subset Y$ such that, for any weakly null sequence $(x_n)_{n=1}^\infty$ in B_X, there exists $n_0 \in \mathbb{N}$ such that*

$$f(x + \tau x_n) \in K + \delta\tau B_Y \quad \textit{for all } n > n_0.$$

Proof. If $\mathrm{Lip}_\infty(f) = 0$, then there exists $\tau > t$ such that $\mathrm{Lip}_\tau(f) < \delta$. Hence, $\omega_f(\tau) < \delta\tau$, and the result follows by letting $x = 0$ and $K = \{f(0)\}$. Indeed, if $z \in B_X$, we have

$$\|f(\tau z) - f(0)\| \le \omega_f(\|\tau z\|) \le \omega_f(\tau) \le \delta\tau.$$

Assume $\mathrm{Lip}_\infty(f) > 0$. In particular, $C = \mathrm{Lip}_s(f) > 0$ for some $s > 0$. Let $c > 0$ be given by Lemma 5.1 applied to X and p. As $q < p$, we can pick $\nu \in (0,1)$ such that $2C(2c)^{1/q}\nu^{1/q-1/p} < \delta$. By Proposition 5.3, there exist $u, v \in X$ such that $\|u - v\| > \max\{s, 2t\nu^{-1/p}\}$ and

$$f(\mathrm{Mid}(u, v, c\nu)) \subset \mathrm{Mid}(f(u), f(v), 2c\nu).$$

Let $x = \frac{1}{2}(u+v)$, and $\tau = \nu^{1/p}\big\|\frac{1}{2}(u-v)\big\|$ (so $\tau > t$). Fix a weakly null sequence $(x_n)_{n=1}^\infty$ in B_X. Then, by Lemma 5.1, there exists $n_0 \in \mathbb{N}$ such that

$x + \tau x_n \in \mathrm{Mid}(u, v, c\nu)$ for all $n > n_0$. So,

$$f(x + \tau x_n) \subset f(\mathrm{Mid}(u, v, c\nu)) \subset \mathrm{Mid}(f(u), f(v), 2c\nu)$$

for all $n > n_0$. Let $K \subset Y$ be given by Lemma 5.2 applied to Y, $f(u), f(v) \in Y$, and $2c\nu$. So,

$$\mathrm{Mid}(f(u), f(v), 2c\nu) \subset K + 2(2c)^{1/q}\nu^{1/q}\frac{\|f(u) - f(v)\|}{2}B_Y.$$

Since $\mathrm{Lip}_s(f) = C$ and $\|u - v\| > s$, we have $\|f(u) - f(v)\| \leq C\|u - v\| = 2C\tau\nu^{-1/p}$. Hence,

$$2(2c)^{1/q}\nu^{1/q}\frac{\|f(u) - f(v)\|}{2} \leq 2C(2c)^{1/q}\nu^{1/q-1/p}\tau < \delta\tau,$$

and we are done. ∎

Remark 5.5. Lemma 5.4 remains valid if we only assume that X has an *equivalent* norm $|||\cdot|||$ with which X becomes asymptotically p-uniformly smooth. Indeed, let $M \geq 1$ be such that $B_{(X,\|\cdot\|)} \subset M \cdot B_{(X,|||\cdot|||)}$. Fix $t > 0$ and $\delta \in (0, 1)$. Applying Lemma 5.4 to $(X, |||\cdot|||)$ with $t' = Mt$ and $\delta' = \delta/M$, we obtain $x \in X$, $\tau' > t'$, and a compact set $K \subset Y$. The result now follows by letting $\tau = \tau'/M$.

Theorem 5.6. *Let $1 \leq q_1 < p < q_2$. Assume that*

(i) *X is an asymptotically p-uniformly smooth Banach space with the p-co-Banach–Saks property, and it does not contain ℓ_1,*
(ii) *Y_1 is a Banach space with a 1-unconditional basis satisfying a lower ℓ_{q_1}-estimate with constant 1, and*
(iii) *Y_2 is a reflexive asymptotically q_2-uniformly smooth Banach space.*

Then X does not coarse Lipschitz embed into $Y_1 \oplus Y_2$.

Proof. Let $Y_1 \oplus_1 Y_2$ denote the space $Y_1 \oplus Y_2$ endowed with the norm defined by $\|(y_1, y_2)\| = \|y_1\| + \|y_2\|$ for all $(y_1, y_2) \in Y_1 \oplus Y_2$. Assume $f = (f_1, f_2) : X \to Y_1 \oplus_1 Y_2$ is a coarse Lipschitz embedding. Then there exists $C > 0$ such that $\rho_f(t) \geq C^{-1}t - C$ and $\omega_{f_2}(t) \leq Ct + C$ for all $t > 0$.

Fix $k \in \mathbb{N}$ and $\delta \in (0, 1)$. Then, by Lemma 5.4, there exists $\tau > k$, $x \in X$, and a compact subset $K \subset Y_1$ such that, for any weakly null sequence $(y_n)_{n=1}^{\infty}$ in B_X, there exists $n_0 \in \mathbb{N}$ such that

$$f_1(x + \tau y_n) \in K + \delta\tau B_{Y_1}$$

for all $n > n_0$.

Since X does not contain ℓ_1, by Rosenthal's ℓ_1-theorem we can pick a normalized weakly null sequence $(x_n)_{n=1}^{\infty}$ in X with $\inf_{n\neq m}\|x_n - x_m\| > 0$. As X has the p-Banach–Saks property (Proposition 2.2), there exists $c > 0$ (independent of k) such that, by going to a subsequence if necessary, we have

$$\|x_{n_1} + \cdots + x_{n_k}\| \leq ck^{1/p}$$

for all $n_1 < \cdots < n_k \in \mathbb{N}$. Define a map $\varphi_{k,\delta} : G_k(\mathbb{N}) \to X$ by letting

$$\varphi_{k,\delta}(n_1,\ldots,n_k) = x + \frac{\tau}{c}k^{-1/p}(x_{n_1} + \cdots + x_{n_k})$$

for all $(n_1,\ldots,n_k) \in G_k(\mathbb{N})$.

As $d((n_1,\ldots,n_k),(m_1,\ldots,m_k)) \le 1$ implies $\|\sum_{j=1}^k x_{n_j} - \sum_{j=1}^k x_{m_j}\| \le 2$, we have $\mathrm{Lip}(f_2 \circ \varphi_{k,\delta}) \le 2\tau C k^{-1/p} c^{-1} + C$. Therefore, by Theorem 3.3, there exists $\mathbb{M}_{k,\delta} \subset \mathbb{N}$ such that

$$\mathrm{diam}(f_2 \circ \varphi_{k,\delta}(G_k(\mathbb{M}_{k,\delta}))) \le 2K\tau C k^{1/q_2 - 1/p} c^{-1} + KCk^{1/q_2}$$

for some $K > 0$ independent of k and δ.

Notice that if $(n_1^j,\ldots,n_k^j)_{j=1}^\infty$ is a sequence in $G_k(\mathbb{M}_{k,\delta})$ with $n_k^j < n_1^{j+1}$ for all $j \in \mathbb{N}$, then $(x_{n_1^j} + \cdots + x_{n_k^j})_{j=1}^\infty$ is a weakly null sequence in $ck^{1/p}B_X$. Therefore,

$$f_1 \circ \varphi_{k,\delta}(n_1^j,\ldots,n_k^j) \in K + \delta\tau B_{Y_1}$$

for large enough j. This argument together with standard Ramsey theory allows us, by passing to a subsequence of $\mathbb{M}_{k,\delta}$, to assume that for all $(n_1,\ldots,n_k) \in G_k(\mathbb{M}_{k,\delta})$,

$$f_1 \circ \varphi_{k,\delta}(n_1,\ldots,n_k) \in K + \delta\tau B_{Y_1}.$$

Therefore, as K is compact, by passing to a further subsequence, we can assume that $\mathrm{diam}(f_1 \circ \varphi_{k,\delta}(G_k(\mathbb{M}_{k,\delta}))) \le 3\delta\tau$ (see [KR, Lemma 4.1]).

We have shown that for all $k \in \mathbb{N}$ and all $\delta \in (0,1)$, there exists a subsequence $\mathbb{M}_{k,\delta} \subset \mathbb{N}$ such that

$$\mathrm{diam}(f \circ \varphi_{k,\delta}(G_k(\mathbb{M}_{k,\delta}))) \le 2K\tau C k^{1/q_2 - 1/p} c^{-1} + KCk^{1/q_2} + 3\delta\tau. \tag{5.1}$$

We may assume that $\mathbb{M}_{k+1,\delta} \subset \mathbb{M}_{k,\delta}$ for all $k \in \mathbb{N}$ and all $\delta \in (0,1)$. For each $\delta \in (0,1)$, let $\mathbb{M}_\delta \subset \mathbb{N}$ diagonalize the sequence $(\mathbb{M}_{k,\delta})_{k=1}^\infty$.

As X has the p-co-Banach–Saks property, arguing similarly to the proof of Theorem 4.1, we find that there exists $d > 0$ (independent of k) such that for all $k \in \mathbb{N}$, there exist $n_1 < \cdots < n_{2k} \in \mathbb{M}_{k,\delta}$ such that

$$\Big\|\sum_{j=1}^k (x_{n_{2j-1}} - x_{n_{2j}})\Big\| \ge dk^{1/p}.$$

Therefore, $\mathrm{diam}(\varphi_{k,\delta}(G_k(\mathbb{M}_\delta))) \ge \tau d/c$, which implies that

$$\mathrm{diam}(f \circ \varphi_{k,\delta}(G_k(\mathbb{M}_\delta))) \ge \tau d(cC)^{-1} - C \tag{5.2}$$

for all $k \in \mathbb{N}$ and all $\delta \in (0,1)$. So, (5.1) and (5.2) give

$$\tau d(cC)^{-1} - C \le 2K\tau C k^{1/q_2 - 1/p} c^{-1} + KCk^{1/q_2} + 3\delta\tau$$

for all $k \in \mathbb{N}$ and $\delta \in (0,1)$. As $\tau > k$, this yields

$$d(cC)^{-1} - Ck^{-1} \leq 2KCk^{1/q_2-1/p}c^{-1} + KCk^{1/q_2-1} + 3\delta$$

for all $k \in \mathbb{N}$ and all $\delta \in (0,1)$. As $q_2 > p > 1$, by letting $k \to \infty$ and $\delta \to 0$ we get a contradiction. ∎

If $T = (T_1, T_2) : X \to Y_1 \oplus Y_2$ is a linear isomorphic embedding, then either $T_1 : X \to Y_1$ or $T_2 : X \to Y_2$ is not strictly singular, i.e., $T_i : X_0 \to Y_i$ is a linear isomorphic embedding for some infinite-dimensional subspace $X_0 \subset X$ and some $i \in \{1,2\}$. Is there an analog of this result for coarse Lipschitz embeddings? Precisely, we ask the following.

Problem 5.7. *Let X, Y_1 and Y_2 be Banach spaces and consider a coarse Lipschitz embedding $f = (f_1, f_2) : X \to Y_1 \oplus Y_2$. Is there an infinite-dimensional subspace $X_0 \subset X$ such that either $f_1 : X_0 \to Y_1$ or $f_2 : X_0 \to Y_2$ is a coarse Lipschitz embedding?*

We can now prove Theorem 1.6, which will be essential in the proof of Theorem 1.7.

Proof of Theorem 1.6. Say $m \in \{1, \dots, n-1\}$ is such that $p \in (p_m, p_{m+1})$ (the other cases have analogous proofs). Then $(T^{p_{m+1}} \oplus \cdots \oplus T^{p_n})_{\ell_\infty}$ is reflexive (see [OSZ, Proposition 5.3(b)8]). Also, it is easy to see that $(T^{p_{m+1}} \oplus \cdots \oplus T^{p_n})_{\ell_\infty}$ is asymptotically p_{m+1}-uniformly smooth. By Theorem 5.6, it is enough to prove the following.

Claim. *Fix $\varepsilon > 0$ such that $p_m + \varepsilon < p$. Then $(T^{p_1} \oplus \cdots \oplus T^{p_m})_{\ell_{p_m}}$ can be renormed so that it has a 1-unconditional basis satisfying a lower $\ell_{p_m+\varepsilon}$-estimate with constant 1.*

For each $k \in \mathbb{N}$ and $p \in [1,\infty)$, denote by $P_k = P_k^p : T^p \to T^p$ the projection on the first k coordinates, and let $Q_k = \mathrm{Id} - P_k$. By [JLS, Proposition 5.6], there exist $M \in [1,\infty)$ and $N \in \mathbb{N}$ such that $Q_N(T^{p_j})$ has an equivalent norm with $(p_j + \varepsilon)$-concavity constant M for all $j \in \{1, \dots, m\}$ (precisely, the modified Tsirelson norm has this property; see [CS] for definition).

As the shift operator on the basis of T^p is an isomorphism onto $Q_1(T^p)$, we find that $T^p \cong Q_k(T^p)$ for all $k \in \mathbb{N}$ and $p \in [1,\infty)$. Therefore, $(T^{p_1} \oplus \cdots \oplus T^{p_m})_{\ell_{p_m}}$ has an equivalent norm with $(p_m + \varepsilon)$-concavity constant M. By [LT, Proposition 1.d.8], we can assume that $M = 1$. As a q-concave basis with constant 1 satisfies a lower ℓ_q-estimate with constant 1, we are done. ∎

To prove Theorem 1.7, we need a lemma. For that, we must introduce some notation. Let $p \in (1,\infty)$. A Banach space X is said to be *as-$\mathcal{L}_p$* if there exists $\lambda > 0$ such that for every $n \in \mathbb{N}$ there is a finite-codimensional subspace $Y \subset X$ such that every n-dimensional subspace of Y is contained in a subspace of X which is λ-isomorphic to $L_p(\mu)$ for some μ. As noticed in [JLS, Proposition 2.4.a], every as-$\mathcal{L}_p$ space is super-reflexive. Also, the p-convexifications T^p are as-$\mathcal{L}_p$ (see [JLS], p. 440).

The following lemma, although not explicitly written, is contained in the proof of [JLS, Proposition 2.7]. For the convenience of the reader, we provide its proof here.

LEMMA 5.8. *Say $1 < p_1 < \cdots < p_n < \infty$ and $X = X^{p_1} \oplus \cdots \oplus X^{p_n}$, where X^{p_j} is as-$\mathcal{L}_{p_j}$ for all $j \in \{1, \ldots, n\}$. Assume that Y is coarsely equivalent to X. Then:*

(i) *There exists a separable Banach space W such that $Y \oplus W$ is Lipschitz equivalent to $\bigoplus_{j=1}^n (X^{p_j} \oplus L_{p_j})$.*
(ii) *Moreover, if $Y = Y^{p_1} \oplus \cdots \oplus Y^{p_n}$, where Y^{p_j} is as-$\mathcal{L}_{p_j}$ for all $i \in \{1, \ldots, n\}$, then $\bigoplus_{j=1}^n (Y^{p_j} \oplus L_{p_j})$ is Lipschitz equivalent to $\bigoplus_{j=1}^n (X^{p_j} \oplus L_{p_j})$.*

Proof. We need some more definitions. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$, and Z be a Banach space. We define the *ultrapower of Z with respect to $\mathcal{U}$* as $Z_{\mathcal{U}} = \{(z_n)_{n=1}^\infty \in Z^{\mathbb{N}} \mid \sup_{i \in \mathbb{N}} \|z_n\| < \infty\}/\sim$, where $(z_n)_{n=1}^\infty \sim (y_n)_{n=1}^\infty$ if $\lim_{n \in \mathcal{U}} \|z_n - y_n\| = 0$. Then $Z_{\mathcal{U}}$ is a Banach space with norm $\|[(z_n)_{n=1}^\infty]\| = \lim_{n \in \mathcal{U}} \|z_n\|$, where $(z_n)_{n=1}^\infty$ is a representative of the class $[(z_n)_{n=1}^\infty] \in Z_{\mathcal{U}}$. Notice that $z \in Z \mapsto [(z)_{n=1}^\infty] \in Z_{\mathcal{U}}$ is a linear isometric embedding. If Z is reflexive, then Z is 1-complemented in the ultrapower $Z_{\mathcal{U}}$ (where the projection is given by $[(z_n)_n] \in Z_{\mathcal{U}} \mapsto w\text{-}\lim_{n \in \mathcal{U}} z_n \in Z$), and we write $Z_{\mathcal{U}} = Z \oplus Z_{\mathcal{U},0}$. Also, we have $(Z \oplus E)_{\mathcal{U}} = Z_{\mathcal{U}} \oplus E_{\mathcal{U}}$.

We can now prove the lemma. For simplicity, assume that $n = 2$.

(i) Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$. As Y is coarsely equivalent to X, $Y_{\mathcal{U}}$ is Lipschitz equivalent to $X_{\mathcal{U}} = X_{\mathcal{U}}^{p_1} \oplus X_{\mathcal{U}}^{p_2}$ (see [K, Proposition 1.6]). As the spaces $X_{\mathcal{U}}^{p_j}$ are reflexive, using the separable complementation property for reflexive spaces (see [FJP, Section 3]), we can pick complemented separable subspaces $W \subset Y_{\mathcal{U},0}$, and $X_{j,0} \subset X_{\mathcal{U},0}^{p_j}$, for $j \in \{1, 2\}$, such that $Y \oplus W$ is Lipschitz equivalent to $(X^{p_1} \oplus X_{1,0}) \oplus (X^{p_2} \oplus X_{2,0})$. By enlarging $X_{j,0}$ and W if necessary, we can assume that $X_{j,0} = L_{p_j}$ for $j \in \{1, 2\}$ (this follows from [JLS, Proposition 2.4.a], [LR, Theorems I(ii) and III(b)]).

(ii) The argument why $X_{1,0} \oplus X_{2,0}$ can be enlarged so that $X_{1,0} \oplus X_{2,0} = L_{p_1} \oplus L_{p_2}$ also shows that W can be assumed to be $L_{p_1} \oplus L_{p_2}$. ∎

We can now prove Theorem 1.7. As mentioned in Section 1, the theorem was proved in [JLS, Theorem 5.8] for the cases $1 < p_1 < \cdots < p_n < 2$ and $2 < p_1 < \cdots < p_n < \infty$. In our proof, Theorem 1.6 will play a similar role to that of [JLS, Corollary 1.7] in their proof. Also, we use ideas in the proof of [KR, Theorem 5.3] in order to unify the cases $1 < p_1 < \cdots < p_n < 2$ and $2 < p_1 < \cdots < p_n < \infty$. For brevity, we will only present the parts of the proof that require Theorem 1.6, and therefore are different from what can be found in the literature.

Sketch of the proof of Theorem 1.7. By [JLS, Proposition 5.7], T^p is uniformly homeomorphic to $T^p \oplus \ell_p$ for all $p \in [1, \infty)$. So, the backward direction follows. Let us prove the forward direction. As uniform homeomorphism implies coarse equivalence, it is enough to assume that Y is coarsely equivalent to X. By Theorem 1.6, Y does not contain ℓ_2. Let $m \in \{1, \dots, n-1\}$ be such that $2 \in (p_m, p_{m+1})$ (if such an m does not exist, the result simply follows from [JLS, Theorem 5.8]).

Claim 1. *$X \oplus \bigoplus_{j=1}^n L_{p_j}$ and $Y \oplus \bigoplus_{j=1}^n L_{p_j}$ are Lipschitz equivalent.*

By Lemma 5.8(i), there exists a separable Banach space W such that $Y \oplus W$ is Lipschitz equivalent to $\bigoplus_{j=1}^n (T^{p_j} \oplus L_{p_j})$. Hence, the image of Y through this Lipschitz equivalence is the range of a Lipschitz projection in $\bigoplus_{j=1}^n (T^{p_j} \oplus L_{p_j})$. Therefore, by [HM, Theorem 2.2], Y is isomorphic to a complemented subspace of $\bigoplus_{j=1}^n (T^{p_j} \oplus L_{p_j})$. Let A be this isomorphic embedding. For each $i \in \{m+1, \dots, n\}$, let $\pi_i : Y \to L_{p_i}$ be the composition of A with the projection $\bigoplus_{j=1}^n (T^{p_j} \oplus L_{p_j}) \to L_{p_i}$. As Y does not contain ℓ_2, π_i factors through ℓ_{p_i} (see [J1]). Hence, Y is isomorphic to a complemented subspace of

$$\bigoplus_{j=1}^{m} (T^{p_j} \oplus L_{p_j}) \oplus \bigoplus_{j=m+1}^{n} (T^{p_j} \oplus \ell_{p_j}).$$

As $Z_1 := \bigoplus_{j=1}^m (T^{p_j} \oplus L_{p_j})$ and $Z_2 := \bigoplus_{j=m+1}^n (T^{p_j} \oplus \ell_{p_j})$ are totally incomparable (i.e., none of their infinite-dimensional subspaces are isomorphic), we have $Y \cong Y_1 \oplus Y_2$, where Y_1 and Y_2 are complemented subspaces of Z_1 and Z_2, respectively (see [EW, Theorem 3.5]). Hence, Y_1^* is complemented in Z_1^*. Notice that, as Y is coarsely equivalent to the super-reflexive space X, also Y is super-reflexive (see [Ri, Theorem 1A]). Hence, Y_1 is super-reflexive, and so is Y_1^*. As Y_1 has cotype 2 (see Remark 4.10) and Y_1^* has nontrivial type (because Y_1^* is super-reflexive), it follows that Y_1^* has type 2 (see [P, the remark below Theorem 1]). So, Y_1^* does not contain a copy of ℓ_2. Indeed, otherwise Y_1^* would contain a complemented copy of ℓ_2 (see [Ma]), contradicting that Y_1 does not contain a copy of ℓ_2.

Proceeding similarly and using the fact that Y_1^* does not contain ℓ_2, we deduce from the main theorem of [J1] that Y_1^* is isomorphic to a complemented subspace of $\bigoplus_{j=1}^m (T^{p_j *} \oplus \ell_{\tilde{p}_j})$, where each $\tilde{p}_j$ is the conjugate of p_j (i.e., $1/p_j + 1/\tilde{p}_j = 1$). Therefore, Y_1 embeds into $\bigoplus_{j=1}^m (T^{p_j} \oplus \ell_{p_j})$ as a complemented subspace. This implies that Y embeds into $\bigoplus_{j=1}^n (T^{p_j} \oplus \ell_{p_j})$ as a complemented subspace.

As the spaces $(T^{p_j} \oplus \ell_{p_j})_{j=1}^n$ are totally incomparable, we can write Y as $Y_{p_1} \oplus \dots \oplus Y_{p_n}$, where each Y_{p_j} is a complemented subspace of $T^{p_j} \oplus \ell_{p_j}$

(see [EW, Theorem 3.5]) and it is an as-$\mathcal{L}_{p_j}$ (see [JLS, Lemma 2.5 and Proposition 2.7]). By Lemma 5.8(ii), $X \oplus \bigoplus_{j=1}^n L_{p_j}$ and $Y \oplus \bigoplus_{j=1}^n L_{p_j}$ are Lipschitz equivalent.

Claim 2. *There exists a quotient W of $L_{p_1} \oplus \cdots \oplus L_{p_n}$ such that $Y \oplus W$ is isomorphic to $X \oplus \bigoplus_{j=1}^n L_{p_j}$.*

The proof of Claim 2 is the same as that of [JLS, Claim in Proposition 2.10], so we do not present it here. We now finish the proof of Theorem 17. As X does not contain any ℓ_s, every operator of X into $\bigoplus_{j=1}^n L_{p_j}$ is strictly singular (see [KM, Theorems II.2 and IV.1]). Therefore, by [EW] (or [LT, Theorem 2.c.13]), $Y \cong Y_X \oplus Y_L$ and $W \cong W_X \oplus W_L$, where Y_X and W_X are complemented subspaces of X, Y_L and W_L are complemented subspaces of $\bigoplus_{j=1}^n L_{p_j}$, and $X \cong Y_X \oplus W_X$. Proceeding as in the proof of Claim 1 above, we find that Y_L is complemented in $\bigoplus_{j=1}^n \ell_{p_j}$. So, Y_L is either finite-dimensional or isomorphic to $\bigoplus_{j\in F} \ell_{p_j}$ for some $F \subset \{1,\dots,n\}$.

Let us show that W_X is finite-dimensional. Suppose this is not the case. As W is a quotient of $\bigoplus_{j=1}^n L_{p_j}$, and W_X is complemented in W, we deduce that W_X^* embeds into $\bigoplus_{j=1}^n L_{\tilde{p}_j}$, where for each j, $\tilde{p}_j$ is the conjugate of p_j. Therefore, W_X^* must contain some ℓ_s (again see [KM, Theorems II.2 and IV.1]). As W_X^* embeds into X^*, and X^* does not contain any ℓ_s, this is a contradiction.

As $X \cong Y_X \oplus W_X$ and $\dim(W_X) < \infty$, we see that $\dim(X/Y_X) < \infty$. Therefore, as X is isomorphic to its hyperplanes, we conclude that $Y_X \cong X$. ∎

Problem 5.9. *Does Theorem* 1.7 *hold if* $2 \in \{p_1,\dots,p_n\}$*?*

Problem 5.10. *What can we say if a Banach space X is either coarsely or uniformly equivalent to the Tsirelson space T?*

Remark 5.11. It is worth noticing that, using Remark 5.5 and adapting the proofs of [KR, Theorems 5.5 and 5.7] to our settings, one can show that $(\bigoplus T_p)_{T_q}$ does not coarse Lipschitz embed into $T_p \oplus T_q$ for all $p, q \in [1,\infty)$ with $p \neq q$.

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