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# On weak Mellin transforms, second degree characters and the Riemann hypothesis

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**1. Introduction.** One of the main driving ideas of number theory is that in order to study an arithmetic problem defined on a global field, one should first try to solve a simpler "localized" form of this problem on various completions of this field. The "modern" approach to the functional equation of the zeta function, developped by John Tate in his thesis [14], is to introduce the ring of adeles $\mathbb{A}_{\mathbb{Q}}$, consider the zeta integral of the function

$$\phi(x) = e^{-\pi x_\infty^2} \otimes_p \mathbf{1}_{\mathbb{Z}_p}(x_p),$$

and show that the functional equation of the zeta function is a consequence of the equality $\mathfrak{F}(\phi) = \phi$ using the Poisson summation formula. According to this approach, the gamma factor $\Gamma(s/2)/\pi^{s/2}$ appearing in the functional equation and the Euler factors $1/(1-p^{-s})$ of the zeta function can be considered as "local zeta functions" associated to each completion of $\mathbb{Q}$. However, these functions do not have any zero, so that it seems difficult to use them to define a localized form of the Riemann hypothesis.

Note however that the gamma factor and the Euler factors are very close to being the Mellin transforms of the "same" function, from an algebraic point of view. More precisely, we consider on $\mathbb{R}$ the function $e^{-\pi i x^2}$, which can be written as $\psi_{\mathbb{R}}(x^2/2)$, where $\psi_{\mathbb{R}}(x) = e^{-2\pi i x}$ is the standard additive character of $\mathbb{R}$. On $\mathbb{Q}_p$, we consider the function $\psi_p(x^2/2)$, where $\psi_p$ is the standard additive character of $\mathbb{Q}_p$ considered as a locally compact abelian group.

The Mellin transforms (or zeta integral) of these functions are not well defined in the usual sense, but it is possible to extend the definition of the

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Mellin transform using the fact that if $f \star g$ is the multiplicative convolution product of $f$ and $g$, then $\mathrm{Mell}(f \star g) = \mathrm{Mell}(f)\,\mathrm{Mell}(g)$. We say that a function $f$ on $\mathbb{R}$ or $\mathbb{Q}_p$ has a well defined "weak Mellin transform" at the character $|x|^s$ if there exists a function $M_f(s)$ such that for any smooth test function $\phi$ with compact support in the multiplicative group $\mathbb{R}^*$ or $\mathbb{Q}_p^*$, we have $\mathrm{Mell}(\phi \star f, s) = \mathrm{Mell}(\phi, s) M_f(s)$. Using this definition, one can prove that the weak Mellin transforms of $\psi_{\mathbb{R}}(x^2/2)$ and $\psi_p(x^2/2)$ are well defined for $\Re(s) > 0$ and:

- $\mathrm{Mell}(\psi_{\mathbb{R}}(x^2/2), s) = e^{-s\pi i/4} \frac{\Gamma(s/2)}{\pi^{s/2}}$,
- $\mathrm{Mell}(\psi_p(x^2/2), s) = \frac{1}{1-p^{-s}}$ for $p \neq 2$,
- $\mathrm{Mell}(\psi_2(x^2/2), s) = \frac{1}{1-2^{-s}}(2^{1-s} - 1 + e^{\pi i/4}(2^s - 1))$.

The function $\mathrm{Mell}(\psi_2(x^2/2), s)$ does have some zeroes, and it is not difficult to check that they all lie on the line $\Re(s) = 1/2$. The main result of this paper is that this "local Riemann hypothesis" can be generalized to all functions of the form $\psi\big(\frac{a}{2}x^2 + bx\big)$ on $\mathbb{R}$ and $\mathbb{Q}_p$ with $a \neq 0$. These functions belong to a class of functions called "second degree characters" by Weil in his celebrated 1964 Acta paper [16]. A continuous function $f$ defined on a locally compact abelian group $G$ with values in the torus $\mathbb{T}$ is called a *second degree character* if the function $f(x+y)f(x)^{-1}f(y)^{-1}$ is a *bicharacter*, i.e. is a group character as a function of $x$ and as a function of $y$. On $\mathbb{R}$ or $\mathbb{Q}_p$, the second degree characters are precisely of the form $\psi\big(\frac{a}{2}x^2 + bx\big)$ with $a$ and $b$ in $\mathbb{R}$ or $\mathbb{Q}_p$.

Weil showed in [16] that if such a second degree character $f$ is *non-degenerate* (i.e. if $a$ is invertible), its Fourier transform can be written as $\gamma_f |a|^{-1/2} \bar{f}(x/a)$ where $\gamma_f$ is some scalar (now called the Weil index) satisfying $|\gamma_f| = 1$. Combining this result with Tate's thesis, we show that if $f$ is a non-degenerate second degree character on $\mathbb{R}$ or $\mathbb{Q}_p$ and $\chi$ a multiplicative unitary character on $\mathbb{R}^*$ or $\mathbb{Q}_p^*$, then the weak Mellin transform $\zeta_f(s, \chi)$ of $f$ at the character $|x|^s \chi(x)$ is well defined for $\Re(s) > 0$ and has an analytic continuation with the following functional equation:

$$\zeta_f(s, \chi) = \rho(s, \chi) \gamma_f |a|^{1/2-s} \bar{\chi}(a) \overline{\zeta_f(1 - \bar{s}, \chi)},$$

where $\rho(s, \chi)$ is the local factor defined by Tate in his thesis. We then prove that if $\zeta_f(s, \chi)$ is not the zero function as a function of $s$, then the zeroes of $\zeta_f(s, \chi)$ satisfy $\Re(s) = 1/2$, using an explicit description of the functions $\zeta_f$ for each locally compact field.

These results can be generalized to any finite extension of $\mathbb{Q}_p$, but not to $\mathbb{C}$: The zeroes of the Mellin transform of $f(z) = \psi_{\mathbb{C}}\big(\frac{a}{2}z^2 + bz\big)$ do not all lie on the line $\Re(s) = 1/2$. For example, if $b = 0$, then all even positive integers are zeroes of $\zeta_f$. However, if we take the second degree character $f(z) = \psi_{\mathbb{C}}\big(\frac{a}{2}|z|^2 + bz\big)$, then all the zeroes of $\zeta_f$ lie on $\Re(s) = 1/2$.

It remains to be seen whether these results can be understood and proved using a representation-theoretic approach. The fact that this "local Riemann hypothesis" fails on $\mathbb{C}$ is not necessarily an obstacle to the existence of such a proof, considering for example that the metaplectic group has a very specific structure on $\mathbb{C}$ (it is split) compared to $\mathbb{R}$, $\mathbb{Q}_p$ or $\mathbb{A}_\mathbb{Q}$.

These results are strikingly similar to those obtained in [2], [3], [8], [11] under the name of "a local Riemann hypothesis". For example, the local Riemann hypothesis of [3] gives good results on $\mathbb{R}$ and $\mathbb{Q}_p$, but Olofsson [11] showed that it also fails on $\mathbb{C}$. Other kinds of "local Riemann hypothesis" can be found in [13, Lemma 2.1] or [10, Section 1.7].

This theory can be generalized to second degree characters defined over finite-dimensional vector spaces. More precisely, if $f$ is any continuous function defined on $L^n$, where $L$ is some non-discrete locally compact field, and if $\phi \in C_c^\infty(\mathrm{GL}_n(L))$, we define the operator $f \mapsto \lambda(\phi)f$ by

$$\lambda(\phi)f(v) = \int\limits_{\mathrm{GL}_n(L)} \phi(g)f(g^{-1}v)\,d^\times g.$$

We show that if $f$ is a non-degenerate second degree character on $L^n$ and $\phi \in C_c^\infty(\mathrm{GL}_n(L))$, then $\lambda(\phi)f$ is a Schwartz function on $L^n$. We then consider the maximal compact subgroups $K = \mathrm{GL}_n(\mathcal{O}_L)$ if $L$ is a local field, and $K = O(n)$ or $U(n)$ if $L$ is $\mathbb{R}$ or $\mathbb{C}$, and the invariant norms on $L^n$ associated to $K$, i.e. $\|v\| = \max_i|v_i|$ or $\|v\| = \sqrt{\sum_i|v_i|^2}$. We then define the Mellin transform of a function on $L^n$ by

$$M(f,s) = \int\limits_{L^n} f(v)\|v\|^s\,\frac{dv}{\|v\|^n},$$

which is well defined for Schwartz functions when $\Re(s) > 0$. If $\phi$ is invariant for the left and right action of $K$, i.e. $\phi(k_1gk_2) = \phi(g)$ for all $k_1, k_2$ in $K$, then $\lambda(\phi)\|v\|^s$ is equal to $\|v\|^s$ up to a scalar factor, which we denote by $\xi_s(\phi)$. It is then immediate that if $\phi$ is a function in $C_c^\infty(\mathrm{GL}_n(L))$ satisfying the same invariance condition, we have

$$M(\lambda(\phi)f,s) = \xi_{s-n}(\phi^*)M(f,s), \quad \text{where} \quad \phi^*(g) = \frac{1}{|\det g|}\phi(g^{-1}).$$

Using this formula, which can be considered as a generalization of $\mathrm{Mell}(\phi \star f) = \mathrm{Mell}(\phi)\,\mathrm{Mell}(f)$, one can give a definition for the weak Mellin transform of a non-degenerate second degree character on $L^n$ similar to the definition given for $n = 1$. If we denote by $\zeta_f(s)$ the weak Mellin transform of a non-degenerate second degree character $f$ defined on $L^n$ and if the endomorphism associated to $f$ is a dilation, then $\zeta_f(s)$ and $\zeta_f(n-s)$ are related by a functional equation, and all the zeroes of $\zeta_f$ lie on the line $\Re(s) = n/2$ if $L$ is a local field or $L = \mathbb{R}$.

We then show how the concept of weak Mellin transform on a vector space can be generalized, replacing the function $\|u\|^s$, which is naturally associated to the trivial representation of $K$, by similar functions $\nu_{\pi,s}(x)$ naturally associated to spherical representations $(\pi, V_\pi)$ of $K$. If we denote by $\zeta_f(s,\pi)$ the weak Mellin transform of a non-degenerate second degree character $f$ defined on $\mathbb{R}^n$ and if the endomorphism associated to $f$ is a dilation, then we prove again that the zeroes of $\zeta_{f_v}(s,\pi)$ lie on the line $\Re(s) = n/2$.

**Notation.** We will usually denote by $L$ a locally compact field, which we always assume to be non-discrete and have characteristic zero, $F$ a number field and $\mathbb{A}_F$ the ring of adeles associated to $F$. We have the usual decomposition $\mathbb{A}_F = \mathbb{A}_\infty \times \mathbb{A}_f$ where $\mathbb{A}_f$ is the restricted product over all the finite places and $\mathbb{A}_\infty$ is the product over the archimedean places.

The standard additive character on $L$ will be denoted by $\psi_L$, or $\psi_p$ when $L = \mathbb{Q}_p$, or $\psi$ if no ambiguity is possible. Explicitly, we have $\psi_\mathbb{R}(x) = e^{-2\pi i x}$, $\psi_\mathbb{C}(z) = e^{-2\pi i(z+\bar{z})}$, $\psi_p(x) = e^{2\pi i \lambda(x)}$ where $\lambda(x) \in \mathbb{Q}$ is any rational number of the form $n/p^k$ satisfying $\lambda(x) - x \in \mathbb{Z}_p$. If $\mathbb{Q}_\mathfrak{p}$ is a local field of residual characteristic $p$, we have $\psi_{\mathbb{Q}_\mathfrak{p}} = \psi_p \circ \mathrm{Tr}_{\mathbb{Q}_\mathfrak{p}/\mathbb{Q}_\mathrm{p}}$. We let $dx$ be the Haar measure of $L$ considered as an additive group, and $d^\times x$ the Haar measure of $L^*$ considered as a multiplicative group. These Haar measures are normalized following Tate's thesis [14]: $dx$ is normalized so that the Fourier inversion formula is valid. The Fourier transform is defined as

$$\mathfrak{F}(f)(y) = \int_L f(x)\psi(xy)\,dx.$$

$d^\times x$ is normalized in the following way: on $\mathbb{R}$, we write $d^\times x = dx/|x|$; on $\mathbb{C}$, we write $d^\times z = dz/|z|_\mathbb{C} = dz/|z|^2$. On $\mathbb{Q}_p$, we denote by $\mathbb{Z}_p$ the ring of integers and $\mathbb{Z}_p^\times$ the group of units. The additive measure of $\mathbb{Z}_p$ is 1, and we normalize the multiplicative measure so that the measure of $\mathbb{Z}_p^\times$ is 1. We then have $d^\times x = \frac{1}{1-1/p}\frac{dx}{|x|}$. If $S$ is any set, we write $\mathbf{1}_S$ for its characteristic function. For example, $\mathbf{1}_{\mathbb{Z}_p}$ is the characteristic function of $\mathbb{Z}_p$.

If $\mathbb{Q}_\mathfrak{p}$ is a general local field of residual characteristic $p$, with group of units $\mathcal{O}_\mathfrak{p}^\times$, we normalize $d^\times x$ so that the measure of $\mathcal{O}_\mathfrak{p}^\times$ is equal to $(\mathcal{N}\mathfrak{d})^{-1/2}$ (where $\mathfrak{d}$ is the different of $\mathbb{Q}_\mathfrak{p}$). The additive measure of the ring of integers $\mathcal{O}_\mathfrak{p}$ is also set to $(\mathcal{N}\mathfrak{d})^{-1/2}$, and we have

$$d^\times x = \frac{1}{1 - 1/\mathcal{N}\mathfrak{p}}\frac{dx}{|x|}.$$

On an $L$-vector space $L^n$, we set $K = \mathrm{GL}_n(\mathcal{O}_L)$ if $L$ is local, $K = O(n)$ if $L$ is real, and $K = U(n)$ if $L$ is complex.

We denote by $\mathbb{A}_F^\times$ the group of ideles of $F$. The Mellin transform of a function $\phi$ defined on $L^\times$ or $\mathbb{A}_F^\times$ (called the zeta integral in Tate's thesis) is defined for $s \in \mathbb{C}$ and a multiplicative unitary character $\chi$ as

$$\mathrm{Mell}(f, s, \chi) = \int_{L^\times} f(x)|x|^s \chi(x)\, d^\times x,$$

resp.

$$\mathrm{Mell}(f, s, \chi) = \int_{\mathbb{A}_F^\times} f(x)|x|^s \chi(x)\, d^\times x.$$

Note that on $\mathbb{R}$, the definition we use is different from the usual definition, which involves an integral from 0 to $\infty$ only. We say that a Mellin transform is *well defined at* $(s, \chi)$ if the associated integral converges absolutely. The functional equations proved by Tate in his thesis can then be written in the following way:

Proposition 1.1 (Tate local functional equation). *If $\phi$ is a Schwartz function on a locally compact field $L$, then for $0 < \Re(s) < 1$ we have*

$$\mathrm{Mell}(\phi, s, \chi) = \rho(s, \chi)\, \mathrm{Mell}(\mathfrak{F}(\phi), 1 - s, \bar{\chi}),$$

*where $\rho(s, \chi)$ is a function of $s$ and $\chi$ but does not depend on $\phi$.*

Proposition 1.2 (Tate global functional equation). *If $\phi$ is a Schwartz function on $\mathbb{A}_F$ and $\chi$ a Hecke character, then $\mathrm{Mell}(\phi, s, \chi)$ is well defined for $\Re(s) > 1$, and has an analytic continuation to $\mathbb{C}$ with possible poles at $0$ and $1$. If we keep the notation $\mathrm{Mell}(\phi, s, \chi)$ for the analytic continuation, we have the equality*

$$\mathrm{Mell}(\phi, s, \chi) = \mathrm{Mell}(\mathfrak{F}(\phi), 1 - s, \bar{\chi}).$$

These results are proved in [14].

If $G$ is a locally compact abelian group, we denote by $S'(G)$ the space of tempered distributions on $G$, i.e. the space of continuous linear functionals on the Schwartz–Bruhat space $S(G)$. If $\mu$ is an element of $S'(L)$, the weak Fourier transform of $\mu$ is defined, following Schwartz, by the usual formula

$$\langle \mathfrak{F}(\mu), \phi \rangle = \langle \mu, \mathfrak{F}(\phi) \rangle.$$

## 2. A connection between Tate's thesis and Weil's 1964 Acta paper

**2.1. Second degree characters.** Let us now recall Weil's [16] definition of a second degree character. A continuous function $f$ defined on a locally compact abelian group $G$ with values in the torus $\mathbb{T}$ is called a *second degree character* if the function $f(x + y) f(x)^{-1} f(y)^{-1}$ is a *bicharacter*, i.e. a group character as a function of $x$ and as a function of $y$. For example, the function $e^{-2\pi i(\frac{a}{2}x^2 + bx)}$ with $a$ and $b$ in $\mathbb{R}$ is a second degree character on $\mathbb{R}$.

To any such function, we can associate a continuous morphism $\varrho$ from $G$ to $G^*$ by the formula

$$f(x+y)f(x)^{-1}f(y)^{-1} = \langle \varrho(y), x \rangle,$$

and it is clear that this morphism has to be symmetric (i.e. $\langle \varrho(y), x \rangle = \langle \varrho(x), y \rangle$). A second degree character is called *non-degenerate* if the associated morphism $\varrho$ is an isomorphism. We will always assume in this paper that the second degree characters considered are non-degenerate and continuous.

Weil [16] gave two formulae describing the weak Fourier transform of a non-degenerate second degree character $f$. Theses formulae will be often used in the following sections. The proofs of these results are available in [16] or [4]. The alternative presentations and proofs we propose below are given in order to show the striking connection between these formulae and Tate's thesis. Indeed, both results can be proved using exactly the same methods: The local formulae are proved using a multiplicity one result for eigendistributions associated to a commutative group of operators, and the global formulae using Poisson summation.

## 2.2. The local functional equation

Proposition 2.1 (Weil local functional equation). *Let $f$ be a non-degenerate second degree character on a locally compact abelian group $G$, and let $\varrho$ be the morphism associated to $f$. Then there exists a complex number $\gamma_f$ satisfying $|\gamma_f| = 1$ such that the weak Fourier transform of $f$ is equal to $\gamma_f |\varrho|^{-1/2} \bar{f}(\varrho^{-1}(x))$.*

Remark: $\gamma_f$ is now usually called the *Weil index* associated to the second degree character $f$.

*Proof.* This proposition can also be written in the following form: for any $\phi \in S(G)$,

$$\int_G f(x)\, \mathfrak{F}(\phi)(x)\, dx = \frac{\gamma_f}{\sqrt{|\varrho|}} \int_{G^*} \bar{f}(\varrho^{-1}(x)) \phi(x)\, dx.$$

Let us first recall that a proof of the Tate local functional equation has been given by Weil [17] using the concept of eigendistribution. It appears that second degree characters can also be considered as eigendistributions: Let $f$ be a second degree character with associated symmetric morphism $\varrho$ so that

$$f(x+y)\bar{f}(x)\bar{f}(y) = \langle \varrho(y), x \rangle.$$

We can write this expression as

$$f(x+y)\langle -\varrho(y), x \rangle = f(x)f(y).$$

Let us introduce for $t$ in $\mathbb{T}$, $u$ in $G$ and $u^*$ in $G^*$ the operator $tU_{u,u^*}$ acting on functions defined on $G$ by the formula $tU_{u,u^*}(f)(x) = tf(x+u)\langle u^*, x \rangle$.

We can then write the functional equation for a second degree character $f$ as

$$tU_{u,-\varrho(u)}(f) = tf(u)f,$$

showing that $f$ is an eigendistribution for the action of the operators $tU_{u,-\varrho(u)}$ for all $u$ in $G$ and $t \in \mathbb{T}$, and that the associated eigenvalue is $tf(u)$.

Recall that the Heisenberg group associated to $G$ can be described as the set $\mathbb{T} \times G \times G^*$ equipped with the group law $(t, u, u^*)(t', v, v^*) = (tt'\langle v^*, u\rangle, u+v, u^*+v^*)$. It is immediate that the map $(t, u, u^*) \mapsto tU_{u,u^*}$ is a representation of this group, which is usually called the *Schrödinger representation*.

We now remark that for $\varrho$ fixed, the set of operators of the form $tU_{u,-\varrho(u)}$ for $t \in \mathbb{T}$ and $u \in G$ is a commutative group (because $\varrho$ is symmetric), and it is not difficult to see that this set is the image of a maximal commutative subgroup of the Heisenberg group associated to $G$. The map which sends the operator $tU_{u,-\varrho(u)}$ to the scalar $tf(u)$ in $\mathbb{T}$ is a character of this commutative group.

This character restricts to the identity on the center $(\mathbb{T}, 0, 0)$ of the Heisenberg group (because $f(0) = f(0 + 0) = f(0)^2$, so that $f(0) = 1$). We can then use the following proposition, attributed to Cartier, which appears in [7], and is a consequence of the Stone–von Neumann theorem:

PROPOSITION 2.2. *Let $A$ be a maximal commutative subgroup of the Heisenberg group, and let $\chi$ be a character of $A$ restricting to the identity on its center. Denote by $\rho$ the Schrödinger representation of the Heisenberg group. Then there exists, up to a scalar factor, a unique distribution $\varDelta$ satisfying the formula*

$$\langle \varDelta, \rho(a)(f)\rangle = \chi(a)\langle \varDelta, f\rangle$$

*for all $a$ in $A$ and all $f$ in the Schwartz space.*

An immediate consequence of this proposition is that any distribution $\varDelta$ in $S'(G)$ satisfying for all $u$ in $G$ the functional equation

$$U_{u,-\varrho(u)}(\varDelta) = f(u)\varDelta$$

is equal to the second degree character $f$ up to a scalar factor.

The Weil local functional equation for second degree characters is then a straightforward consequence of the commutation relations between the Fourier transform and the operators $U_{u,u^*}$.

**2.3. The global functional equation.** If the second degree character is constant on some subgroup (for example, if a second degree character defined on an adele ring $\mathbb{A}_F$ is trivial on $F$), Weil also proved the following result:

PROPOSITION 2.3 (Weil global functional equation). *Let $f$ be a nondegenerate second degree character on $G$, suppose that $f$ is equal to* 1 *on* *a closed subgroup $\Gamma$ of $F$, and assume that the symmetric morphism $\varrho$ associated to $f$ is an isomorphism from $(G,\Gamma)$ to $(G^*,\Gamma^*)$. Then $\gamma(f)=1$.*

This functional equation can be proved in a straightforward way similar to Tate's thesis if $G$ is an adele ring $\mathbb{A}_F$ and $f$ is trivial on $F$: We write an element $y$ of $\mathbb{A}_F$ as $y = x+\delta$ with $x\in F$ and $\delta\in D$, where $D$ is some fundamental domain for the (additive) action of $F$ on $\mathbb{A}_F$. Since $f$ is trivial on $F$, we have

$$f(x+\delta) = f(x)f(\delta)\psi(x\varrho(\delta)) = f(\delta)\psi(x\varrho(\delta)).$$

Let us then consider the integral

$$\int\limits_{\mathbb{A}_F} f(y)\phi(y)\,dy = \int\limits_{D} f(\delta)\Big\{\sum_{x\in F}\psi(x\varrho(\delta))\phi(x+\delta)\Big\}\,d\delta.$$

Applying the Poisson summation formula to the inner sum leads to

$$\int\limits_{\mathbb{A}_F} f(y)\phi(y)\,dy = \int\limits_{D} f(\delta)\Big\{\sum_{x\in F}\psi(-\delta(\varrho(\delta)+x))\,\mathfrak{F}(\phi)(x+\varrho(\delta))\Big\}\,d\delta.$$

The definition of the second degree character $f$ allows us to write

$$\psi(-\delta(\varrho(\delta)+x)) = \bar f(\delta)\bar f(-\delta-\varrho^{-1}(x))f(-\varrho^{-1}(x)).$$

Since $f(-\varrho^{-1}(x)) = 1$ for $x\in F$, and $\varrho$ maps a fundamental domain of $\mathbb{A}_F$ for the action of $F$ to another fundamental domain, the above integral becomes

$$\int\limits_{\mathbb{A}_F} \bar f(\varrho^{-1}(y))\,\mathfrak{F}(\phi)(-y)\,dy. \ \blacksquare$$

## 3. The weak Mellin transform of second degree characters defined over locally compact fields

**3.1. Definition of the weak Mellin transform.** The definition of the weak Mellin transform that we will use is different from the one given for the weak Fourier transform and uses the properties of the convolution product. If $L$ is a non-discrete locally compact field, we denote by $C_c^\infty(L^\times)$ the space of smooth functions with compact support on $L^\times$ considered as a multiplicative group. Here "smooth" means as usual $C^\infty$ if $L$ is $\mathbb{R}$ or $\mathbb{C}$, and locally constant if $L$ is a local field.

It is clear that $C_c^\infty(L^\times)$ is a commutative algebra for the convolution product and that if $g$ and $h$ lie in $C_c^\infty(L^\times)$, we have

$$\mathrm{Mell}(g\star h,s,\chi) = \mathrm{Mell}(g,s,\chi)\,\mathrm{Mell}(h,s,\chi).$$

This motivates the following definition:

Definition 3.1. Let $f$ be a function defined on $L^\times$ and assume that $\phi\star f$ is well defined for all $\phi\in C_c^\infty(L^\times)$. We say that the function $M_f(s,\chi)$

is a *weak Mellin transform* of $f$ at $(s,\chi)$ if for any function $\phi$ in $C_c(L^\times)$, the Mellin transform of $\phi\star f$ is well defined at $(s,\chi)$ and satisfies

$$\operatorname{Mell}(\phi\star f,s,\chi)=M_f(s,\chi)\operatorname{Mell}(\phi,s,\chi).$$

It is immediate that this definition extends the usual definition of the Mellin transform, so that we will use the same notation for Mellin transforms and weak Mellin transforms.

**3.2. Description of second degree characters on a locally compact field.** Let us now describe more explicitly the non-degenerate second degree characters when $G$ is a non-discrete locally compact field $L$ in characteristic zero. The field $L$ is then a finite extension of $\mathbb{Q}_p$ or $\mathbb{R}$, which we call the *base field* $L_0$ of $L$. The group characters of $G$ are simply the functions of the form $\psi(ax)$ with $a$ in $L$. Any function of the form $\psi\big(\frac{1}{2}\alpha(x)x\big)$ where $\alpha$ is a continuous homomorphism of $L$ considered as an additive group is clearly a second degree character, and any function of the form $\langle x,\varrho(x)\rangle$ can be written in this form using the isomorphism between $L$ and $L^*$ given by $a\mapsto\chi_a:\chi_a(x)=\psi(ax)$. All second degree characters $f$ can thus be written in the form

$$f(x)=\psi\big(\tfrac{1}{2}\alpha(x)x+bx\big), \tag{3.1}$$

where $\alpha$ is any continuous $\mathbb{Z}$-module homomorphism from $L$ to $L$ satisfying $\psi(\alpha(x)y)=\psi(x\alpha(y))$. Since we assume that $f$ is non-degenerate, $\alpha$ is also $\mathbb{Q}$-linear. Using continuity and the fact that the closure of $\mathbb{Q}$ in $L$ is equal to $L_0$, we see that $\alpha$ has to be $L_0$-linear.

**3.3. The existence of the weak Mellin transform of a second degree character.** Let now $f$ be a second degree character defined on $L$ and suppose that $\alpha$ and $b$ are as in (3.1). We have a natural left action $\lambda$ of the multiplicative group $L^\times$ on $f$ by

$$\lambda(x)f(y)=f(x^{-1}y).$$

The integrated form of this action can be written, for $\phi\in C_c^\infty(L)$, as

$$\lambda(\phi)f(y)=\int\limits_{L^\times}\phi(x)f(x^{-1}y)\,d^\times x.$$

We do not use the notation $\phi\star f$ because the domain of $\lambda(\phi)f$ is $L$, not $L^*$.

PROPOSITION 3.2. *If $f$ is a non-degenerate second degree character and $\phi\in C_c^\infty(L^\times)$, then $\lambda(\phi)f$ is a Schwartz function on $L$.*

*Proof.* First suppose that $L$ is a local field, $L=\mathbb{Q}_\mathfrak{p}$. We have to show that $\lambda(\phi)f$ has compact support (the continuity is clear). Assuming $y\neq 0$

and using the commutativity of the convolution product, we get

$$\lambda(\phi)(f)(y) = \int_{\mathbb{Q}_{\mathfrak{p}}^{\times}} f(x)\phi(x^{-1}y)\, d^{\times}x.$$

We observe that the integral on $\mathbb{Q}_{\mathfrak{p}}^{\times}$ can be written as an integral on $\mathbb{Q}_{\mathfrak{p}}$, using the relation $d^{\times}x = \frac{1}{1-1/\mathcal{N}\mathfrak{p}}\frac{dx}{|x|}$. Write $\phi^*$ for the function $|x|^{-1}\phi(1/x)$, which is also in $C_c^{\infty}(\mathbb{Q}_{\mathfrak{p}}^{\times})$, and set $\phi^*(0) = 0$, so that $\phi^*$ can also be considered as a Schwartz function on $\mathbb{Q}_{\mathfrak{p}}$. We get

$$\lambda(\phi)(f)(y) = \frac{1}{1-1/\mathcal{N}\mathfrak{p}}\,\frac{1}{|y|}\int_{\mathbb{Q}_{\mathfrak{p}}} f(x)\phi^*(y^{-1}x)\, dx.$$

We now use the local Weil functional equation (Proposition 2.1) to see that

$$\lambda(\phi)(f)(y) = \frac{1}{1-1/\mathcal{N}\mathfrak{p}}\,\frac{\gamma_f}{\sqrt{|\alpha|}}\int_{\mathbb{Q}_{\mathfrak{p}}} \bar{f}(\alpha^{-1}(z))\,\mathfrak{F}(\phi^*)(-zy)\, dz.$$

Since $\mathfrak{F}(\phi^*)$ is Schwartz, it has compact support on $\mathbb{Q}_p$. Suppose that this support is contained in a ball of radius $R$. We also know that $f\circ\alpha^{-1}$ is continuous and equal to 1 near zero, so that there exists some $\epsilon$ such that if $|z| < \epsilon$, then $\bar{f}\circ\alpha^{-1}(z) = 1$. It is then immediate that if $|y| > R/\epsilon$, then the integral vanishes, which shows that $\lambda(\phi)f$ has compact support.

Let now $L = \mathbb{R}$. The proposition can be considered as a simple application of the method of stationary phase, but can also be proved directly using a method fully similar to the local field case.

We consider the integral

$$\lambda(\phi)(f)(y) = \int_{\mathbb{R}^*} \phi(x)f(x^{-1}y)\, d^{\times}x.$$

We write again $\phi^*(x) = |x|^{-1}\phi(1/x)$, $\phi^*(0) = 0$ so that the integral becomes

$$\int_{\mathbb{R}} f(xy)\phi^*(x)\, dx = \frac{\gamma_f}{|y|\sqrt{|\alpha|}}\int_{\mathbb{R}} \bar{f}\circ\alpha^{-1}(x/y)\,\mathfrak{F}(\phi^*)(-x)\, dx.$$

As $(\phi^*)^{(n)}(0) = 0$ for all $n$, we then remark that if $P$ is any polynomial, this expression is equal to

$$\frac{\gamma_f}{\sqrt{|a|}\,|y|}\int_{\mathbb{R}} \big(\bar{f}\circ\alpha^{-1}(x/y) - P(x/y)\big)\,\mathfrak{F}(\phi^*)(-x)\, dx.$$

We then choose $P$ to be the polynomial of order $n$ associated to the Taylor expansion of $\bar{f}\circ\alpha^{-1}$ at zero to get the result using elementary estimates.

The proof for $L = \mathbb{C}$ is similar. ∎

This proposition can be extended without difficulty to division rings but not to split simple algebras (i.e. $\mathrm{GL}_n(D)$ where $D$ is a division ring and $n \geq 2$).

Proposition 3.3. *If $f$ is a non-degenerate second degree character defined on a locally compact field $L$, then the weak Mellin transform of $f$ is well defined for $\Re(s) > 0$.*

*Proof.* Using the fact that $\lambda(\phi)(f)$ is a Schwartz function, it is immediate that the Mellin transform of $\phi \star f$ is well defined for $\Re(s) > 0$.

It is then enough to check that if $\phi, \mu \in C_c^\infty(L^\times)$, then for $\Re(s) > 0$ we have

$$\mathrm{Mell}(\mu, \chi, s)\,\mathrm{Mell}(\phi \star f, \chi, s) = \mathrm{Mell}(\phi, \chi, s)\,\mathrm{Mell}(\mu \star f, \chi, s).$$

Since all the Mellin transforms appearing in this equality are well defined for $\Re(s) > 0$, this is a straightforward consequence of $\mu \star (\phi \star f) = \phi \star (\mu \star f)$. ∎

If $f$ is a non-degenerate second degree character, we will denote by $\zeta_f(s, \chi)$ the weak Mellin transform of $f$ at the multiplicative character $|x|^s \chi(x)$.

**3.4. The functional equation of $\zeta_f$.** Let us first give an elementary equality for $\zeta_f$. Using the formula $\mathrm{Mell}(\phi(ax), s, \chi) = |a|^{-s}\bar{\chi}(a)\,\mathrm{Mell}(\phi, s, \chi)$ which is valid for $\phi \in C_c^\infty(L^\times)$, it is immediate that

$$\zeta_{f(ax)}(s, \chi) = |a|^{-s}\bar{\chi}(a)\zeta_f(s).$$

Let us now show that $\zeta_f$ has an analytic continuation:

Proposition 3.4. *Let $\phi \in C_c^\infty(L^\times)$ and $f$ a second degree character defined on $L$. Then the Fourier transform of the Schwartz function $\lambda(\phi)f$ is equal to the Schwartz function $\gamma_f|\alpha|^{-1/2}\lambda(\phi^*)(\bar{f} \circ \alpha^{-1})$ where $\phi^*(x) = |x|^{-1}\phi(1/x)$.*

*Proof.* Since $\lambda(\phi)f$ is Schwartz, it is enough to show that the conclusion is true in the weak sense. Indeed, it is not difficult to check, using the Weil functional equation, that for any Schwartz function $\varphi$,

$$\int_L \lambda(\phi)f(y)\,\mathfrak{F}(\varphi)(y)\,dy = \frac{\gamma_f}{\sqrt{|\alpha|}} \int_L \lambda(\phi^*)(\bar{f} \circ \alpha^{-1})(y)\varphi(y)\,dy. \ ∎$$

This leads to the following formula:

Proposition 3.5. *If $f$ is a non-degenerate second degree character on $L$, then for $0 < \Re(s) < 1$,*

$$\zeta_f(s, \chi) = \frac{\gamma_f}{\sqrt{|\alpha|}}\rho(s, \chi)\zeta_{\bar{f}\circ\alpha^{-1}}(1 - s, \bar{\chi}),$$

*where $\rho(s, \chi)$ is the local factor appearing in Tate's local functional equation.*

*Proof.* We know from Tate's thesis that if $\varphi$ is a Schwartz function, then $\operatorname{Mell}(\varphi, s, \chi) = \rho(s, \chi) \operatorname{Mell}(\mathfrak{F}(\varphi), 1-s, \bar{\chi})$ for $0 < \Re(s) < 1$, so that

$$\operatorname{Mell}(\lambda(\phi)(f), s, \chi) = \rho(s, \chi) \operatorname{Mell}\left(\frac{\gamma_f}{\sqrt{|\alpha|}} \lambda(\phi^*)(\bar{f} \circ \alpha^{-1})(y), 1-s, \bar{\chi}\right).$$

Using the definition of weak Mellin transforms, we find that for any $\phi \in C_c^\times(L^\times)$,

$$\operatorname{Mell}(\phi, s, \chi)\zeta_f(s, \chi) = \rho(s, \chi) \frac{\gamma_f}{\sqrt{|\alpha|}} \operatorname{Mell}(\phi, s, \chi)\zeta_{\bar{f}\circ\alpha^{-1}}(1-s, \bar{\chi}),$$

and we get the result by choosing $\phi$ so that $\operatorname{Mell}(\phi, s, \chi) \neq 0$. ∎

It is then immediate that $\zeta_f$ has an analytic continuation to $\Re(s) \leq 0$ with possible poles at the poles of $\rho(s, \chi)$. The previous formula is not really a functional equation since $\zeta_f$ and $\zeta_{\bar{f}\circ\alpha^{-1}}$ are not the same function. We can however get true functional equations from this under some additional hypothesis on $f$. For example, suppose that $f$ is of the form

$$f(x) = \psi\left(\frac{a}{2}x^2 + bx\right).$$

Then $\alpha(x) = ax$ and

$$\bar{f} \circ \alpha^{-1}(x) = \psi\left(-\frac{1}{2a}x^2 - \frac{b}{a}x\right) = \bar{f}\left(\frac{x}{a}\right),$$

so that

$$\zeta_{\bar{f}\circ\alpha^{-1}}(s, \chi) = |a|^s \chi(a) \bar{\zeta}_f(\bar{s}, \bar{\chi}),$$

which leads to the functional equation

$$\zeta_f(s, \chi) = \gamma_f \rho(s, \chi) |a|^{1/2-s} \bar{\chi}(a) \bar{\zeta}_f(1-\bar{s}, \chi). \tag{3.2}$$

One can also consider an element $\sigma$ of the Galois group of $L$ satisfying $\sigma^2 = \mathrm{Id}$ and a function $f$ of the form

$$f(x) = \psi\left(\frac{a}{2}\sigma(x)x + bx\right)$$

with $\sigma(a) = a$ and $\sigma(b) = b$. Since $\psi(\sigma(y)) = \psi(y)$ (because that of $\sigma(y)$ is equal to that of $y$), we get $f(x+y)\bar{f}(x)\bar{f}(y) = \psi(a\sigma(x)y)$, so that we can take $\alpha(x) = a\sigma(x)$ and $\alpha^{-1}(x) = \sigma^{-1}(x/a)$ to obtain

$$\begin{aligned}\bar{f} \circ \alpha^{-1}(x) &= \bar{\psi}\left(\frac{a}{2}\,\frac{x}{a}\sigma^{-1}\left(\frac{x}{a}\right) + b\sigma^{-1}\left(\frac{x}{a}\right)\right) \\ &= \bar{\psi}\left(\frac{1}{2a}\sigma(x)x + \sigma(b)\frac{x}{a}\right) = \bar{f}\left(\frac{x}{a}\right),\end{aligned}$$

which leads to the same functional equation.

**3.5. The weak Mellin transform of $\psi_p(x^2/2)$ on $\mathbb{Q}_p$.** Let us now prove the results given in the introduction for the Mellin transform of $\psi_p(x^2/2)$ on $\mathbb{Q}_p$ for $p$ a rational prime:

PROPOSITION 3.6. *The weak Mellin transform of $\psi_p(x^2/2)$ at the character $|x|^s$ is equal to:*

- $\frac{1}{1-p^{-s}}$ *if* $p \neq 2$,
- $\frac{1}{1-2^{-s}}(2^{1-s} - 1 + e^{\pi i/4}(2^s - 1))$ *if* $p = 2$.

*Proof.* We take $\mathbf{1}_{\mathbb{Z}_p^\times}$ as a test function, note that its Mellin transform at any unramified character is equal to 1 for any value of $s$, compute explicitly $\lambda(\mathbf{1}_{\mathbb{Z}_p^\times})(f)$ and take the Mellin transform of the result:

For $p \neq 2$, consider the integral

$$\lambda(\mathbf{1}_{\mathbb{Z}_p^\times})f(y) = \int_{\mathbb{Q}_p^\times} \mathbf{1}_{\mathbb{Z}_p^\times}(x)\psi_p\left(\frac{1}{2}\frac{y^2}{x^2}\right) d^\times x = \int_{\mathbb{Q}_p^\times} \mathbf{1}_{\mathbb{Z}_p^\times}(x)\psi_p\left(\frac{1}{2}y^2x^2\right) d^\times x.$$

If the valuation of $y$ is positive or zero, it is immediate that the result is equal to 1 because $\psi_p$ is equal to 1 on $\mathbb{Z}_p$. If the valuation of $y$ is negative, we replace the integral on $\mathbb{Q}_p^\times$ by an integral on $\mathbb{Q}_p$ and use the local Weil formula (Proposition 2.1) to see that

$$\begin{aligned}\lambda(\mathbf{1}_{\mathbb{Z}_p^\times})f(y) &= \frac{1}{1-1/p}\int_{\mathbb{Q}_p} \psi_p\left(\frac{1}{2}y^2x^2\right)(\mathbf{1}_{\mathbb{Z}_p}(x) - \mathbf{1}_{p\mathbb{Z}_p}(x))\,dx \\ &= \frac{1}{1-1/p}\frac{\gamma_f}{|y|}\int_{\mathbb{Q}_p} \psi_p\left(-\frac{1}{2}\frac{x^2}{y^2}\right)\left(\mathbf{1}_{\mathbb{Z}_p}(x) - \frac{1}{p}\mathbf{1}_{\frac{1}{p}\mathbb{Z}_p}(x)\right) dx.\end{aligned}$$

The restriction of the function $\psi_p\left(-\frac{1}{2}\frac{x^2}{y^2}\right)$ to $\mathbb{Z}_p$ and $\frac{1}{p}\mathbb{Z}_p$ is equal to 1 because the valuation of $y$ is negative. The integral is then equal to zero. Hence

$$\lambda(\mathbf{1}_{\mathbb{Z}_p^\times})f = \mathbf{1}_{\mathbb{Z}_p},$$

so that, taking the Mellin transform, we get

$$\zeta_f(s) = \frac{1}{1-p^{-s}}.$$

Let us now consider the case $p = 2$. We have to compute the integral

$$\lambda(\mathbf{1}_{\mathbb{Z}_2^\times})f(y) = \int_{\mathbb{Q}_2^\times} \psi_2\left(\frac{1}{2}y^2x^2\right)\mathbf{1}_{\mathbb{Z}_2^\times}(x)\,d^\times x.$$

If the valuation of $y$ is $\geq 1$, the result is 1 because the function $\psi_2\left(\frac{1}{2}y^2x^2\right)$ remains equal to 1. Suppose now that the valuation of $y$ is zero. We can suppose that $y = 1$ because $\lambda(\mathbf{1}_{\mathbb{Z}_p^\times})(f)$ is clearly unramified (i.e. invariant under the action of $\mathbb{Z}_p^\times$). We thus have to compute the integral

$$\int\limits_{\mathbb{Z}_2^\times} \psi_2(x^2/2)\, d^\times x.$$

For $x \in \mathbb{Z}_2^\times$, we can write $x = 1 + 2z$ with $z \in \mathbb{Z}_2$, and it is immediate that $\psi(x^2/2) = \psi(1/2) = e^{\pi i} = -1$, so that the integral is $-1$. Now suppose that the valuation of $y$ is $-1$, for example $y = 1/2$. We then have to compute the integral

$$\int\limits_{\mathbb{Z}_2^\times} \psi_2(x^2/8)\, d^\times x.$$

Any element of $\mathbb{Z}_2^\times$ can be written as $x = k + 4z$ with $z \in \mathbb{Z}_2$ and $k$ equal to 1 or 3. We have

$$\psi_2(x^2/8) = \psi_2(k^2/8) = \psi_2(1/8) = e^{\pi i/4},$$

so that the integral is $e^{\pi i/4}$. If the valuation of $y$ is $-2$ or less, we use the local Weil formula in the same way as for $p \neq 2$ to find that the result is zero. We can then write

$$\lambda(\mathbf{1}_{\mathbb{Z}_2^\times})f = \mathbf{1}_{2\mathbb{Z}_2} - \mathbf{1}_{\mathbb{Z}_2^\times} + e^{\pi i/4}\mathbf{1}_{\frac{1}{2}\mathbb{Z}_2^\times}.$$

Taking the Mellin transform, we get

$$\zeta_f(s) = \frac{2^{-s}}{1-2^{-s}} - 1 + e^{\pi i/4}2^s = \frac{1}{1-2^{-s}}\big(2^{1-s}(1-2^{s-1}) + e^{\pi i/4}2^s(1-2^{-s})\big). \blacksquare$$

### 3.6. The value of $\zeta_f(1)$

Proposition 3.7. *Let $f$ be a non-degenerate second degree character defined on a locally compact field $L$, and $\alpha$ the associated endomorphism. On $\mathbb{Q}_\mathfrak{p}$ we have*

$$\zeta_f(1) = \frac{1}{1 - 1/\mathcal{N}\mathfrak{p}} \frac{\gamma_f}{\sqrt{|\alpha|}},$$

*and on $\mathbb{R}$ and $\mathbb{C}$,*

$$\zeta_f(1) = \gamma_f/\sqrt{|\alpha|}.$$

Remark: 1 is then never a zero of $\zeta_f(s)$.

*Proof.* Consider $L = \mathbb{Q}_\mathfrak{p}$, some test function $\phi$ and compute

$$\begin{aligned}\mathrm{Mell}(\lambda(\phi)f, 1) &= \int\limits_{\mathbb{Q}_\mathfrak{p}} \lambda(\phi)f(x)|x|\, d^\times x \\ &= \frac{1}{1-1/\mathcal{N}\mathfrak{p}} \int\limits_{\mathbb{Q}_\mathfrak{p}} \lambda(\phi)f(x)\, dx = \frac{1}{1-1/\mathcal{N}\mathfrak{p}}\, \mathfrak{F}(\lambda(\phi)f)(0).\end{aligned}$$

Using Proposition 3.4, we get

$$\mathrm{Mell}(\lambda(\phi)f,1)=\frac{1}{1-1/\mathcal{N}\mathfrak{p}}\,\frac{\gamma_f}{\sqrt{|\alpha|}}\lambda(\phi^*)(\bar{f}\circ\alpha^{-1})(0).$$

As we have seen that $\bar{f}\circ\alpha^{-1}(0)=1$, we get

$$\mathrm{Mell}(\lambda(\phi)f,1)=\frac{1}{1-1/\mathcal{N}\mathfrak{p}}\,\frac{\gamma_f}{\sqrt{|\alpha|}}\int\limits_{\mathbb{Q}_\mathfrak{p}^\times}\frac{1}{|x|}\phi\left(\frac{1}{x}\right)d^\times x$$
$$=\frac{1}{1-1/\mathcal{N}\mathfrak{p}}\,\frac{\gamma_f}{\sqrt{\alpha}}\,\mathrm{Mell}(\phi,1).$$

The proof is the same for $\mathbb{R}$ and $\mathbb{C}$. ■

The Weil indices associated to second degree characters of the form $\psi(ax^2)$ are explicitly described in [12] for all locally compact fields. For example, the Weil index of the function $\psi_\mathbb{R}(x^2/2)=e^{-\pi i x^2}$ is equal to $e^{-\pi i/4}$. Let $\gamma_a$ be the Weil index of the second degree character $\psi\big(\frac{a}{2}x^2\big)$, and consider more general second degree characters:

PROPOSITION 3.8. *The Weil index of the second degree character $f(x)=\psi\big(\frac{a}{2}x^2+bx\big)$ is equal to $\gamma_f=\gamma_a\psi(-b^2/(2a))$.*

*Proof.* We consider a Schwartz function $\phi$ and the integral

$$\int\limits_L f(x)\phi(x)\,dx=\int\limits_L\psi\left(\frac{a}{2}x^2+bx\right)\phi(x)\,dx.$$

Let $\phi_b(x)=\phi(x)\psi(bx)$. This is again a Schwartz function. We use the Weil local functional equation associated to the second degree character $\psi\big(\frac{a}{2}x^2\big)$:

$$\int\limits_L f(x)\phi(x)\,dx=\int\limits_L\psi\left(\frac{a}{2}x^2\right)\phi_b(x)\,dx$$
$$=\frac{\gamma_a}{\sqrt{|a|}}\int\limits_L\psi\left(-\frac{1}{2a}x^2\right)\mathfrak{F}(\phi)(x+b)\,dx.$$

Writing $-y=x+b$, we get

$$\int\limits_L f(x)\phi(x)\,dx=\frac{\gamma_a}{\sqrt{|a|}}\psi\left(-\frac{b^2}{2a}\right)\int\limits_L\bar{\psi}\left(\frac{a}{2}\left(\frac{y}{a}\right)^2+b\frac{y}{a}\right)\mathfrak{F}(\phi)(-y)\,dy.\ ■$$

**3.7. The function $\zeta_{a,b}(s)$ considered as a function of $b$.** Consider a family of second degree characters of the form $f_{a,b}(x)=\psi\big(\frac{1}{2}ax^2+bx\big)$, and let $\zeta_{a,b}(s)$ be the weak Mellin transform at $s$ of $f_{a,b}$. In this subsection we consider $\zeta_{a,b}(s)$ as a function of $b$. Let $D_s$ denote the distribution on $S(L)$

defined for $\Re(s) > 0$ by

$$\langle D_s, \phi \rangle = \int_{L^\times} \phi(x)|x|^s \, d^\times x.$$

We have the following alternative definition of $\zeta_{a,b}(s)$:

PROPOSITION 3.9. *The function $\zeta_{a,b}(s)$ considered as a function of $b$ (or, more precisely, as a distribution in the variable $b$) is equal to the weak Fourier transform of the distribution $\psi\big(\frac{a}{2}x^2\big)D_s$.*

Remark: The Fourier transform and the map $\varphi \mapsto \psi\big(\frac{a}{2}x^2\big)\varphi$ both belong to a group of unitary operators called the metaplectic group (cf. [16]). The function $\zeta_{a,b}(s)$ considered as a function of $b$ is then the image of $D_s$ under the action of a metaplectic operator. As $D_s$ can be defined, up to a scalar factor, by being an eigendistribution for the dilation group (cf. [17]), which is also a subgroup of the metaplectic group, we see that $\zeta_{a,b}(s)$, considered as a function of $b$, can also be defined, up to a scalar factor, as an eigendistribution for a subgroup of the metaplectic group conjugate to the dilation group. Note also that a consequence of this proposition is that $\zeta_{a,b}(s)$ considered as a function of $b$ is never a square integrable function, and never the zero function.

*Proof.* We have to prove that if $\varphi$ is a Schwartz function on $L$, then

$$\int_{b\in L} \zeta_{a,b}(s)\varphi(b)\, db = \int_{y\in L^\times} \psi\left(\frac{a}{2}y^2\right) \mathfrak{F}(\varphi)(y)|y|^s \, d^\times y.$$

The right hand side is simply the Mellin transform of the Schwartz function $\psi\big(\frac{a}{2}y^2\big)\,\mathfrak{F}(\varphi)(y)$ at $s$.

Choose $\phi \in C_c^\infty(L^\times)$ so that $\mathrm{Mell}(\phi, s) \neq 0$, and consider the product

$$\begin{aligned}\Big(\int_{b\in L} \zeta_{a,b}(s)\varphi(b)\, db\Big) \mathrm{Mell}(\phi, s) &= \int_{b\in L} \mathrm{Mell}(\lambda(\phi) f_{a,b}, s)\varphi(b)\, db \\ &= \int_{b\in L} \Big(\int_{x\in L^\times} \lambda(\phi) f_{a,b}(x)|x|^s \, d^\times x\Big)\varphi(b)\, db.\end{aligned}$$

We remark that this double integral is absolutely convergent for $0 < \Re(s) < 1$ since

$$|\lambda(\phi) f_{a,b}(y)| \le K,$$

and, using the Weil formula,

$$|\lambda(\phi) f_{a,b}(y)| \le K'/|y|,$$

where $K$ and $K'$ do not depend on $b$. As a consequence, we can exchange the integration signs and get

$$\int_{x\in L^\times} \Big(\int_{b\in L} \lambda(\phi) f_{a,b}(x)\varphi(b)\, db\Big)|x|^s \, d^\times x. \tag{3.3}$$

We now remark that the inner integral can be written as

$$\int\limits_{b\in L}\int\limits_{y\in L^\times}\phi(y)\psi\Big(\frac{1}{2}a(y^{-1}x)^2+by^{-1}x\Big)\varphi(b)\,d^\times y\,db$$
$$=\int\limits_{y\in L^\times}\phi(y)\psi\Big(\frac{a}{2}(y^{-1}x)^2\Big)\,\mathfrak{F}(\varphi)(y^{-1}x)\,d^\times y=\lambda(\phi)\Big(\psi\Big(\frac{a}{2}x^2\Big)\,\mathfrak{F}(\varphi)(x)\Big)(x).$$

If we insert this in (3.3), we get as expected

$$\operatorname{Mell}(\phi,s)\operatorname{Mell}\Big(\psi\Big(\frac{a}{2}x^2\Big)\,\mathfrak{F}(\varphi)(x)\Big),s\Big).$$

If $\Re(s)\notin\,]0,1[$, we use the unicity of the analytic continuation since both expressions are analytic in $s$. ∎

**3.8. The zeroes of $\zeta_f(s,\chi)$ on a local field.** In this section, we will study the zeroes of $\zeta_f(s,\chi)$ on a local field $L=\mathbb{Q}_\mathfrak{p}$. We split the study into two parts: $\chi=1$ on the unit group (unramified character) and $\chi\neq 1$ on the unit group.

**3.8.1.** *Unramified character.* Consider a local field $\mathbb{Q}_\mathfrak{p}$ and an unramified character on $\mathbb{Q}_\mathfrak{p}$, i.e. a character of the form $|x|^s$.

Let $\varpi$ be a uniformizer, $\mathcal{O}$ the ring of integers, $\mathcal{O}^\times$ the group of units, and $q=\mathcal{N}\mathfrak{p}=|\varpi|^{-1}$. We also denote by $\mathfrak{d}$ the different ideal and define $d$ by the formula $\mathfrak{d}=\mathfrak{p}^d$ so that $\mathcal{N}\mathfrak{d}=q^d$.

Theorem 3.10. *Let $f$ be a non-degenerate second degree character on $\mathbb{Q}_\mathfrak{p}$ of the form $f(x)=\psi\big(\frac{1}{2}ax^2+bx\big)$. Then all the zeroes of $\zeta_f(s)$ lie on the line $\Re(s)=1/2$.*

*Proof.* Since the zeroes of $\zeta_f$ do not change if we replace $f(x)$ by $f(cx)$ with $c\neq 0$, we can suppose that $|a|$ is $q^d$ or $q^{d-1}$.

It is immediate that $\operatorname{Mell}(\mathbf{1}_{\mathcal{O}^\times},s,\chi)$ is equal to the measure of $\mathcal{O}^\times$, i.e. $(\mathcal{N}\mathfrak{d})^{-1/2}$, so that

$$\zeta_f(s,\chi)=(\mathcal{N}\mathfrak{d})^{1/2}\operatorname{Mell}(\lambda(\mathbf{1}_{\mathcal{O}^\times})f,s).$$

Let us then compute

$$\lambda(\mathbf{1}_{\mathcal{O}^\times})f(y)=\int\limits_{\mathbb{Q}_\mathfrak{p}^\times}\mathbf{1}_{\mathcal{O}^\times}(x)f(y/x)\,d^\times x.$$

Writing $z=1/x$, we get

$$\lambda(\mathbf{1}_{\mathcal{O}^\times})f(y)=\frac{1}{1-1/q}\int\limits_{\mathbb{Q}_\mathfrak{p}}\mathbf{1}_{\mathcal{O}^\times}(z)f(yz)\,dz.$$

An element of $\mathbb{Q}_\mathfrak{p}$ is in $\mathcal{O}^\times$ if and only if it is in $\mathcal{O}$ but not in $\varpi\mathcal{O}$. Hence

$$\lambda(\mathbf{1}_{\mathcal{O}^\times})f(y) = \frac{1}{1-1/q}\int_{\mathbb{Q}_\mathfrak{p}} f(yz)\big(\mathbf{1}_{\mathcal{O}}(z) - \mathbf{1}_{\mathcal{O}}(z/\varpi)\big)\,dz$$
$$= \frac{1}{1-1/q}\int_{\mathbb{Q}_\mathfrak{p}} \left(f(yz) - \frac{1}{q}f(\varpi yz)\right)\mathbf{1}_{\mathcal{O}}(z)\,dz.$$

Define

$$\theta_f(y) = \int_{\mathbb{Q}_\mathfrak{p}} f(yx)\mathbf{1}_{\mathcal{O}}(x)\,dx.$$

Then

$$\lambda(\mathbf{1}_{\mathcal{O}^\times})f(y) = \frac{1}{1-1/q}\left(\theta_f(y) - \frac{1}{q}\theta_f(y\varpi)\right),$$

so that if the Mellin transform of $\theta$ is well defined at some s with $\Re(s) > 0$, we have

$$\operatorname{Mell}(\lambda(\mathbf{1}_{\mathcal{O}^\times})f, s) = \frac{1}{1-1/q}(1-q^{s-1})\operatorname{Mell}(\theta_f, s).$$

As 1 is never a zero of $\zeta_f$, we see that the zeroes of $\zeta_f$ are the same as the zeroes of $\operatorname{Mell}(\theta_f, s)$.

Let us now give an explicit description of $\theta_f$ and $\operatorname{Mell}(\theta_f, s)$. We know by the definition of the local different and our choice of the Haar measure that the Fourier transform of $\mathbf{1}_{\mathcal{O}}$ is $(\mathcal{N}\mathfrak{d})^{-1/2}\mathbf{1}_{\mathfrak{d}^{-1}}$, so that using the local Weil formula (Proposition 2.1) we get

$$\theta_f(y) = \frac{\gamma_f}{\sqrt{|a|}}\frac{1}{|y|}\int_{\mathbb{Q}_\mathfrak{p}} \bar{f}\left(\frac{x}{ay}\right)(\mathcal{N}\mathfrak{d})^{-1/2}\mathbf{1}_{\mathfrak{d}^{-1}}(x)\,dx.$$

We write $z = x\varpi^d$ so that $dz = (\mathcal{N}\mathfrak{d})^{-1}\,dx$. Hence

$$\theta_f(y) = (\mathcal{N}\mathfrak{d})^{1/2}\frac{\gamma_f}{\sqrt{|a|}}\frac{1}{|y|}\int_{\mathbb{Q}_\mathfrak{p}} \bar{f}\left(\frac{z}{ya\varpi^d}\right)\mathbf{1}_{\mathcal{O}}(z)\,dz.$$

We then have the functional equation

$$\theta_f(y) = (\mathcal{N}\mathfrak{d})^{1/2}\frac{\gamma_f}{\sqrt{|a|}}\frac{1}{|y|}\bar{\theta}_f\left(\frac{1}{ya\varpi^d}\right).$$

Now first suppose that the valuation of $a$ is $-d$ so that $|a| = q^d = \mathcal{N}\mathfrak{d}$. The functional equation becomes, since $\theta_f$ is unramified (i.e. invariant under the action of $\mathcal{O}^\times$),

$$\theta_f(y) = \gamma_f\frac{1}{|y|}\bar{\theta}_f\left(\frac{1}{y}\right).$$

In order to compute $\theta_f$, we can then suppose that the valuation of $y$ is $\geq 0$. The function $g(x) = f(yx)$, if restricted to $x \in \mathcal{O}$, is an additive character:

If $x$ and $z$ are in $\mathcal{O}$, we have

$$g(x+z)\bar{g}(x)\bar{g}(z) = f(yx+yz)\bar{f}(yx)\bar{f}(yz) = \psi(ay^2xz) = 1.$$

Indeed, $|ay^2xz| \le |a| = q^d$, which shows that $ay^2xz \in \mathfrak{d}^{-1}$. By the definition of $\theta_f$ and the fact that $\mathcal{O}$ is an additive group, we find that $\theta_f(y) = (\mathcal{N}\mathfrak{d})^{-1/2}$ if and only if $f(xy) = 1$ for all $x \in \mathcal{O}$, because the integral of an additive character on a compact abelian group is zero if the character is not trivial on the group, or the measure of the group if the character is trivial. It is then immediate that if $\theta_f(y) = (\mathcal{N}\mathfrak{d})^{-1/2}$, then $\theta_f(z) = (\mathcal{N}\mathfrak{d})^{-1/2}$ for all $z$ having a higher valuation than $y$. We can thus write the restriction of $\theta$ to $\mathcal{O}$ as $(\mathcal{N}\mathfrak{d})^{-1/2}\mathbf{1}_{\varpi^k\mathcal{O}}(y)$ for some $k \ge 0$. Using the functional equation, we see that if the valuation of $y$ is $\le 0$, we have

$$\theta_f(y) = \gamma_f \frac{1}{|y|}(\mathcal{N}\mathfrak{d})^{-1/2}\mathbf{1}_{\varpi^k\mathcal{O}}\left(\frac{1}{y}\right).$$

Suppose first that $k = 0$. Then $\gamma_f = 1$ (by the functional equation with $y = 1$). Avoiding double counting for $|y| = 1$, we thus have the following description of $\theta$:

$$\theta_f(y) = (\mathcal{N}\mathfrak{d})^{-1/2}\left(\mathbf{1}_{\mathcal{O}}(y) + (1-\mathbf{1}_{\mathcal{O}}(y))\frac{1}{|y|}\right).$$

We see that the Mellin transform of $\theta_f$ is well defined for $0 < \Re(s) < 1$, and compute that

$$\mathrm{Mell}(\theta_f, s) = (\mathcal{N}\mathfrak{d})^{-1}\left(\frac{1}{1-q^{-s}} + \frac{q^{s-1}}{1-q^{s-1}}\right) = \frac{(\mathcal{N}\mathfrak{d})^{-1}}{(1-q^{-s})(1-q^{s-1})}\left(1-\frac{1}{q}\right).$$

Hence

$$\begin{aligned}\zeta_f(s,\chi) &= (\mathcal{N}\mathfrak{d})^{1/2}\,\mathrm{Mell}(\lambda(\mathbf{1}_{\mathcal{O}^\times})f, s)\\ &= (\mathcal{N}\mathfrak{d})^{1/2}\frac{1}{1-1/q}(1-q^{s-1})\,\mathrm{Mell}(\theta_f, s) = (\mathcal{N}\mathfrak{d})^{-1/2}\frac{1}{1-q^{-s}},\end{aligned}$$

and there is no zero.

Now suppose that $k \ge 1$. Then

$$\theta_f(y) = (\mathcal{N}\mathfrak{d})^{-1/2}\left(\mathbf{1}_{\varpi^k\mathcal{O}}(y) + \gamma_f\frac{1}{|y|}\mathbf{1}_{\varpi^k\mathcal{O}}\left(\frac{1}{y}\right)\right),$$

and the Mellin transform is

$$\mathrm{Mell}(\theta_f, s) = (\mathcal{N}\mathfrak{d})^{-1}\left(\frac{q^{-ks}}{1-q^{-s}} + \gamma_f\frac{q^{k(s-1)}}{1-q^{s-1}}\right).$$

The zeroes of $\zeta_f(s)$ are then the roots of the equation

$$q^{-ks}(1-q^{s-1}) + \gamma_f q^{k(s-1)}(1-q^{-s}) = 0.$$

Let us write $X = q^{s-1/2}$ so that $|X| = 1$ if and only if $\Re(s) = 1/2$. The equation becomes

$$X^{-k}\left(1 - \frac{X}{\sqrt{q}}\right) + \gamma_f X^k \left(1 - \frac{1}{X}\frac{1}{\sqrt{q}}\right) = 0,$$

or

$$\gamma_f X^{2k} - \frac{\gamma_f}{\sqrt{q}} X^{2k-1} - \frac{X}{\sqrt{q}} + 1 = 0.$$

The number of solutions of this polynomial equation cannot be greater than $2k$. Let us show that we have exactly $2k$ solutions on the unit circle: we write $X = e^{i\phi}$, so that if we choose some square root of $\gamma_f$, the equation becomes, after multiplication by $\frac{1}{\sqrt{\gamma_f}} e^{-ik\phi}$,

$$\sqrt{\gamma_f} e^{ik\phi} - \frac{\sqrt{\gamma_f}}{\sqrt{q}} e^{i(k-1)\phi} - \frac{\overline{\sqrt{\gamma_f}}}{\sqrt{q}} e^{i(1-k)\phi} + \overline{\sqrt{\gamma_f}} e^{-ik\theta} = 0.$$

This expression is twice the real part of $\sqrt{\gamma_f} e^{ik\phi} - \frac{\sqrt{\gamma_f}}{\sqrt{q}} e^{i(k-1)\phi}$, and it is clear (because $q > 1$) that when $\phi$ moves from zero to $2\pi$, this real expression has $2k$ sign changes, so that it has $2k$ distinct zeroes, which completes the proof for the case $|a| = q^d$.

Now suppose that $|a| = q^{d-1}$. The functional equation of $\theta$ becomes

$$\theta(y) = \gamma_f \sqrt{q} \frac{1}{|y|} \bar{\theta}\left(\frac{1}{\varpi y}\right).$$

The same argument shows that there exists $k \geq 0$ such that for $y \in \mathcal{O}$,

$$\theta(y) = (\mathcal{N}\mathfrak{d})^{-1/2} \mathbf{1}_{\varpi^k \mathcal{O}}(y).$$

Using the functional equation, we get the following expression for $\theta$, valid for any $k \geq 0$:

$$\theta(y) = (\mathcal{N}\mathfrak{d})^{-1/2} \left(\mathbf{1}_{\varpi^k \mathcal{O}}(y) + \gamma_f \sqrt{q} \frac{1}{|y|} \mathbf{1}_{\varpi^k}\left(\frac{1}{\varpi y}\right)\right).$$

Hence the Mellin transform of $\theta$ is well defined for $0 < \Re(s) < 1$, and we have

$$\mathrm{Mell}(\theta, s) = (\mathcal{N}\mathfrak{d})^{-1} \left(\frac{q^{-ks}}{1 - q^{-s}} + \frac{\gamma_f}{\sqrt{q}} q^s \frac{q^{k(1-s)}}{1 - q^{s-1}}\right).$$

A similar argument proves that all the roots of this equation satisfy $\Re(s) = 1/2$. ■

**3.8.2.** *Ramified characters.* We consider a ramified character $\chi$ on the unit group of $\mathbb{Q}_{\mathfrak{p}}^{\times}$ and extend it to a character of $\mathbb{Q}_{\mathfrak{p}}$ by writing $\chi(\varpi) = 1$ (cf. Tate's thesis). We then have the following theorem:

THEOREM 3.11. *Let $f$ be a non-degenerate second degree character on $\mathbb{Q}_\mathfrak{p}$ of the form $\psi\big(\frac{a}{2}x^2+bx\big)$ and assume that $\zeta_f(s,\chi)$ is not identically zero as a function of $s$. Then all the zeroes of $\zeta_f(s,\chi)$ lie on the line $\Re(s)=1/2$.*

Remark: It is clear that if $\chi$ is odd, then the weak Mellin transform of $\psi\big(\frac{a}{2}x^2\big)$ at $(s,\chi)$ is always equal to zero.

*Proof of Theorem 3.11.* In order to compute $\zeta_f(s,\chi)$, we will use the test function $\phi_{\bar\chi}(x)=\bar\chi(x)\mathbf{1}_{\mathcal{O}^\times}(x)$ which satisfies $\operatorname{Mell}(\phi_{\bar\chi},s,\chi)=(\mathcal{N}\mathfrak{d})^{-1/2}$ for all $s$. We denote by $\mathfrak{f}$ the conductor of $\chi$ and define $n$ so that $\mathfrak{f}=\mathfrak{p}^n$.

We recall from Tate's thesis [14, p. 322] that if $\chi$ is ramified, the local factor $\rho(\chi,s)$ appearing in Tate's local functional equation is given by

$$\rho(s,\chi)=(\mathcal{N}(\mathfrak{f}\mathfrak{d}))^{s-1/2}\rho_0(\chi),$$

where the term $\rho_0(\chi)$ satisfies $|\rho_0(\chi)|=1$ and is given by

$$\rho_0(\chi)=(\mathcal{N}\mathfrak{f})^{-1/2}\sum_{\epsilon}\chi(\epsilon)\psi(\epsilon/\varpi^{d+n}),$$

where $\{\epsilon\}$ is a set of representatives of the cosets of $1+\mathfrak{f}$ in $\mathcal{O}^\times$.

PROPOSITION 3.12. *The Fourier transform of $\phi_\chi(x)=\chi(x)\mathbf{1}_{\mathcal{O}^\times}(x)$ is equal to $\chi(-1)\rho_0(\chi)q^{-(n+d)/2}\phi_{\bar\chi}(\varpi^{n+d}x)$.*

*Proof.* Since $\mathcal{O}^\times$ is a compact multiplicative group, the Mellin transform of $\phi_\chi$ at the character $\chi'$ is equal to zero if $\chi'\neq\bar\chi$ on $\mathcal{O}^\times$, and to $(\mathcal{N}\mathfrak{d})^{-1/2}$ if $\chi'=\bar\chi$ on $\mathcal{O}^\times$. Using Tate's local functional equation, we then see that the Mellin transform of $\mathfrak{F}(\phi_\chi)$ at $\chi'(x)|x|^s$ is zero for $\chi'\neq\chi$. For $\chi'=\chi$, we get

$$\begin{aligned}\operatorname{Mell}(\mathfrak{F}(\phi_\chi),\chi,s)&=\rho(s,\chi)\operatorname{Mell}(\phi_\chi(-x),\bar\chi,1-s),\\&=(\mathcal{N}(\mathfrak{f}\mathfrak{d}))^{s-1/2}\rho_0(\chi)(\mathcal{N}\mathfrak{d})^{-1/2}\chi(-1),\\&=q^{(n+d)(s-1/2)}q^{-d/2}\rho_0(\chi)\chi(-1)=\chi(-1)\rho_0(\chi)q^{-d/2-(n+d)/2+s(n+d)}.\end{aligned}$$

Since the function $\chi(-1)\rho_0(\chi)q^{-(n+d)/2}\phi_{\bar\chi}(\varpi^{n+d}x)$ has the same Mellin transform as $\mathfrak{F}(\phi_\chi)$ for all multiplicative characters $\chi$, these two functions have to be equal on $\mathbb{Q}_\mathfrak{p}^\times$. The equality for $x=0$ is immediate. ■

Now consider a second degree character of the form

$$f(x)=\psi\left(\frac{a}{2}x^2+bx\right)$$

with $a\neq0$. We have

$$\lambda(\phi_{\bar\chi})f(y)=\int_{\mathbb{Q}_\mathfrak{p}^\times}f(y/x)\phi_{\bar\chi}(x)\,d^\times x.$$

Writing $z = 1/x$, we get

$$\lambda(\phi_{\bar{\chi}})f(y) = \frac{1}{1-1/q}\int_{\mathbb{Q}_\mathfrak{p}} f(yz)\phi_\chi(z)\,dz.$$

Using the local Weil formula and Proposition 3.12, we get

$$\lambda(\phi_{\bar{\chi}})f(y) \\ = \frac{1}{1-1/q}\frac{\gamma_f}{\sqrt{|a|}}\frac{1}{|y|}q^{-(n+d)/2}\chi(-1)\rho_0(\chi)\int_{\mathbb{Q}_\mathfrak{p}} \bar{f}\left(\frac{z}{ya}\right)\phi_{\bar{\chi}}(-\varpi^{n+d}z)\,dz.$$

With $x = \varpi^{n+d}z$, the expression becomes

$$\lambda(\phi_{\bar{\chi}})f(y) = \frac{1}{1-1/q}\frac{\gamma_f}{\sqrt{|a|}}\frac{1}{|y|}q^{(n+d)/2}\rho_0(\chi)\int_{\mathbb{Q}_\mathfrak{p}} \bar{f}\left(\frac{x}{ya\varpi^{n+d}}\right)\phi_{\bar{\chi}}(x)\,dx,$$

so that if we set $\theta_{f,\bar{\chi}}(y) = \lambda(\phi_{\bar{\chi}})f(y)$, we have the functional equation

$$\theta_{f,\bar{\chi}}(y) = \gamma_f\rho_0(\chi)\frac{q^{(n+d)/2}}{\sqrt{|a|}}\frac{1}{|y|}\bar{\theta}_{f,\bar{\chi}}\left(\frac{1}{ya\varpi^{n+d}}\right).$$

We can again suppose without loss of generality that the valuation of $a$ is $-n-d$ or $-n-d+1$.

We write $a = u\varpi^{-n-d+\delta}$ with $|u| = 1$ and $\delta$ equal to 0 or 1, so that $|a| = q^{n+d-\delta}$ and the functional equation becomes

$$\theta_{f,\bar{\chi}}(y) = \gamma_\chi\rho_0(\chi)\frac{q^{\delta/2}}{|y|}\bar{\theta}_{f,\bar{\chi}}\left(\frac{1}{uy\varpi^{\delta}}\right). \tag{3.4}$$

In order to compute $\theta_{f,\bar{\chi}}(y)$, we can thus suppose that the valuation of $y$ is $\geq 0$. Considering the integral

$$\theta_{f,\bar{\chi}}(y) = \frac{1}{1-1/q}\int_{\mathcal{O}_\mathfrak{p}^\times} f(yz)\chi(z)\,dz,$$

we split $\mathcal{O}_\mathfrak{p}^\times$ into cosets modulo the subgroup $1+\mathfrak{f}$, so that $\chi$ is constant on each coset, and choose a representative $z_i$ of each coset. Hence

$$\theta_{f,\bar{\chi}}(y) = \frac{1}{1-1/q}\sum_i\int_{\mathfrak{f}} f(yz_i(1+x))\chi(z_i)\,dx.$$

As $f$ is a second degree character, for $x \in \mathfrak{f}$ we have

$$f(yz_i + yz_ix)\bar{f}(yz_i)\bar{f}(yz_ix) = \psi(axy^2z_i^2) = 1.$$

Indeed, the valuation of $x$ is $\geq n$, so that the valuation of $ax$ is $\geq -d$, and the valuations of $y$ and $z_i$ are $\geq 0$. Therefore

$$\theta_{f,\bar{\chi}}(y) = \frac{1}{1-1/q}\sum_i\Big(f(yz_i)\chi(z_i)\int_{\mathfrak{f}} f(yz_ix)\,dx\Big).$$

The change of variable $x' = z_i x$ in the last integral shows that it is a constant as a function of $i$, so that

$$\theta_{f,\bar{\chi}}(y) = \frac{1}{1-1/q}\Big(\sum_i \chi(z_i) f(z_i y)\Big)\Big(\int_{\mathfrak{f}} f(yx)\,dx\Big).$$

We then remark that if $x_1$ and $x_2$ are in $\mathfrak{f}$, then the valuations of $x_1$ and $x_2$ are higher than $n$, so that the valuation of $ax_1$ is $\geq -d$ and we have again

$$f(x_1+x_2)\bar{f}(x_1)\bar{f}(x_2) = \psi(ax_1x_2) = 1.$$

Since the restriction of $f(x)$ to $\mathfrak{f}$ is then an additive character, the restriction of $f(yx)$ to $\mathfrak{f}$ is also an additive character if the valuation of $y$ is $\geq 0$. As a consequence, there exists a unique $k \in \mathbb{Z}$ such that $f(yx)$ is equal to 1 for all $x \in \mathfrak{f}$ if the valuation of $y$ is $\geq k$, and not constant on $\mathfrak{f}$ if the valuation of $y$ is $< k$. We then have $\theta_{f,\bar{\chi}}(y) = 0$ if the valuation of $y$ is $< k$. If $k < 0$, then $\theta_{f,\bar{\chi}}(y) = 0$ thanks to the functional equation and there is nothing else to prove.

We next suppose $k \geq 0$. We show that $\theta_{f,\bar{\chi}}(y)$ is also zero if the valuation of $y$ is $\geq k+1$.

Consider the sum $\sum_i \chi(z_i)f(z_iy)$. If $f(zy)$ is constant as a function of $z$ for $z \in \mathcal{O}^\times$, it is immediate that this sum is zero because $\chi$ is assumed to be ramified. More generally, we can compare $f(z_iy)$ and $f(z_jy)$ using the functional equation of the second degree character $f$:

$$f(yz_j)\bar{f}(y(z_j-z_i))\bar{f}(yz_i) = \psi(ay^2z_i(z_j-z_i)).$$

Now suppose that the valuation of $y$ is $\geq k+1$. Then the definition of $k$ shows that $f(y(z_i-z_j)) = 1$ if $z_i - z_j \in (1/\varpi)\mathfrak{f}$. Under the same conditions we also have $\psi(ay^2z_i(z_j-z_i)) = 1$: The valuation of $y^2$ is $\geq 2$, the valuation of $a$ is $\geq -n-d$, the valuation of $z_i$ is 0 and the valuation of $z_j - z_i$ is assumed to be $> n-1$, so that the valuation of $ay^2z_i(z_j-z_i)$ is $> -d$.

We see that if $z_i$ and $z_j$ are in the same coset modulo $1+(1/\varpi)\mathfrak{f}$, then $f(z_iy) = f(z_jy)$. Let us renumber the $z_i$, writing $z_i = z_{j,k}$ where $j$ indicates to which coset of $1+(1/\varpi)\mathfrak{f}$ it belongs. Then

$$\sum_i \chi(z_i)f(z_iy) = \sum_{j,k}\chi(z_{j,k})f(z_{j,k}y) = \sum_j \alpha_j\Big(\sum_k \chi(z_{j,k})\Big),$$

with $\alpha_j = f(z_{j,k}y)$ for any choice of $k$. However, the character $\chi$ is constant on the subgroup $1+\mathfrak{f}$ but not on $1+(1/\varpi)\mathfrak{f}$ (this is the definition of $\mathfrak{f}$), and for $j$ fixed, the set of $z_{j,k}$ is a full set of representatives of cosets modulo $1+\mathfrak{f}$ inside a coset modulo $1+\mathfrak{f}/\varpi$. The sum over $k$ is then equal, up to a scalar, to an integral of a non-constant multiplicative character on a compact subgroup, so it is zero.

We thus see that if the valuation of $y$ is $\geq k+1$, then $\theta_{f,\bar{\chi}}(y) = 0$, and if it is lower than $k$ and positive, then also $\theta_{f,\bar{\chi}}(y) = 0$.

As the restriction of $\theta_{f,\bar\chi}$ to $y \in \varpi^k\mathcal{O}^\times$ has to be of the form $C\bar\chi(y)$ for some constant $C = \theta_{f,\bar\chi}(\varpi^k)$ (because it is immediate from the definition of $\theta_{f,\bar\chi}$ that if $w$ is a unit, then $\theta_{f,\bar\chi}(wx) = \bar\chi(w)\theta_{f,\bar\chi}(x)$), if the valuation of $y$ is $\geq 0$ we can write

$$\theta_{f,\bar\chi}(y) = C\bar\chi(y)\mathbf{1}_{\varpi^k\mathcal{O}^\times}(y).$$

By the functional equation (3.4), for $y \in \mathbb{Q}_\mathfrak{p}$ we then have

$$\theta(y) = C\bar\chi(y)\mathbf{1}_{\varpi^k\mathcal{O}^\times}(y) + \gamma_f\bar\rho_0(\chi)\frac{q^{\delta/2}}{|y|}\chi\left(\frac{1}{uy}\right)\bar C\mathbf{1}_{\varpi^k\mathcal{O}^\times}\left(\frac{1}{uy\varpi^\delta}\right).$$

If $\frac{1}{uy\varpi^\delta}$ is in $\varpi^k\mathcal{O}^\times$, then $|y| = q^{k+\delta}$, so that also

$$\theta(y) = \bar\chi(y)C\left(\mathbf{1}_{\varpi^k\mathcal{O}^\times}(y) + \gamma_f\bar\rho_0(\chi)\bar\chi(u)\frac{\bar C}{C}q^{-k-\delta/2}\mathbf{1}_{\varpi^{-k-\delta}\mathcal{O}^\times}(y)\right).$$

Set $\omega = \gamma_f\bar\rho_0(\chi)\bar\chi(u)\bar C/C$. It is clear that $|\omega| = 1$ We get

$$\mathrm{Mell}(\theta, s, \chi) = (\mathcal{N}\mathfrak{d})^{-1/2}C(q^{-ks} + \omega q^{-k-\delta/2}q^{(k+\delta)s}),$$

and it is immediate that the zeroes of this function lie on the line $\Re(s) = 1/2$. ∎

**3.9. The function $\zeta_f(s,\chi)$ on $\mathbb{R}$.** If a second degree character $f$ is defined on $\mathbb{R}$, then the automorphism $\alpha$ associated to $f$ has to be $\mathbb{R}$-linear, so that it can be written as $\alpha(x) = ax$. Any non-degenerate second degree character $f$ on $\mathbb{R}$ is thus of the form

$$f(x) = \psi\left(\frac{a}{2}x^2 + bx\right) = e^{-2\pi i(\frac{a}{2}x^2+bx)}$$

with $a \in \mathbb{R}^\times$ and $b \in \mathbb{R}$, and the functional equation

$$\zeta_f(s,\chi) = \gamma_f\rho(s,\chi)|a|^{1/2-s}\bar\chi(a)\bar\zeta_f(1-\bar s,\chi)$$

is always valid. With this description of $f$, we introduce the notation $\zeta_f(s,\chi) = \zeta_{a,b}(s,\chi)$. We have only two unitary characters $\chi$ on the unit group: the identity and the sign function $\mathrm{sgn}(x)$, which we denote $\pm(x)$. Since $e^{-2\pi i(\frac{a}{2}x^2)}$ is even, we have $\zeta_{a,0}(s,\pm) = 0$ for all $s$. We have the following description of the weak Mellin transform of a second degree character:

PROPOSITION 3.13. *The weak Mellin transform of the second degree character $\psi\big(\frac{a}{2}x^2 + bx\big)$ at the character $|x|^s$ for $a > 0$ is*

$$\zeta_{a,b}(s) = \frac{e^{-s\pi i/4}}{\sqrt{a}^s}\frac{\Gamma(s/2)}{\pi^{s/2}}{}_1F_1(s/2, 1/2, \pi i b^2/a).$$

Remark: It can be checked that the functional equation of $\zeta_{a,b}$ is consistent with Kummer's formula

$$e^x{}_1F_1(a,b,-x) = {}_1F_1(b-a,b,x).$$

*Proof of Proposition 3.13.* We remark that the formula

$$\mathfrak{F}(e^{-2\pi i(\frac{1}{2}ax^2+bx)}) = \frac{e^{\pi i b^2/a}}{\sqrt{ai}} e^{2\pi i(\frac{1}{2}a(x/a)^2+bx/a)}$$

is valid not only for $a \in \mathbb{R}_+^*$ and $b \in \mathbb{R}$, but also for $a \in \mathbb{C}$ with $\Im(a) < 0$ and $\Re(a) > 0$ and $b \in \mathbb{C}$. We can then define $\zeta_{a,b}(s)$ using the same method for any $a \in \mathbb{C}$ with $\Im(a) < 0$, $\Re(a) > 0$ and $b \in \mathbb{C}$, and looking at the proof of Proposition 3.2, it is not difficult to see that $\zeta_{a,b}(s)$ is a continuous function of $a$ provided $a$ does not cross the lines $\Re(a) = 0$ or $\Im(a) = 0$. It is however not difficult to compute $\zeta_{a,0}(s)$ when $\Im(a) < 0$ and $\Re(a) > 0$: this is a regular Mellin transform, and we get

$$\zeta_{a,0}(s) = \frac{1}{\sqrt{a}^s} e^{-s\pi i/4} \frac{\Gamma(s/2)}{\pi^{s/2}}.$$

Thanks to the continuity of $\zeta_{a,0}(s)$ as a function of $a$, this formula is also valid for $a \in \mathbb{R}_+^*$.

Now suppose that $a$ is fixed in $\mathbb{R}_+^*$ and consider $\zeta_{a,b}$ as a function of $b$. We observe that $\zeta_{a,b}$ considered as a function of $b$ is the limit of the functions $\zeta_{a',b}$, which are analytic in $b \in \mathbb{C}$, when $a' \in \mathbb{C}$, $\Im(a') < 0$ converges to $a \in \mathbb{R}_+^*$, and that the convergence is uniform if $b$ stays in a compact set.

As a consequence, $\zeta_{a,b}$ is an analytic function of $b$ (as a uniform limit of complex analytic functions of $b$), so that we can describe it using its Taylor expansion at zero:

$$\zeta_{a,b}(s) = \sum_{k\geq 0} \frac{\partial^k}{\partial b^k} \zeta_{a,b}(s)\bigg|_{b=0} \frac{b^k}{k!}.$$

We remark that $\zeta_{a,b}(s)$ is an even function of $b$, so that all the odd derivatives at zero are zero. In order to evaluate the even derivatives, we use the following proposition:

PROPOSITION 3.14. *For $s$ fixed, $\zeta_{a,b}(s)$ satisfies the equations*

$$\begin{aligned}
&\frac{\partial}{\partial a}\zeta_f(s) = -\pi i \zeta_f(s+2),\\
&\frac{\partial^2}{\partial b^2}\zeta_f(s) = (-2\pi i)^2 \zeta_f(s+2),\\
&\frac{\partial}{\partial a}\zeta_{a,b}(s) + \frac{1}{4\pi i}\frac{\partial^2}{\partial b^2}\zeta_{a,b}(s) = 0.
\end{aligned}$$

*Proof.* The function $e^{-2\pi i(\frac{a}{2}x^2+bx)}$ satisfies the equations

$$\begin{aligned}
&\frac{\partial}{\partial a} e^{-2\pi i(\frac{a}{2}x^2+bx)} = (-\pi i)x^2 e^{-2\pi i(\frac{a}{2}x^2+bx)},\\
&\frac{\partial^2}{\partial b^2} e^{-2\pi i(\frac{a}{2}x^2+bx)} = (-2\pi i)^2 x^2 e^{-2\pi i(\frac{a}{2}x^2+bx)},
\end{aligned}$$

so that we mainly have to prove that we can exchange integration and differentation. Consider for example the first equation. We have to prove that the Mellin transform of $\frac{\partial}{\partial a}\lambda(\phi)e^{-2\pi i(\frac{a}{2}x^2+bx)}$ converges absolutely and find a uniform bound for the associated absolute integral. We remark that this expression is equal to

$$\lambda(\phi)\left(\frac{\partial}{\partial a}e^{-2\pi i(\frac{a}{2}x^2+bx)}\right).$$

The function $g(x) = \frac{\partial}{\partial a}e^{-2\pi i(\frac{a}{2}x^2+bx)} = -\pi i x^2 e^{-2\pi i(\frac{a}{2}x^2+bx)}$ is $C^\infty$; its Fourier transform is also $C^\infty$ and can be computed explicitly using the commutation relation for the Fourier transform and differential operators.

We can then use the proof of Proposition 3.2 in the real case, and show that the Mellin transform of $\lambda(\phi)g$ is well defined for $\Re(s) > 0$ with an absolute bound which remains finite if $a$ stays in a compact set in $\mathbb{R}^*$ which does not contain 0. As a consequence, the Mellin transform of the complete expression is also well defined for $\Re(s) > 0$, and we have a uniform bound for the integral defining the Mellin transform, which allows us to exchange differentiation and Mellin integration. ∎

Thus for $\Re(a) > 0$,

$$\begin{aligned}\frac{\partial^{2k}}{\partial b^{2k}}\zeta_{a,b}(s)\bigg|_{b=0} &= (-4\pi i)^k\frac{d^k}{da^k}\zeta_{a,b}(s)\bigg|_{b=0}\\ &= (-4\pi i)^k e^{-s\pi i/4}\frac{\Gamma(s/2)}{\pi^{s/2}}\frac{d}{da^k}\left(\frac{1}{\sqrt{a}^s}\right) = \frac{e^{-s\pi i/4}}{\sqrt{a}^s}\frac{\Gamma(s/2)}{\pi^{s/2}}\frac{(4\pi i)^k(s/2)_k}{a^k}.\end{aligned}$$

The Taylor expansion of $\zeta_{a,b}(s)$ becomes

$$\zeta_{a,b}(s) = \frac{e^{-s\pi i/4}}{\sqrt{a}^s}\frac{\Gamma(s/2)}{\pi^{s/2}}\sum_{k\geq 0}\left(\frac{s}{2}\right)_k\frac{(4\pi i)^k b^{2k}}{a^k(2k)!}.$$

Writing $(2k)! = (2\cdot 4\cdot 6\cdots 2k)(1\cdot 3\cdot 5\cdots(2k-1)) = (4^k)k!\left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right)\cdots\left(\frac{1}{2}+k-1\right) = 4^k k!\left(\frac{1}{2}\right)_k$, we recognize a Kummer confluent hypergeometric function ${}_1F_1$:

$$\begin{aligned}\zeta_{a,b}(s) &= \frac{e^{-s\pi i/4}}{\sqrt{a}^s}\frac{\Gamma(s/2)}{\pi^{s/2}}\sum_{k=0}^{\infty}\frac{(s/2)_k}{(k)!(1/2)_k}\left(\frac{\pi i b^2}{a}\right)^k\\ &= \frac{e^{-s\pi i/4}}{\sqrt{a}^s}\frac{\Gamma(s/2)}{\pi^{s/2}}\,{}_1F_1(s/2, 1/2, \pi i b^2/a).\ ∎\end{aligned}$$

Proposition 3.15. *The weak Mellin transform of* $\psi_{\mathbb{R}}\left(\frac{a}{2}x^2+bx\right)$ *at the character* $\operatorname{sgn}(x)|x|^s$ *is*

$$\zeta_{a,b}(s,\pm) = -2\pi i b\frac{e^{-(s+1)\pi i/4}}{\sqrt{a}^{s+1}}\frac{\Gamma((s+1)/2)}{\pi^{(s+1)/2}}\,{}_1F_1\left(\frac{s+1}{2},\frac{3}{2},\frac{\pi i b^2}{a}\right).$$

*Proof.* We start from the equality

$$\frac{\partial}{\partial b} e^{-2\pi i(\frac{a}{2}x^2+bx)} = -2\pi i x e^{-2\pi i(\frac{a}{2}x^2+bx)},$$

which leads, after exchanging integration and differentiation, to

$$\frac{\partial}{\partial b}\zeta_{a,b}(s) = -2\pi i \zeta_{a,b}(s+1,\pm),$$

so that

$$\begin{aligned}\zeta_{a,b}(s,\pm) &= -\frac{1}{2\pi i}\frac{\partial}{\partial b}\zeta_{a,b}(s-1)\\ &= -\frac{1}{2\pi i}\frac{e^{-(s-1)\pi i/4}}{\sqrt{a}^{s-1}}\frac{\Gamma((s-1)/2)}{\pi^{(s-1)/2}}\frac{\partial}{\partial b}\,{}_1F_1\left(\frac{s-1}{2},\frac{1}{2},\frac{\pi i b^2}{a}\right).\end{aligned}$$

We then use the elementary formula

$$\frac{\partial}{\partial z}\,{}_1F_1\,(\alpha,\beta,z) = \sum_{k\geq 0}\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\frac{z^k}{k!} = \frac{\alpha}{\beta}\,{}_1F_1(\alpha+1,\beta+1,z)$$

to get the result. ■

Let us now look at the location of the zeroes of $\zeta_{a,b}$.

PROPOSITION 3.16. *Suppose that* ${}_1F_1(u,v,z)=0$ *with* $z$ *imaginary and* $v\in\mathbb{R}_+^*$. *Then* $\Re(u)=v/2$.

*Proof.* This result is essentially due to H. Weber [15]. We include a short proof for completeness.

It is well known (cf. [1] for example) that ${}_1F_1(u,v,z)$ considered as a function of $z$ is a solution of the confluent hypergeometric equation

$$zf''(z)+(v-z)f'(z)-uf=0.$$

We can suppose without loss of generality that the zero of ${}_1F_1(u,v,z)$ has a positive imaginary part, so that we can write it as $z=it_0^2$ with $t_0\in\mathbb{R}$.

For $t>0$ we consider the function

$$\Phi(t)=t^\alpha e^{-it^2/2}\,{}_1F_1(u,v,it^2)\quad\text{with}\quad \alpha=v-1/2.$$

Elementary calculations show that the confluent hypergeometric equation becomes

$$\Phi''(t)=\Phi(t)\big(-\alpha/t^2+\alpha^2/t^2-t^2+2i(2u-v)\big).$$

For $t>0$ consider the function

$$W(t)=\Phi(t)\bar{\Phi}'(t)-\bar{\Phi}(t)\Phi'(t)\in i\mathbb{R}.$$

Then

$$W'(t)=\Phi(t)\bar{\Phi}''(t)-\bar{\Phi}(t)\Phi''(t)=-4i|\Phi(t)^2|(2\Re(u)-v).$$

Hence

$$W(t_2) - W(t_1) = \int_{t_1}^{t_2} W'(t)\,dt = -4i(2\Re(u) - v) \int_{t_1}^{t_2} |\Phi(t)^2|\,dt.$$

Since $\Phi(t)$ is square integrable on $[0, t_0]$ (because $v > 0$), it is immediate that $W(t)$ converges to some value as $t \to 0$, and it is not difficult to check that this value is zero. As also $W(t_0) = 0$, we get

$$0 = W(t_0) - W(0) = -4i(2\Re(u) - v) \int_0^{t_0} |\Phi(t)^2|\,dt,$$

which shows that $2\Re(u) - v = 0$. ■

THEOREM 3.17. *Let $f$ be a non-degenerate second degree character defined on $\mathbb{R}$, $\chi$ a unitary character on $\{-1, 1\}$, and assume that $\zeta_f(s, \chi)$ is not the zero function as a function of $s$. Then all the zeroes of $\zeta_f(s, \chi)$ lie on the line $\Re(s) = 1/2$.*

*Proof.* This is an immediate consequence of the previous propositions. ■

**3.10. The function $\zeta_f(s, \chi)$ on $\mathbb{C}$.** On $\mathbb{C}$ we will study the second degree characters of the form $\psi_{\mathbb{C}}\big(\frac{a}{2}z^2 + bz\big)$ and $\psi_{\mathbb{C}}\big(\frac{a}{2}|z|^2 + bz\big)$. The characters on the unit group of $\mathbb{C}^*$ are of the form $c_n(z) = (z/|z|)^n$ with $n \in \mathbb{Z}$. We will frequently use the Wirtinger derivatives on $\mathbb{C}$ defined for $\phi(z) = \phi(x + iy)$ as

$$\frac{\partial \phi}{\partial z} = \frac{1}{2}\left(\frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}\right), \qquad \frac{\partial \phi}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial \phi}{\partial x} + i\frac{\partial \phi}{\partial y}\right).$$

**3.10.1.** *Second degree characters of the form $\psi_{\mathbb{C}}\big(\frac{a}{2}|z|^2 + bz\big)$.* We consider second degree characters which can be written as $f(z) = \psi\big(\frac{a}{2}|z|^2 + bz\big)$ with $a \in \mathbb{R}_+^*$ and $b \in \mathbb{C}$.

We denote by $Z_{a,b}(s, n)$ the weak Mellin transform of $\psi\big(\frac{a}{2}|z|^2 + bz\big)$ at the character $|z|_{\mathbb{C}}^s c_n(z)$.

PROPOSITION 3.18. *The weak Mellin transform of $\psi_{\mathbb{C}}\big(\frac{a}{2}|z|^2 + bz\big)$ at the character $|z|_{\mathbb{C}}^s c_n(z)$ is, for $n = 0$,*

$$Z_{a,b}(s, 0) = \frac{e^{-\pi i s/2}}{a^s} \frac{\Gamma(s)}{(2\pi)^{s-1}} \, {}_1F_1(s, 1, 2\pi i |b|^2/a),$$

*and for $n > 0$,*

$$Z_{a,b}(s, n)$$
$$= (-1)^n \frac{e^{-\frac{\pi i}{2}(s - n/2)}}{a^s} \frac{\Gamma(s + n/2)}{(2\pi)^{s-1}} \frac{1}{n!} \left(\frac{\sqrt{2\pi}}{\sqrt{a}}\bar{b}\right)^n {}_1F_1(s + n/2, 1 + n, \, 2\pi i |b|^2/a).$$

Remark: As a consequence of Proposition 3.16, the zeroes of $Z_{a,b}(s, n)$ lie on the axis $\Re(s) = 1/2$.

*Proof.* We have, writing $z = x + iy$,

$$\psi\left(\frac{a}{2}|z|^2 + bz\right) = e^{-2\pi i(a|z|^2+bz+\bar{b}\bar{z})} = e^{-2\pi i(a|z|^2+2\Re(b)x-2\Im(b)y)}.$$

We make the same observation as in the real case: We consider the right expression, and remark that if we take $a \in \mathbb{C}$ with $\Im(a) < 0$, and replace $\Re(b)$ and $\Im(b)$ with complex values, then using the same method we can still define its weak Mellin transform, which is continuous as a function of $a$, $\Re(b)$ and $\Im(b)$. However, if we suppose that $\Im(a) < 0$, then it has a well defined regular Mellin transform, which is analytic in $\Re(b)$ and $\Im(b)$. We can then use the same method as before: first compute $Z_{a,b}(s,n)$ for $b = 0$ and then use the Taylor expansion of $Z_{a,b}$ at $b = 0$.

Let us first remark that for $n \neq 0$, we have $Z_{a,0}(s,n) = 0$: the function $\psi\left(\frac{a}{2}|z|^2\right)$ is invariant if we replace $z$ by $uz$ with $|u| = 1$. For $n = 0$, the computation of $Z_{a,0}(s,0)$ is straightforward and gives

$$Z_{a,0}(s,0) = \mathrm{Mell}(e^{-2\pi i a|z|^2}, |z|^{2s}, c_0) = \frac{e^{-s\pi i/2}}{a^s}\frac{\Gamma(s)}{(2\pi)^{s-1}}.$$

Now suppose that $b \neq 0$ and $n = 0$. We consider the equality (using the Wirtinger operator $\frac{\partial}{\partial b}$)

$$\frac{\partial}{\partial b}e^{-2\pi i(\frac{1}{2}a|z|^2+bz+\frac{1}{2}\bar{a}|z|^2+\bar{b}\bar{z})} = -2\pi i z e^{-2\pi i(\frac{1}{2}a|z|^2+bz+\frac{1}{2}\bar{a}|z|^2+\bar{b}\bar{z})},$$

which leads, using the same kind of argument as in the real case, to

$$\frac{\partial}{\partial b}Z_{a,b}(s,n) = -2\pi i Z_{a,b}(s+1/2, n+1).$$

We also have

$$\frac{\partial}{\partial \bar{b}}Z_{a,b}(s,n) = -2\pi i Z_{a,b}(s+1/2, n-1).$$

Let us now write the Wirtinger Taylor expansion of $Z_{a,b}(s)$ near zero:

$$Z_{a,b}(s) = \sum_{n,p\geq 0}\frac{\partial^{p+n}}{\partial^p b\partial^n \bar{b}}Z_{a,b}(s)\bigg|_{b=0}\frac{b^p\bar{b}^n}{p!n!}.$$

The Wirtinger derivatives with $p \neq n$ cancel so that we get

$$\begin{aligned}Z_{a,b}(s) &= \sum_{n\geq 0}\frac{\partial^{2n}}{\partial^n b\partial^n \bar{b}}Z_{a,b}(s)\bigg|_{b=0}\frac{|b|^{2n}}{(n!)^2}\\ &= \sum_{n\geq 0}(-2\pi i)^{2n}Z_{a,0}(s+n)\frac{|b|^{2n}}{(n!)^2}\\ &= \frac{e^{-\pi i s/2}}{a^s}\frac{\Gamma(s)}{(2\pi)^{s-1}}\sum_{n\geq 0}\left(\frac{2\pi i}{a}\right)^n (s)_n\frac{|b|^{2n}}{(n!)^2}.\end{aligned}$$

We recognize again a confluent hypergeometric function:

$$Z_{a,b}(s) = \frac{e^{-\pi is/2}}{a^s}\,\frac{\Gamma(s)}{(2\pi)^{s-1}}\,{}_1F_1(s,1,2\pi i|b|^2/a).$$

In order to compute $Z_{a,b}(s,n)$ for $n>0$, we use the formula

$$Z_{a,b}(s,n) = \frac{1}{(-2\pi i)^n}\,\frac{\partial^n}{\partial b^n} Z_{a,b}(s-n/2,0). \ ∎$$

**3.10.2.** *Second degree characters of the form $\psi(\frac{a}{2}z^2+bz)$.* We denote by $\zeta_{a,b}(s,n)$ the weak Mellin transform of the second degree character $f_{a,b}(z) = \psi_{\mathbb{C}}\big(\frac{a}{2}z^2+bz\big)$ at the character $|z|^s_{\mathbb{C}}c_n(z) = |z|^{2s}c_n(z)$. In order to compute $\zeta_{a,b}(s,n)$, we use the same method as for the real case or for $Z_{a,b}(s)$: compute $\zeta_{a,b}(s)$ for $b=0$, then use a Taylor expansion to compute $\zeta_{a,b}(s)$ for all $b$. Let us first compute $\zeta_{a,b}(s,n)$ for $b=0$:

Proposition 3.19. *If $n$ is odd, then*

$$\zeta_{a,0}(s,n) = 0.$$

*If $n$ is even, then*

$$\zeta_{a,0}(s,n) = |a|^{-s}c_{-n/2}(a)(-i)^{|n/2|}\pi^{1-s}\frac{\Gamma(s/2+|n|/4)}{\Gamma((1-s/2)+|n|/4)}.$$

Remark: for $n=0$, $\zeta_{a,0}(s,0)$ vanishes for even positive integer values of $s$, so that the zeroes of $\zeta_{a,0}$ are not all on the line $\Re(s)=1/2$.

*Proof.* The function $\psi(z^2)$ is even, and $c_n$ is odd if $n$ is odd, which proves the first formula, because the multiplicative convolution of any function with an even function gives an even function. If $n$ is even, we use the fact that $\mathbb{C}$ is quadratically closed, i.e. $z\mapsto z^2$ is onto. A consequence is that for any Schwartz function $\phi$ on $\mathbb{C}$ and $n$ even, we have

$$\begin{aligned}\mathrm{Mell}(\phi(x^2),s,n) &= \int_{\mathbb{C}^*} \phi(x^2)|x|^s_{\mathbb{C}}(x/|x|)^n\,d^\times x \\ &= 2\int_{\mathbb{C}^*}\phi(y)|y|^{s/2}_{\mathbb{C}}(y/|y|)^{n/2}\,\frac{d^\times y}{4} = \frac{1}{2}\,\mathrm{Mell}(\phi,s/2,n/2),\end{aligned}$$

and this equality can easily be generalized to weak Mellin transforms. We observe, however, that for $n$ even and $0<\Re(s)<1$, the weak Mellin transform of $\psi_{\mathbb{C}}(z)$ is well defined and equal to the function $\rho(c_n|\,|^s)$ appearing in Tate's local functional equation (cf. [14, p. 319]):

$$\mathrm{Mell}(\psi_{\mathbb{C}}(z),s,n) = \rho(c_n|\,|^s) = (-i)^{|n|}\frac{(2\pi)^{1-s}\Gamma(s+|n|/2)}{(2\pi)^s\Gamma((1-s)+|n|/2)}.$$

Indeed, let $\phi\in C_c^\infty(\mathbb{C}^*)$. We want to compute the Mellin transform of

$$\lambda(\phi)\psi_{\mathbb{C}}(z) = \int_{\mathbb{C}^*}\phi(x)\psi_{\mathbb{C}}(x^{-1}z)\,d^\times x.$$

We can, however, rewrite this integral as

$$\int_{\mathbb{C}} \frac{1}{|y|_{\mathbb{C}}} \phi\left(\frac{1}{y}\right) \psi(yz)\, dy = \mathfrak{F}(\phi^*)(z),$$

where $\phi^*\ (y) := \frac{1}{|y|_{\mathbb{C}}}\phi(1/y)$ and $\phi^*$ is a Schwartz function on $\mathbb{C}$. However, we know from Tate's thesis that

$$\begin{aligned}\mathrm{Mell}(\phi^*, s, n) &= \rho(c_n\|^s)\,\mathrm{Mell}(\mathfrak{F}(\phi^*), 1-s, -n)\\ &= \rho(c_n\|^s)\,\mathrm{Mell}(\lambda(\phi)\psi, 1-s, 1-n).\end{aligned}$$

Hence the weak Mellin transform of $\psi$ is well defined and

$$\mathrm{Mell}(\psi, 1-s, -n) = \frac{1}{\rho(c_n\|^s)} = \rho(c_{-n}||^{1-s}),$$

which shows that $\rho(c_n\|^s)$ is indeed the weak Mellin transform of $\psi(z)$. It is then immediate that the weak Mellin transform of $\psi(az)$ for $n$ even is

$$\mathrm{Mell}(\psi(az), s, n) = |a|_{\mathbb{C}}^{-s} c_{-n}(a)\rho(c_n\|^s),$$

so that

$$\begin{aligned}\mathrm{Mell}\left(\psi\left(\frac{a}{2}z^2\right), s, n\right) &= \frac{1}{2}\,\mathrm{Mell}\left(\psi\left(\frac{a}{2}z\right), \frac{s}{2}, \frac{n}{2}\right)\\ &= \frac{1}{2}\left|\frac{a}{2}\right|_{\mathbb{C}}^{-s/2} c_{-n/2}(a)\rho(c_{n/2}||^{s/2}),\end{aligned}$$

which proves the formula for $n$ even. ∎

Let us now consider the case $b \neq 0$.

PROPOSITION 3.20. *The function $\zeta_{a,b}(s,n)$ satisfies the equations*

$$\begin{aligned}\frac{\partial \zeta_{a,b}(s,n)}{\partial b} &= -2\pi i \zeta_{a,b}(s+1/2, n+1),\\ \frac{\partial \zeta_{a,b}(s,n)}{\partial \bar{b}} &= -2\pi i \zeta_{a,b}(s+1/2, n-1).\end{aligned}$$

*Proof.* These equations can be considered to be the Mellin transform of the formulae

$$\begin{aligned}\frac{\partial}{\partial b}\psi\left(\frac{a}{2}z^2 + bz\right) &= -2\pi i z \psi\left(\frac{a}{2}z^2 + bz\right),\\ \frac{\partial}{\partial \bar{b}}\psi\left(\frac{a}{2}z^2 + bz\right) &= -2\pi i \bar{z} \psi\left(\frac{a}{2}z^2 + bz\right).\end{aligned}$$

The exchange of differentiation and integration is justified as in Proposition 3.14. ∎

Let us now give an explicit description of the weak Mellin transform of $\psi_{\mathbb{C}}(\frac{a}{2}z^2 + bz)$. Since $\zeta_{a,b}(s) = |a|^{-s}c_{-n/2}(a)\zeta_{1,b/\sqrt{a}}(s)$, we can suppose that $a = 1$.

Proposition 3.21. *The weak Mellin transform of* $\psi_{\mathbb{C}}\big(\frac{1}{2}z^2+bz\big)$ *at the character* $|z|_{\mathbb{C}}^s$ *is equal to*

$$\zeta_{1,b}(s)=\pi^{1-s}\bigg\{\frac{\Gamma(s/2)}{\Gamma(1-s/2)}\,{}_1F_1(s/2,1/2,i\pi b^2)\,{}_1F_1(s/2,1/2,i\pi\bar{b}^2)\\ -4\pi|b|^2\frac{\Gamma((s+1)/2)}{\Gamma(1-(s+1)/2)}\,{}_1F_1\bigg(\frac{s+1}{2},\frac{3}{2},\pi i b^2\bigg)\,{}_1F_1\bigg(\frac{s+1}{2},\frac{3}{2},\pi i\bar{b}^2\bigg)\bigg\}.$$

*Proof.* Let us first prove that $\zeta_{1,b}(s)$ is analytic as a function of $\Re(b)$ and $\Im(b)$. Note that

$$\psi_{\mathbb{C}}\big(\tfrac{1}{2}z^2+bz\big)=e^{-2\pi i(\Re(z^2)+2\Re(b)\Re(z)-2\Im(b)\Im(z))}.$$

For $\beta_1,\beta_2\in\mathbb{C}$ let

$$\psi_{\beta_1,\beta_2}(z)=e^{-2\pi i(\Re(z^2)+2\beta_1\Re(z)-2\beta_2\Im(z))}.$$

This function does not define a tempered distribution on $\mathbb{C}$ if $\Im(\beta_1)\neq 0$ or $\Im(\beta_2)\neq 0$, so that we cannot consider its weak Fourier transform using the usual definition. However, the integral of this function against any gaussian function $g(z)=\lambda e^{-a|z|^2+\beta z+\gamma\bar{z}}$ with $a>0$ and $\beta,\gamma,\lambda\in\mathbb{C}$ is well defined, and by restricting our set of test functions to those gaussian functions (this set is stable under the Fourier transform), we can still consider the weak Fourier transform of $\psi_{\beta_1,\beta_2}(z)$ and define the associated weak Mellin transform for $0<\Re(s)<1$ using the same method as in Section 3 and test functions for the Mellin transform of the form $\phi(z)=g^*(z)=|z|_{\mathbb{C}}^{-1}g(1/z)$ (note that we just need one test function $\phi$ satisfying $\mathrm{Mell}(\phi,s)\neq 0$ in order to define the weak Mellin transform). This weak Mellin transform is then clearly an analytic function of $\beta_1$ and $\beta_2$ (because it is holomorphic), so that its restriction to $\beta_1$ and $\beta_2$ real is real-analytic.

We then consider $\zeta_{1,b}(s)$ as an analytic function of $\Re(b)$ and $\Im(b)$ and use the Wirtinger Taylor expansion

$$\begin{aligned}\zeta_{1,b}(s)&=\sum_{n,p\geq 0}\frac{\partial^{p+n}}{\partial^p b\partial^n\bar{b}}\zeta_{1,b}(s)(0)\frac{b^p\bar{b}^n}{p!n!}\\ &=\sum_{n,p\geq 0}\frac{(-2\pi ib)^p(-2\pi i\bar{b})^n}{p!n!}\zeta_{1,0}\bigg(s+\frac{p+n}{2},p-n\bigg)\\ &=\pi^{1-s}\sum_{n,p\geq 0,\,p+n\text{ even}}\frac{(2\pi ib)^p(2\pi i\bar{b})^n}{p!n!}\,\frac{(-i)^{|p-n|/2}}{\pi^{(p+n)/2}}\,\frac{\Gamma\big(\frac{s}{2}+\frac{p+n}{4}+\frac{|p-n|}{4}\big)}{\Gamma\big(1-\frac{s}{2}-\frac{p+n}{4}+\frac{|p-n|}{4}\big)}\\ &=\pi^{1-s}\sum_{n,p\geq 0,\,p+n\text{ even}}(-i)^{|p-n|/2}\frac{(2\sqrt{\pi}ib)^p(2\sqrt{\pi}i\bar{b})^n}{p!n!}\,\frac{\Gamma\big(s/2+\max(p,n)/2\big)}{\Gamma\big(1-s/2-\min(p,n)/2\big)}\\ &=\pi^{1-s}(S_1+S_2),\end{aligned}$$

where $S_1$ is the sum over $n$ and $p$ even, and $S_2$ is over $n$ and $p$ odd. The first sum $S_1$ becomes, after writing $p = 2k$ and $n = 2l$,

$$S_1 = \sum_{k,l\geq 0} (-i)^{|k-l|} \frac{(2\sqrt{\pi}ib)^{2k}(2\sqrt{\pi}i\bar{b})^{2l}}{(2k)!(2l)!} \frac{\Gamma(s/2+\max(k,l))}{\Gamma(1-s/2-\min(k,l))}.$$

We now use the elementary formulae involving the Pochhammer symbol:

$$\Gamma(s/2+\max(k,l)) = (s/2)_{\max(k,l)}\Gamma(s/2),$$

$$\Gamma(1-s/2) = (-1)^{\min(k,l)}(s/2)_{\min(k,l)}\Gamma\big(1-s/2-\min(k,l)\big).$$

We also have the identities

$$(s/2)_{\max(k,l)}(s/2)_{\min(k,l)} = (s/2)_k(s/2)_l,$$

$$(-i)^{|k-l|}(-1)^{\min(k,l)} = (-i)^{|k-l|}(-i)^{k+l-|k-l|} = (-i)^{k+l},$$

so that we get

$$S_1 = \frac{\Gamma(s/2)}{\Gamma(1-s/2)} \sum_{k,l\geq 0} (-i)^{k+l} \frac{(2\sqrt{\pi}ib)^{2k}(2\sqrt{\pi}i\bar{b})^{2l}}{(2k)!(2l)!}(s/2)_k(s/2)_l.$$

This expression can be factored as

$$S_1 = \frac{\Gamma(s/2)}{\Gamma(1-s/2)}\bigg(\sum_{k\geq 0}\frac{(4i\pi b^2)^k}{(2k)!}(s/2)_k\bigg)\bigg(\sum_{l\geq 0}\frac{(4i\pi\bar{b}^2)^l}{(2l)!}(s/2)_l\bigg),$$

and we recognize the product of two confluent hypergeometric functions, writing again $(2k)! = 4^k k!(1/2)_k$:

$$S_1 = \frac{\Gamma(s/2)}{\Gamma(1-s/2)}\,{}_1F_1(s/2,1/2,i\pi b^2)\,{}_1F_1(s/2,1/2,i\pi\bar{b}^2).$$

The computations for $S_2$ are similar. ∎

**4. The weak Mellin transform of second degree characters defined on adele rings.** We consider a number field $F$ and the associated adele ring $\mathbb{A}_F$.

**4.1. Factorizable second degree characters on $\mathbb{A}_F$.** We know that the continuous characters of $\mathbb{A}_F$ are of the form $\psi(bx)$ with $b \in \mathbb{A}_F$, so that we can write any second degree character on $\mathbb{A}_F$ as

$$f(x) = \psi\big(\tfrac{1}{2}\alpha(x)x + bx\big),$$

with $b \in \mathbb{A}_F$ and $\alpha$ a continuous morphism of additive groups from $\mathbb{A}_F$ to $\mathbb{A}_F$ such that $\alpha^{-1}$ is also continuous.

We say that a second degree character is *factorizable* if it can be written as a tensor product (denoting by $P_F$ the set of places of $F$)

$$f = \bigotimes_{v\in P_F} f_v,$$

so that if an element of $\mathbb{A}_F$ is written as $a = (a_v)_{v\in P_F}$, we have $f(a) = \prod_{v\in P_F} f_v(a_v)$. For example, the second degree character $\psi\big(\frac{a}{2}x^2 + bx\big)$ is factorizable, but if $\sigma$ is an automorphism of $F$, and if we keep the notation $\sigma$ for its natural action on $\mathbb{A}_F$, then the second degree character $\psi\big(\frac{a}{2}\sigma(x)x+bx\big)$ is not in general factorizable.

If $f$ is factorizable, then also $\alpha = \bigotimes_{v\in P_F} \alpha_v$, and the continuity of $\alpha$ and $\alpha^{-1}$ means that there exists a finite set $S$ of valuations such that if $v \notin S$, then $\alpha_v(\mathcal{O}_v^\times) = \mathcal{O}_v^\times$, so that $|\alpha_v| = 1$. It is clear that on $\mathbb{A}_\mathbb{Q}$, all second degree characters are factorizable (because there is no continuous non-trivial additive map from $\mathbb{Q}_p$ to $\mathbb{Q}_{p'}$ with $p \neq p'$).

**4.2. The existence of the weak Mellin transform.** On an adele ring, the weak Mellin transform is defined as follows:

Definition 4.1. We say that a function $f$ defined on $\mathbb{A}^\times$ has a well defined weak Mellin transform at the character $|x|^s\chi(x)$ if there exists a function $M_f(s,\chi)$ such that for any test function $\phi \in C_c^\infty(\mathbb{A}_F^\times)$, we have

$$\mathrm{Mell}(\phi\star f, s,\chi) = \mathrm{Mell}(\phi, s,\chi)M_f(s,\chi).$$

Let us first prove the existence of the weak Mellin transform of a factorizable non-degenerate second degree character defined on an adele ring. We consider a second degree character $f$, a function $\phi \in C_c^\infty(\mathbb{A}_F^\times)$ and the map

$$\lambda(\phi)f(y) = \int_{\mathbb{A}_F^\times} \phi(x)f(x^{-1}y)\, d^\times x,$$

where $y \in \mathbb{A}_F$. We then have the following proposition:

Proposition 4.2. *If $f$ is factorizable and $\phi \in C_c^\infty(\mathbb{A}^\times)$, then $\lambda(\phi)f$ is a Schwartz function on $\mathbb{A}$.*

*Proof.* It is well known that a Schwartz function on $\mathbb{A}_F = \mathbb{A}_\infty \times \mathbb{A}_f$ can be written as a finite sum of functions of the form $\phi_\infty \otimes \phi_f$ where $\phi_\infty$ is a Schwartz function on $\mathbb{A}_\infty$ and $\phi_f$ is a factorizable Schwartz function on $\mathbb{A}_f$ (see for example [5, p. 25]).

With exactly the same proof, it is not difficult to find that if $\phi$ is in $C_c^\infty(\mathbb{A}^\times)$, we can write $\phi$ as a finite sum of functions of the form $\phi_\infty \otimes \phi_f$, where $\phi_\infty \in C_c^\infty(\mathbb{A}_\infty^\times)$ and $\phi_f$ is a factorizable function in $C_c^\infty(\mathbb{A}_f^\times)$, $\phi_f = \bigotimes \phi_v$ with nearly all the $\phi_v$ equal to $\mathbf{1}_{\mathcal{O}_v^\times}$. Let us now prove the proposition for such a function. It is immediate that

$$\lambda(\phi)(f) = \lambda(\phi_\infty) f_\infty \bigotimes_{\text{finite}\, v} \lambda(\phi_v) f_v.$$

We have already proved that all the functions $\lambda(\phi_v)f_v$ are in $S(F_v)$ for $v$ finite. The remaining two points to prove are that $\lambda(\phi_\infty)f_\infty$ is Schwartz on $\mathbb{A}_\infty$ and that nearly all the functions $\lambda(\phi_v)f_v$ are equal to $\mathbf{1}_{\mathcal{O}_v}$. Let us first prove the former. Suppose for example that we have two real places to consider, so that the second degree character $f_\infty$ can be decomposed as $f_1 \otimes f_2$ where $f_1$ and $f_2$ are non-degenerate second degree characters defined over $\mathbb{R}$. We then consider the function

$$\lambda(\phi_\infty) f_\infty(y_1, y_2) = \int_{\mathbb{R}^{*2}} \phi_\infty(x_1, x_2) f_1(x_1^{-1} y_1) f_2(x_2^{-1} y_2)\, d^\times x_1\, d^\times x_2.$$

Write $\phi_\infty^*(x_1, x_2) = \frac{1}{|x_1 x_2|}\phi_\infty(1/x_1, 1/x_2)$. We extend this function to $\mathbb{R}^2$ by writing $\phi_\infty^*(0, x_2) = \phi_\infty^*(x_1, 0) = 0$, so that $\phi_\infty^*$ is a Schwartz function and the integral becomes

$$\lambda(\phi_\infty) f_\infty(y_1, y_2) = \int_{\mathbb{R}^2} \phi_\infty^*(z_1, z_2) f_1(z_1 y_1) f_2(z_2 y_2)\, dz_1\, dz_2.$$

To prove for example that this function is fast decreasing as a function of $y_1$, we simply apply the local Weil functional equation to the integral over $z_1$, choose a polynomial $P$ to be the Taylor expansion of $\bar{f}_1$, and proceed as in the one-dimensional case.

Let us now show that nearly all the functions $\lambda(\phi_v)f$ are equal to $\mathbf{1}_{\mathcal{O}_v}$. We remark that

- nearly all the $\phi_v$ are equal to $\mathbf{1}_{\mathcal{O}_v^\times}$,
- nearly all the $\alpha_v$ satisfy $\alpha_v(\mathcal{O}_v) = \mathcal{O}_v$ (because $\alpha$ is continuous and its inverse is also continuous),
- nearly all the $b_v$ satisfy $|b|_v \le 1$ ($b$ is an adele),
- the local different $\mathfrak{d}_v$ is equal to $\mathcal{O}_v$ for nearly all valuations,
- for nearly all $v$, we have $|2|_v = 1$.

It is then enough to prove that for $v \in P_F$ satisfying $|\alpha_v| = 1$, $|b_v| \le 1$, $\mathfrak{d}_v = \mathcal{O}_v$ and $|2|_v = 1$, the function $\lambda(\mathbf{1}_{\mathcal{O}_v^\times})(f_v)$ is equal to $\mathbf{1}_{\mathcal{O}_v}$. This can be proved using exactly the same method as in Proposition 3.6 for $p \ne 2$. ∎

PROPOSITION 4.3. *If $f$ is a factorizable non-degenerate second degree character defined on $\mathbb{A}_F$, then the weak Mellin transform of $f$ is well defined for $\Re(s) > 1$.*

*Proof.* The proof is the same as for the local case (replace $> 0$ by $> 1$). ∎

We will use the notation $\Xi_f(s, \chi)$ for the weak Mellin transform of a second degree character $f$ defined on an adele ring at the character $|x|^s \chi(x)$.

**4.3. The functional equation of $\Xi_f$**

PROPOSITION 4.4. *For $\phi \in S(\mathbb{A}^\times)$ and $f$ a non-degenerate second character defined on $\mathbb{A}$, the Fourier transform of the Schwartz function $\lambda(\phi)f(y)$ is equal to $\gamma_f|\alpha|^{-2}\lambda(\phi^*)\bar f\circ\alpha(y)$, where $\phi^*(x) = |x|^{-1}\phi(1/x)$.*

*Proof.* The proof is exactly the same as for the local case. ∎

PROPOSITION 4.5. *Let $\chi$ be a Hecke character on $\mathbb{A}_F^\times$. Then the function $\Xi_f(s,\chi)$ considered as a function of $s$ has an analytic continuation to $\mathbb{C}$, with possible poles at 0 and 1 if $\chi$ is unramified. If we keep the notation $\Xi_f(s,\chi)$ for the analytic continuation, we have the equality*

$$\Xi_f(s,\chi) = \frac{\gamma_f}{\sqrt{|\alpha|}}\Xi_{\bar f\circ\alpha^{-1}}(1-s,\chi).$$

*Proof.* The proof is the same as for the local case, but we have to replace the Tate local functional equation by the global functional equation. ∎

**4.4. The connection with Hecke $L$-functions.** We recall that if $\chi = \bigotimes\chi_v$ is a Hecke character defined on $\mathbb{A}_F^\times$, the Hecke $L$-function $L(s,\chi)$ is defined for $\Re(s) > 1$ as

$$L(s,\chi) = \prod_{v\text{ finite},\,\chi_v\text{ unramified at }v} \frac{1}{1-\chi_v(\varpi_v)(\mathcal{N}\mathfrak{p}_v)^{-s}}.$$

THEOREM 4.6. *Let $F$ be a number field, $\chi$ a unitary Hecke character and $L(\chi,s)$ the associated Hecke $L$-function. Let $f$ be a factorizable non-degenerate second degree character and $\Xi_f(s,\chi)$ the Mellin transform of $f$ at $(s,\chi)$. Then $(s,\chi)$ is a zero of $\Xi_f$ if and only if it is either a non-trivial zero of $L(s,\chi)$, or a zero of one of the local functions $\zeta_{f_v}(s)$.*

*Proof.* We have already computed that for nearly all valuations $v$,

$$\lambda(\mathbf{1}_{\mathcal{O}_v^\times})f_v = \mathbf{1}_{\mathcal{O}_v},$$

so that

$$\zeta_{f_v}(s,\chi)\,\mathrm{Mell}(\mathbf{1}_{\mathcal{O}_v}^\times, s,\chi) = \mathrm{Mell}(\mathbf{1}_{\mathcal{O}_v}, s,\chi).$$

Nearly all these valuations satisfy the condition that $\chi_v$ is unramified at $v$. Consider the following two finite sets: $S$ is the set of all valuations which are either infinite, or finite with $\chi_v$ ramified, or satisfy $\lambda(\mathbf{1}_{\mathcal{O}_v^\times})f_v \neq \mathbf{1}_{\mathcal{O}_v}$, or satisfy $\mathfrak{d}_v \neq \mathcal{O}_v$, or satisfy $|2|_v \neq 1$; and $T$ is the set of all valuations which are either infinite, or finite with $\chi_v$ ramified. It is clear that $T \subset S$.

Let $v \notin S$. Since $\chi$ is unramified and $\mathcal{N}\mathfrak{d}_v = 1$, we have $\mathrm{Mell}(\mathbf{1}_{\mathcal{O}_v^\times}, s,\chi) = 1$ so that

$$\zeta_{f_v}(s,\chi) = \mathrm{Mell}(\mathbf{1}_{\mathcal{O}_v}, s,\chi) = \int_{F_v^\times} \mathbf{1}_{\mathcal{O}_v}(x)\chi(x)|x|^s\,d^\times x.$$

We split the integral according to the valuation of $x$ to get

$$1+\chi(\varpi_v)\mathcal{N}\mathfrak{p}^{-s}+\cdots=\frac{1}{1-\chi(\varpi_v)\mathcal{N}\mathfrak{p}^{-s}}.$$

We can then write

$$\begin{aligned}\Xi_f(s,\chi)&=\prod_{v\in S}\zeta_{f_v}(s,\chi_v)\prod_{v\notin S}\frac{1}{1-\chi(\varpi_v)(\mathcal{N}\mathfrak{p}_v)^{-s}}\\&=\Big(\prod_{v\in S}\zeta_{f_v}(s,\chi_v)\Big)L(s,\chi)\prod_{v\in S,\,v\notin T}(1-\chi(\varpi_v)\mathcal{N}\mathfrak{p}^{-s}),\end{aligned}$$

which gives the result, since $\zeta_{f_v}(s,\chi_v)$ has no pole for $\Re(s)>0$, the only pole of $L(s,\chi)$ is at $\chi(x)|x|^s=|x|$, and 1 is never a zero of $\zeta_f$ for unramified characters. ∎

**5. Weak Mellin transforms and second degree characters defined on vector spaces.** Let us now come back to Riemann's proof of the functional equation of $\zeta$, which is based on the Poisson summation formula on $\mathbb{Z}$, i.e. to the fact that the distribution $\delta_{\mathbb{Z}}$ is equal to its Fourier transform. We know that this Poisson summation formula can be generalized to distributions of the form $\delta_{\mathbb{Z}^n}$ defined on $\mathbb{R}^n$, leading to the functional equation of Epstein zeta functions or Eisenstein series. The idea of this section is to perform a similar generalization, replacing a second degree character defined on a field by a second degree character defined on a vector space. The main result of this section is that the natural generalizations of the local functions $\zeta_{f_v}$ to vector spaces have their zeroes on the line $\Re(s)=n/2$ under reasonable conditions.

**5.1. A local functional equation on vector spaces.** In order to generalize our results to second degree characters defined on vector spaces, we first need a generalization of Tate's local functional equation to Schwartz functions defined on vector spaces. We define the following maximal compact subgroups $K_L$ of $\mathrm{GL}_n(L)$: for $L=\mathbb{R}$, we write $K_{\mathbb{R}}=O(n)$. For $L=\mathbb{C}$, we write $K_{\mathbb{C}}=U(n)$. If $L$ is a local field, we write $K_L=\mathrm{GL}_n(\mathcal{O}_L)$. We define the generalized norm of an element of the vector space $L^n$ as follows: if $L$ is a local field, $\|x\|=\sup(|x_1|,\dots,|x_n|)$. If $L$ is equal to $\mathbb{R}$, $\|x\|$ is the usual norm. If $L$ is equal to $\mathbb{C}$, we take the square of the usual norm, i.e. $\|x\|_{\mathbb{C}}=\|x\|^2$ (note that in this case, $\|x\|_{\mathbb{C}}$ is not a norm under the usual definition). It is not difficult to check that $\|x\|$ is invariant under the action of the compact group $K_L$.

For any Schwartz function $\varphi$ defined on $L^n$ and $\Re(s)>0$ set

$$M(\varphi,s)=\int_{L^n}\varphi(x)\|x\|^s\,\frac{dx}{\|x\|^n}.$$

This integral is well defined: The convergence near zero is a consequence of the fact that $\varphi$ is continuous. The convergence for $\|x\|$ large is a consequence of the fact that $\varphi$ is Schwartz. We call this integral the *Mellin transform* of $\varphi$. For example, it is not difficult to compute that the Mellin transform of the function $\mathbf{1}_{\mathcal{O}^n}$ defined on $\mathbb{Q}_{\mathfrak{p}}^n$ is equal to $(\mathcal{N}\mathfrak{d})^{-n/2}(1-1/q^n)1/(1-q^{-s})$ (using the usual notation $q = |\varpi|^{-1}$).

If a function $\varphi$ defined on $L^n$ is invariant under the action of $K$, we say that it is a *radial function*.

PROPOSITION 5.1. *For any Schwartz function $f$, and every $s \in \mathbb{C}$ with $0 < \Re(s) < n$, the Mellin transforms of $f$ and $\mathfrak{F}(f)$ are related by the formula*

$$M(f,s) = \rho_n(s) M(\mathfrak{F}(f), n-s)$$

*for some scalar $\rho_n(s)$ which does not depend on $f$.*

*Proof.* First suppose that $L = \mathbb{R}$. Then this proposition simply states that the Fourier transform of $\|x\|^{s-n}$ considered as a distribution is equal to $\|x\|^{-s}$ up to a scalar factor. This is a well known result in the theory of homogeneous distributions (cf. [6, Th. 2.4.6]). Since a radial homogeneous distribution on $\mathbb{C}^n$ can also be considered as a radial homogeneous distribution on $\mathbb{R}^{2n}$, the result is also true for $L = \mathbb{C}$.

Now let $L = \mathbb{Q}_{\mathfrak{p}}$. First consider Schwartz functions which are radial (i.e. invariant under the action of $K$). Such a function $\phi$ can be written as a finite sum $\sum_k a_k \mathbf{1}_{(\varpi^k \mathcal{O})^n}$ (because $\phi$ has compact support and is continuous at zero), its Fourier transform is $\mathfrak{F}(\phi) = (\mathcal{N}\mathfrak{d})^{-n/2} \sum_k a_k q^{-nk} \mathbf{1}_{(\varpi^{-k}\mathfrak{d}^{-1}\mathcal{O})^n}$, and elementary computations show that writing $\mathfrak{d} = \mathfrak{p}^d$ we have

$$M(\phi,s) = q^{d(s-n/2)} \frac{1-q^{s-n}}{1-q^{-s}} M(\mathfrak{F}(\phi), n-s).$$

Now suppose that $\phi$ is not radial. If $\phi_K$ is the radial function obtained by averaging $\phi$ under the action of $K$, we have $M(\phi,s) = M(\phi_K,s)$. As this averaging action commutes with the Fourier transform, we get the result for all $\phi$. ■

**5.2. From second degree characters on vector spaces to Schwartz functions.** We also need a generalization of Proposition 3.2 to vector spaces:

THEOREM 5.2. *Let $f$ be a non-degenerate second degree character on an $L$-vector space $L^n$ of finite dimension $n$, where $L$ is a locally compact field. Let $\phi \in C_c^\infty(\mathrm{GL}_n(L))$ and define*

$$\lambda(\phi)(f)(v) = \int_{\mathrm{GL}_n(L)} \phi(x) f(x^{-1}v)\, d^\times x.$$

*Then $\lambda(\phi)(f)$ is a Schwartz function on $L^n$.*

*Proof.* Let us first prove an elementary proposition.

PROPOSITION 5.3. *Any element $v \in L^n \setminus \{0\}$ can be written as $kv_1$ where $k$ is in $K$ and the only non-zero coordinate of $v_1$ is the first one* (*i.e.* $v_1 = (x_1, 0, 0, \ldots)$ *for some $x_1$ in $L$*).

*Proof.* This is clear for $\mathbb{R}^n$ and $\mathbb{C}^n$. So suppose that $L$ is local and $\varpi$ is a uniformizer, with $|\varpi| = 1/q$. It is enough to show that any vector $x$ in $L^n$ satisfying $\|x\| = q^{-m}$ is in the orbit of the vector $v_1 = (\varpi^m, 0, 0, \ldots)$ under the action of $K_L$ for any $m \in \mathbb{Z}$. After multiplication by $\varpi^{-m}\,\mathrm{Id}$, which commutes with $K$, we can suppose that $m = 0$.

As $\|x\| = \max_i |x_i| = 1$, all the coordinates of $x$ are in $\mathcal{O}_L$ and at least one of the coordinates of $x$ is a unit. Since the map exchanging the basis vectors $e_1$ and $e_i$ is in $K$, for all $i$, we can suppose that the first coordinate $x_1$ is a unit. We can then write

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} x_1 & 0 & 0 & \cdots \\ x_2 & 1 & 0 & \cdots \\ x_3 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ldots \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

and the square matrix is in $\mathrm{GL}_n(\mathcal{O}_L)$. ■

Let us now prove the theorem. Suppose that $L$ is local. It is immediate that $\lambda(\phi)f$ is locally constant, so that we have to show that it has a compact support. In order to simplify notation, we suppose that $n = 2$, but the proof remains the same for all $n$. We first convert the integral on $\mathrm{GL}_2(L)$,

$$\lambda(\phi)(f)(v) = \int\limits_{\mathrm{GL}_2(L)} \phi(x) f(x^{-1}v)\, d^\times x,$$

into an integral on $M_2(L)$: If $dx$ is the standard Haar measure on $M_2(L)$, $dx/|\det x|^2$ is a Haar measure on $\mathrm{GL}_2(L)$. Then $\lambda(\phi)(v)$ is equal, up to a constant scalar factor, to the integral

$$\int\limits_{M_2(L)} \phi(x) f(x^{-1}v)\, \frac{dx}{|\det x|^2}.$$

Write $\phi^\times(x) = \phi(x^{-1})|\det x|^{-2}$ (not to be confused with $\phi^* = \phi(x^{-1})|\det x|^{-1}$) for $x \in \mathrm{GL}_2(L)$ and $\phi^\times(x) = 0$ for $x \notin \mathrm{GL}_2(L)$ the integral becomes

$$\int\limits_{M_2(L)} \phi^\times(x) f(xv)\, dx,$$

where $\phi^\times$ is a Schwartz function on $M_2(L)$.

First suppose that the vector $v$ is of the form $(y, 0)$. Let us write the matrix $x$ as $x = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$, so that $xv = (\alpha y, \gamma y) = y(\alpha, \gamma)$ and the integral can be rewritten as

$$\int_{\beta,\delta\in\mathbb{Q}_p}\left\{\int_{\alpha,\gamma\in\mathbb{Q}_p} f(\alpha y,\gamma y)\phi^\times\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\right)d\alpha\,d\gamma\right\}d\beta\,d\delta =: I.$$

We have assumed that the function $f(\alpha,\gamma)$ is a non-degenerate second degree character on $L^2$. If we denote by $\varrho$ the morphism from $V$ to $V^*$ associated to $f$, and identify $V$ with $V^*$ using the standard additive character $\psi$, then the weak Fourier transform of $f(\alpha,\gamma)$ is equal to $(\gamma_f/\sqrt{|\varrho|})\bar{f}(\varrho^{-1}(\alpha,\gamma))$ (cf. Weil [16]). The Fourier transform of $f(y(\alpha,\beta))$ is then equal to $|y|^{-2}\frac{\gamma_f}{\sqrt{|\varrho|}}\bar{f}(y^{-1}\varrho^{-1}(\alpha,\gamma))$. We then get the equality, with $\mathfrak{F}_{\alpha,\gamma}(\phi^\times)$ standing for the Fourier transform of $\phi^\times(\alpha,\beta,\gamma,\delta)$ considered as a function of $\alpha$ and $\gamma$ only,

$$I = \frac{1}{|y|^2}\frac{\gamma_f}{\sqrt{\varrho}} \times \int_{\beta,\delta\in\mathbb{Q}_p}\Big(\int_{\alpha,\gamma\in\mathbb{Q}_p}\bar{f}(-\varrho^{-1}(\alpha/y,\gamma/y))\,\mathfrak{F}_{\alpha,\gamma}(\phi^\times)(\alpha,\beta,\gamma,\delta)\,d\alpha\,d\gamma\Big)\,d\beta\,d\delta.$$

We then remark that the function $\mathfrak{F}_{\alpha,\gamma}(\phi^\times)(\alpha,\beta,\gamma,\delta)$ has compact support (it is a Schwartz function on $L^4$). As a consequence, we can suppose that its support is included in a ball of radius $R$ (using the sup norm $\|\alpha,\beta,\gamma,\delta\| = \max(|\alpha|,|\beta|,|\gamma|,|\delta|)$). We also know that $f\circ\varrho^{-1}$ is continuous and equal to 1 near zero, so that there exists some $\epsilon$ such that if $|\alpha|<\epsilon$ and $|\gamma|<\epsilon$, then $\bar{f}\circ\varrho^{-1}(\alpha,\gamma)=1$. It is then immediate that if $|y|>R/\epsilon$, then the integral becomes zero: The expressions $|\alpha/y|$ and $|\gamma/y|$ are always $<\epsilon$ if $\alpha$ and $\gamma$ are in the support of $\mathfrak{F}_{\alpha,\gamma}(\phi^\times)$ so that the integral becomes, since $\left(\begin{smallmatrix}0 & \beta\\ 0 & \delta\end{smallmatrix}\right)\notin \mathrm{GL}_2(L)$,

$$\frac{1}{|y|^2}\frac{\gamma_f}{\sqrt{\varrho}}\int_{\beta,\delta\in\mathbb{Q}_p}\Big(\int_{\alpha,\gamma\in\mathbb{Q}_p}\mathfrak{F}_{\alpha,\gamma}(\phi^\times)(\alpha,\beta,\gamma,\delta)\,d\alpha\,d\gamma\Big)\,d\beta\,d\delta = \frac{1}{|y|^2}\frac{\gamma_f}{\sqrt{\varrho}}\int_{\beta,\delta\in\mathbb{Q}_p}\phi^\times\left(\begin{pmatrix}0 & \beta\\ 0 & \delta\end{pmatrix}\right)d\beta\,d\delta = 0.$$

Now suppose that the vector $v$ is not of the form $(y,0)$. We have seen that it is always possible to write $v=kv'$ with $k\in\mathrm{GL}_2(\mathcal{O})$ and $v'=(y',0)$ for some $y'\in L$. The sup norm $\|v\|$ is equal to $|y'|$. It is immediate that

$$\lambda(\phi)(v) = \lambda(\phi)(kv') = \lambda(\phi(kx))(v').$$

If the support of $\phi$ is included in a ball of radius $R$, then the support of $\phi(kx)$ is included in the same ball, since $\|kx\|=\|x\|$ for all $k\in\mathrm{GL}_2(\mathcal{O})$ and $x\in\mathrm{GL}_2(L)$. The function is then equal to zero if $|y'|=\|v\|>R/\epsilon$, which proves the theorem for $L$ local.

Now consider the case $L = \mathbb{R}$ or $\mathbb{C}$, say $L = \mathbb{R}$. If $\phi \in C_c^\infty(\mathrm{GL}_2(\mathbb{R}))$, we write again $\phi^\times(x) = \phi(x^{-1})|\det(x)|^{-2}$. The support of $\phi^\times$ is also compact in $M_2(\mathbb{R})$ for the topology of $M_2(\mathbb{R})$ (because the inclusion map from $\mathrm{GL}_2(\mathbb{R})$ to $M_2(\mathbb{R})$ is continuous) so that if $x \in M_2(\mathbb{R}) \setminus \mathrm{GL}_2(\mathbb{R})$, then $g$ is zero in a neighborhood of $x$ so that all its derivatives at $x$ vanish.

It is immediate that $\phi^\times$ is Schwartz on $M_2(\mathbb{R})$ because it is $C^\infty$ with compact support. We first consider the case $v = (y, 0)$. The same computation as for $L$ ultrametric leads to the integral

$$\frac{1}{|y|^2}\frac{\gamma_f}{\sqrt{\varrho}}\int_{\beta,\delta\in\mathbb{Q}_p}\Big\{\int_{\alpha,\gamma\in\mathbb{Q}_p}\bar{f}(\varrho^{-1}(\alpha/y,\gamma/y))\,\mathfrak{F}_{\alpha,\gamma}(\phi^\times)\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\right)d\alpha\,d\gamma\Big\}\,d\beta\,d\delta.$$

We then observe that for $n, m \geq 0$, the integral

$$\int_{\alpha,\gamma\in\mathbb{Q}_p}\alpha^n\gamma^m\,\mathfrak{F}_{\alpha,\gamma}(\phi^\times)\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\right)d\alpha\,d\gamma$$

is equal, up to a constant, to

$$\int_{\alpha,\gamma\in\mathbb{Q}_p}\mathfrak{F}_{\alpha,\gamma}\left(\frac{\partial^{n+m}}{\partial\alpha^n\partial\gamma^m}\phi^\times\right)\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\right)d\alpha\,d\gamma = \frac{\partial^{n+m}}{\partial\alpha^n\partial\gamma^m}\phi^\times\begin{pmatrix}0 & \beta\\ 0 & \delta\end{pmatrix} = 0.$$

If $P(\alpha,\beta)$ is the polynomial associated to the Taylor expansion of degree $n$ of $\bar{f}(\varrho^{-1}(\alpha,\gamma))$, the integral is equal to

$$\frac{1}{|y|}\frac{\gamma_f}{\sqrt{\varrho}}\int_{\beta,\delta\in\mathbb{Q}_p}\Big\{\int_{\alpha,\gamma\in\mathbb{Q}_p}\big(\bar{f}(\varrho^{-1}(\alpha/y,\gamma/y)) - P(\alpha/y,\gamma/y)\big)\times\mathfrak{F}_{\alpha,\gamma}(\phi^\times)\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\right)d\alpha\,d\gamma\Big\}\,d\beta\,d\delta,$$

and the remainder of the proof is similar to the one-dimensional case.

The proof for $L = \mathbb{C}$ is similar. ∎

This theorem can be extended without difficulty to vector spaces defined over locally compact division rings, since the commutativity of the field has not been used in the proofs.

**5.3. The weak Mellin transform of a second degree character defined on a vector space.** On a locally compact field, we have defined the weak Mellin transform thanks to the formula $\mathrm{Mell}(\lambda(\phi)f, s) = \mathrm{Mell}(\phi, s)\,\mathrm{Mell}(f, s)$, which is valid for all Schwartz functions with $\Re(s) > 0$. This formula can also be written as $\lambda(\phi)|x|^{s-1} = \mathrm{Mell}(\phi, 1-s)|x|^{s-1}$: the function $|x|^{s-1}$ on $\mathbb{R}^*$ is stable, up to a scalar factor, under the action of the multiplicative group.

To generalize this formula to vector spaces defined on locally compact fields, we consider the function $\|x\|^s$ on $L^n \setminus \{0\}$ and the natural left action

$\lambda$ of $\mathrm{GL}_n(L)$ on this function. If $\phi$ is a general element of $C_c^\infty(\mathrm{GL}_n(L))$, then $\lambda(\phi)\|x\|^s$ is not equal to $\|x\|^s$ up to a scalar factor. We show however that this is the case if $\phi$ is invariant under the action of $K$, which allows us to define the weak radial Mellin transform.

PROPOSITION 5.4. *Let $\nu_s$ be a function defined on $L^n \setminus \{0\}$, invariant under the action of $K$ and satisfying $\nu_s(\lambda x) = |\lambda|^s \nu_s(x)$ for all $\lambda \in L$. Then $\nu_s$ is equal to $\|x\|^s$ up to a scalar factor.*

*Proof.* Let $C$ be the value of $\nu_s(x)$ on $e_1 = (1, 0, \ldots)$. We then write, using the decomposition $x = kv_1$ given in Proposition 5.3,

$$\nu_s(x) = \nu_s(k(x_1, 0, 0, \ldots)) = |x_1|^s \nu_s(1, 0, 0, \ldots) = C\|x\|^s. \blacksquare$$

Let $\mathcal{H}(\mathrm{GL}_n(L))$ be the *spherical Hecke algebra* of $\mathrm{GL}_n(L)$, i.e. the algebra of functions in $C_c^\infty(\mathrm{GL}_n(L))$ invariant under the left and right actions of $K$.

PROPOSITION 5.5. *Let $\phi \in \mathcal{H}(\mathrm{GL}_n(L))$. Then for all $s \in \mathbb{C}$, there exists a scalar $\xi_s(\phi)$ such that*

$$\lambda(\phi)\|x\|^s = \xi_s(\phi)\|x\|^s \quad \text{for all } x \in L^n \setminus \{0\}.$$

*Proof.* Set $f(x) = \|x\|^s$. We have

$$\lambda(\phi)(f)(kv) = \int_{\mathrm{GL}_n(L)} \phi(x) f(x^{-1}kv)\, d^\times x.$$

Writing $x^{-1}k = y^{-1}$, so that $ky = x$, we get

$$\int_{\mathrm{GL}_n(L)} \phi(ky) f(y^{-1}v)\, d^\times y = \lambda(\phi)(f)(v).$$

It is then immediate that $\lambda(\phi)\|x\|^s$ satisfies the conditions of the previous proposition, so that it is equal to $\|x\|^s$ up to a scalar factor on $\mathbb{R}^n \setminus \{0\}$. $\blacksquare$

PROPOSITION 5.6. *The function $\xi_s$ is a character of the Hecke algebra $\mathcal{H}(\mathrm{GL}_n(L))$: If $\phi_1, \phi_2 \in \mathcal{H}(\mathrm{GL}_n(L))$, then*

$$\xi_s(\phi_1 \star \phi_2) = \xi_s(\phi_1)\xi_s(\phi_2).$$

*Proof.* Immediate consequence of $\lambda(\phi_1 \star \phi_2) = \lambda(\phi_1)\lambda(\phi_2)$. $\blacksquare$

PROPOSITION 5.7. *Let $\varphi$ be a function defined on $L$ such that $M(\varphi, s)$ is well defined. Then $M(\lambda(\phi)\varphi, s)$ is well defined, and for $\phi \in \mathcal{H}(\mathrm{GL}_n(L))$ we have*

$$M(\lambda(\phi)\varphi, s) = \xi_{s-n}(\phi^*)M(\varphi, s) \quad \text{with} \quad \phi^*(g) = \frac{1}{|\det g|}\phi(g^{-1}).$$

*Proof.* We have

$$M(\lambda(\phi)\varphi, s) = \int_{x\in L^n} \int_{g\in \mathrm{GL}_n(L)} \phi(g)\varphi(g^{-1}x)\|x\|^s \, d^\times g \, \frac{dx}{\|x\|^n}$$
$$= \int_{y\in L^n} \varphi(y)(\lambda(\phi^*)\|x\|^{s-n})(y)\, dy.$$

By Proposition 5.5, this equals $\xi_{s-n}(\phi^*) \int_{g\in \mathrm{GL}_n(L)} \varphi(y)\|y\|^{s-n}\, dy$. ∎

We are now in a position to extend the definition of the Mellin transform to second degree characters defined on vector spaces.

Definition 5.8. Let $f$ be a non-degenerate second degree character defined on $L^n$. Choose $\phi \in \mathcal{H}(\mathrm{GL}_n(L))$ so that $\xi_{s-n}(\phi^*) \neq 0$. We define the weak Mellin transform $M(f, s)$ of $f$ by the formula

$$M(\lambda(\phi)f, s) = \xi_{s-n}(\phi^*)M(f, s).$$

This quantity does not depend on the choice of $\phi$.

*Proof.* The proof is the same as for Proposition 3.3 thanks to the commutativity of the Hecke algebra $\mathcal{H}(\mathrm{GL}_n(L))$. ∎

Proposition 5.9. *If $\phi \in C_c^\infty(\mathrm{GL}_n(L))$, and $f$ is a Schwartz function, then the Fourier transform of the function $\lambda(\phi)f$ is equal to $\lambda(\phi^c)\,\mathfrak{F}(f)$ where $\phi^c$ is defined by the formula $\phi^c(g) = \phi^*(g^t) = |\det g|^{-1}\phi((g^t)^{-1})$.*

*Proof.* We write (using the notation $x.y = \sum_i x_i y_i$)

$$\mathfrak{F}(\lambda(\phi)f)(y) = \int_{x\in L^n} \Big( \int_{g\in \mathrm{GL}_n(L)} \phi(g)f(g^{-1}x)\, d^\times g\Big)\psi(x.y)\, dx.$$

Setting $g^{-1}x = z$, we get

$$\mathfrak{F}(\lambda(\phi)f)(y) = \int_{g\in \mathrm{GL}_n(L)} \phi(g)|\det g|\, \mathfrak{F}(f)(g^t y)\, d^\times g.$$

Setting $h = (g^t)^{-1}$ yields

$$\mathfrak{F}(\lambda(\phi)f)(y) = \int_{h\in \mathrm{GL}_n(L)} \phi((h^t)^{-1})\frac{1}{|\det h|}\, \mathfrak{F}(f)(h^{-1}y)\, d^\times h. \quad ∎$$

Proposition 5.10. *The proposition is also valid if $f$ is a non-degenerate second degree character.*

*Proof.* Since $\lambda(\phi)f$ is a Schwartz function, it is enough to prove this in the weak sense, using the same method as for Proposition 3.4. The computation is straightforward. ∎

Proposition 5.11. *The weak Mellin transform of a second degree character $f$ satisfies for $0 < \Re(s) < n$ the equation*

$$\zeta_f(s) = \rho_n(s)\zeta_{\mathfrak{F}(f)}(n-s) = \rho_n(s)\frac{\gamma_f}{\sqrt{|\varrho|}}\zeta_{\bar{f}(\varrho^{-1}x)}(n-s)$$

*where the scalar factor $\rho_n(s)$ has been defined in Proposition* 5.1.

*Proof.* First, let $\varphi$ be a Schwartz function $\varphi$ and $\phi \in \mathcal{H}(\mathrm{GL}_n(L), \pi)$ so that $\xi_s(\phi^*) \neq 0$. Then $\lambda(\phi)\varphi$ is again a Schwartz function, so that by the local functional equation (Proposition 5.1),

$$M(\lambda(\phi)\varphi, s) = \rho_n(s)M(\mathfrak{F}(\lambda(\phi)\varphi), n-s).$$

Using the previous proposition, we get

$$M(\lambda(\phi)\varphi, s) = \rho_n(s)M(\lambda(\phi^c)\,\mathfrak{F}(\varphi), n-s).$$

If $\phi \in \mathcal{H}(\mathrm{GL}_n(L))$, then also $\phi^c \in \mathcal{H}(\mathrm{GL}_n(L))$, so that we can use Proposition 5.7 to get, introducing the notation $\phi^t(g) = (\phi^c)^*(g) = \phi(g^t)$,

$$\xi_{s-n}(\phi^*)M(\varphi, s) = \rho_n(s)\xi_{-s}(\phi^t)M(\mathfrak{F}(\varphi), n-s).$$

If we compare this with the local functional equation for $\varphi$, we conclude that $\xi_{s-n}(\phi^*) = \xi_{-s}(\phi^t)$ for any $\phi \in \mathcal{H}(\mathrm{GL}_n(L))$. Let now $f$ be a non-degenerate second degree character. We know that $\lambda(\phi)f$ is a Schwartz function, so that

$$M(\lambda(\phi)f, s) = \rho_n(s)M(\mathfrak{F}(\lambda(\phi)f), n-s).$$

Assuming that $\phi$ is in $\mathcal{H}(\mathrm{GL}_n(L))$, using the previous propositions and the definition of the weak Mellin transform, this becomes

$$\xi_{s-n}(\phi^*)\zeta_f(s) = \rho_n(s)\xi_{-s}(\phi^t)\zeta_{\mathfrak{F}(f)}(n-s),$$

and finally $\zeta_f(s) = \rho_n(s)\zeta_{\mathfrak{F}(f)}(n-s)$. ∎

This formula shows that $\zeta_f(s)$ has an analytic continuation, but this is not really a functional equation, since $\zeta_f$ and $\zeta_{\mathfrak{F}(f)}$ are not the same function and do not necessarily have related zeroes. We note again (cf. (3.2)), however, that if $\varrho$ is scalar, i.e. of the form $\varrho = a\,\mathrm{Id}$, then we get a true functional equation.

### 5.4. The zeroes of $\zeta_f$ for second degree characters defined on $\mathbb{Q}_p^n$

Theorem 5.12. *Let $f$ be a non-degenerate second degree character on $\mathbb{Q}_p^n$ and assume that the associated map $\varrho$ is a dilation, $\varrho = a\,\mathrm{Id}$. Then the zeroes of the weak Mellin transform of $f$ are on the line $\Re(s) = n/2$.*

*Proof.* Suppose that

$$f(x_1, \dots, x_n) = \psi\left(\frac{1}{2}a\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} b_i x_i\right)$$

with $a \neq 0$ and $b_i \in \mathbb{Q}_p$. For some vector $v$, let us compute

$$\lambda(\mathbf{1}_{\mathrm{GL}_n(\mathbb{Z}_p)})f(v) = \int\limits_{\mathrm{GL}_n(\mathbb{Q}_p)} f(gv)\mathbf{1}_{\mathrm{GL}_n(\mathbb{Z}_p)}(g)\, d^\times g.$$

We know, thanks to the unicity of the Haar measure, that $d^\times g$ is equal to $|\det g|^{-n} dg$ up to some scalar factor, say $\mu$. Then, noting that the determinant of any element of $\mathrm{GL}_n(\mathbb{Z}_p)$ has to be a unit,

$$\lambda(\mathbf{1}_{\mathrm{GL}_n(\mathbb{Z}_p)})f(v) = \mu \int\limits_{M_n(\mathbb{Q}_p)} f(gv)\mathbf{1}_{\mathrm{GL}_n(\mathbb{Z}_p)}(g)dg. \tag{5.1}$$

Let $g_{i,j}$ be the matrix coefficients of the matrix $g$. We can assume that $v = re_1 = (r, 0, \ldots, 0)$ for some $r \in \mathbb{Q}_p$ because the result is a radial function (i.e. invariant under the action of $\mathrm{GL}_n(\mathbb{Z}_p)$). We have $gv = g(r, 0, \ldots, 0) = (rg_{1,1}, \ldots, rg_{n,1})$ and

$$f(gv) = \psi\bigg(\frac{1}{2}ar^2 \sum_{i=1}^{n} g_{i,1}^2 + r\sum_{i=1}^{n} b_i g_{i,1}\bigg).$$

This expression is independent of the $g_{i,j}$ with $j \neq 1$, so that the integral can be simplified. We use the following proposition:

PROPOSITION 5.13. *Let $A \in M_n(\mathbb{Z}_p)$ and assume the matrix elements $a_{i,1}$ of the first column of $A$ are not all in $p\mathbb{Z}_p$ and are fixed, while the other matrix elements are considered as variables. Then the additive measure of the set of $(a_{i,j})_{j\neq 1}$ satisfying $(a_{i,j}) \in \mathrm{GL}_n(\mathbb{Z}_p)$ is equal to the measure of $\mathrm{GL}_{n-1}(\mathbb{Z}_p)$.*

*Proof.* We can suppose without loss of generality that the valuation of $a_{1,1}$ is zero. Suppose for example that $n = 3$ and write

$$\begin{pmatrix} a_{1,1} & a_{1,1}x & a_{1,1}y \\ a_{2,1} & a_{2,1}x + z & a_{2,1}y + t \\ a_{3,1} & a_{3,1}x + u & a_{3,1}y + v \end{pmatrix} = \begin{pmatrix} a_{1,1} & 0 & 0 \\ a_{2,1} & 1 & 0 \\ a_{3,1} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & y \\ 0 & z & t \\ 0 & u & v \end{pmatrix}.$$

Since $\left(\begin{smallmatrix} a_{1,1} & 0 & 0 \\ a_{2,1} & 1 & 0 \\ a_{3,1} & 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_3(\mathbb{Z}_p)$, the left matrix is in $\mathrm{GL}_3(\mathbb{Z}_p)$ if and only if $x, y \in \mathbb{Z}_p$ and $\left(\begin{smallmatrix} z & t \\ u & v \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}_p)$. As the additive measure of $\mathbb{Z}_p$ is 1, the measure of the possible vectors $(x, y, z, t, u, v)$ is equal to the additive measure of $\mathrm{GL}_2(\mathbb{Z}_p)$. We then observe that the determinant of the map $(x, y, z, t, u, v) \mapsto (a_{1,1}x, a_{1,1}y, a_{2,1}x + z, a_{2,1}y + t, a_{3,1}x + u, a_{3,1}y + v)$ is equal to $a_{1,1}^2$, which is a unit. The proof for general $n$ is the same. ∎

Let $\kappa$ be the measure of $\mathrm{GL}_{n-1}(\mathbb{Z}_p)$. The integral in (5.1) becomes

$$\mu\kappa \int\limits_{(g_{1,1},\ldots,g_{n,1})\in D} \psi\bigg(\frac{1}{2}ar^2 \sum_{i=1}^{n} g_{i,1}^2 + r\sum_{i=1}^{n} b_i g_{i,1}\bigg)\, dg_{1,1}\ldots dg_{n,1},$$

where $D$ is the set of all vectors $(g_{i,1},\dots,g_{i,n})\in\mathbb{Z}_p^n$ so that at least one $g_{i,1}$ is a unit. This domain can be expressed as $\mathbb{Z}_p^n\setminus(p\mathbb{Z}_p)^n$, so that the integral can be written as

$$\mu\kappa\int_{(g_{1,1},\dots,g_{n,1})\in\mathbb{Z}_p^n}\psi\Big(\frac{1}{2}ar^2\sum_{i=1}^n g_{i,1}^2+r\sum_{i=1}^n b_ig_{i,1}\Big)\,dg_{1,1}\dots dg_{n,1}$$
$$-\mu\kappa\int_{(g_{1,1},\dots,g_{n,1})\in(p\mathbb{Z}_p)^n}\psi\Big(\frac{1}{2}ar^2\sum_{i=1}^n g_{i,1}^2+r\sum_{i=1}^n b_ig_{i,1}\Big)\,dg_{1,1}\dots dg_{n,1}.$$

It is then natural to introduce the function

$$\theta_f(r)=\int_{(g_{1,1},\dots,g_{n,1})\in\mathbb{Z}_p^n}\psi\Big(\frac{1}{2}ar^2\sum_{i=1}^n g_{i,1}^2+r\sum_{i=1}^n b_ig_{i,1}\Big)\,dg_{1,1}\dots dg_{n,1},\tag{5.2}$$

so that the integral can be written as

$$\mu\kappa\Big(\theta(r)-\frac{1}{p^n}\theta(pr)\Big).$$

We thus get the equality

$$\lambda(\mathbf{1}_{\mathrm{GL}_2(\mathbb{Z}_p)})f(v)=\mu\kappa\Big(\theta(r)-\frac{1}{p^n}\theta(pr)\Big).$$

Since $\mathbf{1}_{\mathrm{GL}_2(\mathbb{Z}_p)}\star\mathbf{1}_{\mathrm{GL}_2(\mathbb{Z}_p)}=\mathbf{1}_{\mathrm{GL}_2(\mathbb{Z}_p)}$ so that $\xi_s(\mathbf{1}_{\mathrm{GL}_2(\mathbb{Z}_p)})=1$ for all values of $s$, the weak Mellin transform of $f$ is simply the Mellin transform of $\lambda(\mathbf{1}_{\mathrm{GL}_2(\mathbb{Z}_p)})f$:

$$\zeta_f(s)=\int_{v\in\mathbb{Q}_p^n}(\lambda(\mathbf{1}_{\mathrm{GL}_2(\mathbb{Z}_p)})f)(v)\|v\|^s\,\frac{dv}{\|v\|^n}.$$

This integral is equal, up to a scalar factor, to

$$\int_{r\in\mathbb{Q}_p}(\lambda(\mathbf{1}_{\mathrm{GL}_n(\mathbb{Z}_p)})f)(re_1)|r|^s\,\frac{dr}{|r|}.$$

This is equal, up to a scalar factor, to

$$\int_{r\in\mathbb{Q}_p^*}\Big(\theta_f(r)-\frac{1}{p^n}\theta_f(pr)\Big)|r|^s\,d^\times r=\Big(1-\frac{1}{p^{n-s}}\Big)\,\mathrm{Mell}(\theta_f,s).$$

Let us now compute the Mellin transform of $\theta_f(r)$. We remark that the integral definition (5.2) of $\theta_f$ naturally splits as the product of one-dimensional integrals $\theta_{f_i}$ which we have already computed in the proof of Theorem 3.10. To complete the proof of the theorem, we have to prove the following proposition:

Proposition 5.14. *Let $f_1,\dots,f_n$ be non-degenerate second degree characters on $\mathbb{Q}_p$ and let $\theta_{f_1},\dots,\theta_{f_n}$ be the associated functions. Assume that* *the endomorphisms $\varrho_i$ associated to $f_i$ are all of the form $\varrho_i(x) = ax$. Then the zeroes of the Mellin transform of the product $\theta_{f_1}(r)\dots\theta_{f_n}(r)$ are on the line $\Re(s) = n/2$.*

*Proof.* Suppose for example that the valuation of $a$ is even. After rescaling $f_i$, we can assume that this valuation is zero. We have seen in the proof of Theorem 3.10 that all the functions $\theta_{f_i}$ are of the form

$$\mathbf{1}_{\mathbb{Z}_p}(x) + (1 - \mathbf{1}_{\mathbb{Z}_p}(x))\frac{1}{|x|},$$

or for $k \geq 1$,

$$\mathbf{1}_{p^k\mathbb{Z}_p}(x) + \gamma_f\frac{1}{|x|}\mathbf{1}_{p^k\mathbb{Z}_p}\left(\frac{1}{x}\right).$$

We remark that in each formula, the left term vanishes for negative valuations of $x$, and the right term vanishes for positive or zero valuations of $x$. If all the terms satisfy $k = 0$, then the proof is immediate. Let $n_k$ be the number of terms $\theta_{f_i}$ associated to some $k$. The product of the functions $\theta_{f_i}$ becomes

$$(\mathbf{1}_{\mathbb{Z}_p}(x))^{n_0}\prod_{k\geq 1}(\mathbf{1}_{p^k\mathbb{Z}_p}(x))^{n_k} + \gamma\left((1 - \mathbf{1}_{\mathbb{Z}_p}(x))\frac{1}{|x|}\right)^{n_0}\prod_{k\geq 1}\left(\frac{1}{|x|}\mathbf{1}_{p^k\mathbb{Z}_p}\left(\frac{1}{x}\right)\right)^{n_k},$$

where $\gamma$ is the product of all the $\gamma_f$. Let $m$ be the maximum of the $k$'s appearing with non-zero $n_k$. The first term simplifies to $\mathbf{1}_{p^m\mathbb{Z}_p}$, and assuming $m \geq 1$ the last term also simplifies as

$$\gamma\frac{1}{|x|^{n_0+\dots+n_m}}\mathbf{1}_{p^m\mathbb{Z}_p}\left(\frac{1}{x}\right).$$

It is immediate that $n_0 + \dots + n_m = n$ so that we get the function

$$\mathbf{1}_{p^m\mathbb{Z}_p} + \gamma\frac{1}{|x|^n}\mathbf{1}_{p^m\mathbb{Z}_p}\left(\frac{1}{x}\right),$$

and it is immediate, using the same method as for $n = 1$, that the zeroes of the Mellin transform of this function are on the line $\Re(s) = n/2$. The proof for odd valuations of $a$ is similar. ∎

Remark: The computations in the proof of Theorem 5.12 can be repeated to give an explicit description of the weak Mellin transform of any second degree character of the form $\psi_p\big(\sum_{i=1}^n\big(\frac{a_i}{2}x_i^2 + b_ix_i\big)\big)$. It is then easy to see that if all the $a_i$ have the same valuation, then the zeroes of the associated weak Mellin transform lie on $\Re(s) = n/2$, but it is not so if the $a_i$ have different valuations.

**5.5. The weak Mellin transform of a second degree character defined on a real vector space.** We give an explicit description of the weak Mellin transform of the function $\psi_{\mathbb{R}}(\frac{a}{2}\|x\|^2 + b.x)$ on $\mathbb{R}^n$.

Proposition 5.15. *The weak Mellin transform of*

$$f(x) = \psi_{\mathbb{R}}\left(\frac{a}{2}\|x\|^2 + b.x\right)$$

*on* $\mathbb{R}^n$ *with* $a > 0$ *and* $b \in \mathbb{R}^n$ *is*

$$\zeta_{a,b}(s) = \frac{e^{-\pi i s/4}}{\sqrt{a}^s} \frac{\pi^{n/2}}{\Gamma(n/2)} \frac{\Gamma(s/2)}{\pi^{s/2}} \, {}_1F_1(s/2, n/2, \pi i \|b\|^2/a).$$

Remark: This function vanishes only for $\Re(s) = n/2$ (cf. Proposition 3.16). It is similar for $n = 2$ to the weak Mellin transform of $\psi_{\mathbb{C}}\left(\frac{a}{2}|z|^2 + bz\right)$ computed in Proposition 3.18. This is not a surprise since a second degree character defined on $\mathbb{C}$ can also be considered as a second degree character on $\mathbb{R}^2$, and the associated weak Mellin transforms considered as functions of $b$ are the weak Fourier transforms of similar distributions (cf. Proposition 3.9), with $|z|_{\mathbb{C}}^s = (\sqrt{x^2 + y^2})^{2s}$ and $dz = 2dxdy$. The same approach shows that the weak Mellin transform of the function $\psi_{\mathbb{R}}(x^2 - y^2)$ at $s$ is equal up to a scalar factor to the weak Mellin transform of $\psi_{\mathbb{C}}(\frac{1}{2}z^2)$ at $s/2$, so that it vanishes for all positive integers divisible by 4 (cf. Proposition 3.19).

*Proof.* The method is the same as for $n = 1$: the same argument shows that $\zeta_{a,b}(s)$ can be considered also with $a \in \mathbb{C}$ with $\Im(a) < 0$ and $b \in \mathbb{C}$, and that with this definition $\zeta_{a,b}$ is continuous in $a$ and analytic in $b$. We first compute $\zeta_{a,b}(s)$ for $b = 0$, and write its Taylor expansion as a function of $b$. Suppose first $b = 0$ and $a > 0$. Write $a' = a - i\epsilon$ with $\epsilon > 0$ and compute

$$I := \int_{\mathbb{R}^n} e^{-2\pi i \frac{a'}{2}\|x\|^2} \|x\|^s \frac{dx}{\|x\|^n}.$$

Write $x = ru$ where $u$ is on the unit sphere $S_{n-1}$. Then

$$I = \int_{S_{n-1}} \int_0^\infty e^{-2\pi i \frac{a'}{2} r^2} r^s r^{n-1} \frac{dr}{r^n} \, du.$$

As the area of $S_{n-1}$ is $2\pi^{n/2}/\Gamma(n/2)$, we get

$$I = \frac{\pi^{n/2}}{\Gamma(n/2)} \frac{1}{\sqrt{a'}^s} e^{-\pi i s/4} \frac{\Gamma(s/2)}{\pi^{s/2}}.$$

We then get the result for $a$ real by letting $\epsilon \to 0$.

Now suppose that $b \neq 0$, and consider $\zeta_{a,b}(s)$ as a function of $b$. It is clear that it is even, and that it is also radial. Hence it can be written as a function of $\|b\| = r$ as

$$\zeta_{a,b}(s) = \sum_{k \geq 0} a_k r^{2k}.$$

Let $\Delta$ denote the laplacian in $\mathbb{R}^n$. The values of $a_k$ can be described thanks to the formula $\Delta r^{2k} = 4k(n/2 + (k-1))r^{2k-2}$:

$$\Delta^k r^{2k} = 4^k k!(n/2)_k$$

so that

$$a_k = \frac{\Delta_b^k \zeta_{a,b}(s)|_{b=0}}{4^k k!(n/2)_k}$$

where $\Delta_b$ is the laplacian with respect to the variables $b_1, \ldots, b_n$. In order to compute $\Delta_b^k \zeta_{a,b}(s)$, we remark that the second degree character $f_{a,b}(x) = e^{-2\pi i(\frac{a}{2}\|x\|^2 + b.x)}$ satisfies the partial differential equations

$$\frac{\partial}{\partial a} f_{a,b} = -\pi i\|x\|^2 f_{a,b}, \qquad \Delta_b f_{a,b} = (-2\pi i)^2\|x\|^2 f_{a,b},$$

so that

$$\Delta_b f_{a,b} = (-4\pi i)\frac{\partial f_{a,b}}{\partial a}.$$

It is not difficult to show, as in Proposition 3.14 (or by taking $a \in \mathbb{C}$ with $\Im(a) < 0$ and letting $a$ converge to a real value), that we can exchange integration and differentiation to get

$$\Delta_b \zeta_{a,b} = (-4\pi i)\frac{\partial \zeta_{a,b}}{\partial a}.$$

For $b = 0$ we have the identity

$$\left.\frac{\partial^k \zeta_{a,b}}{\partial a^k}\right|_{b=0} = \frac{\pi^{n/2}}{\Gamma(n/2)}(-1)^k (s/2)_k \frac{1}{a^{s/2+k}} e^{-\pi i s/4} \frac{\Gamma(s/2)}{\pi^{s/2}},$$

so that

$$\zeta_{a,b}(s) = \frac{e^{-\pi i s/4}}{\sqrt{a}^s} \frac{\pi^{n/2}}{\Gamma(n/2)} \frac{\Gamma(s/2)}{\pi^{s/2}} \sum_{k\geq 0} \frac{1}{4^k k!(n/2)_k}(-1)^k(-4\pi i)^k (s/2)_k \frac{1}{a^k} r^{2k}.$$

We again recognize a confluent hypergeometric function, so

$$\zeta_{a,b}(s) = \frac{e^{-\pi i s/4}}{\sqrt{a}^s} \frac{\pi^{n/2}}{\Gamma(n/2)} \frac{\Gamma(s/2)}{\pi^{s/2}}\, {}_1F_1(s/2, n/2, \pi i r^2/a). \ \blacksquare$$

**5.6. Weak Mellin transforms associated to non-trivial representations of $K$.** The Mellin transform can be used to decompose a radial function as an integral of functions of the form $\|x\|^s$. It is clear that if one wants to get a complete decomposition of a function defined on $\mathbb{R}^n \setminus \{0\}$, one has also to consider elementary functions which are non-constant on the unit sphere, and the most natural way to do this is to use the theory of spherical harmonics, i.e. to consider scalar integrals of the form

$$\int_{L^n} \varphi(x) Y\left(\frac{x}{\|x\|}\right)\|x\|^s \frac{dx}{\|x\|^n},$$

where $Y$ is some spherical harmonic on the sphere. Using these scalar integrals does not allow one to define a weak Mellin transform, but we know that spherical harmonics are associated to a special class of representations of $SO(n)$: these representations $(V_\pi, \pi)$ have the property that if $e$ is a vector in $L^n$, then there is, up to a scalar factor, a unique vector $v$ in $V_\pi$ invariant under the action of the stabilizer of $e$ in the group $SO(n)$ (cf. [9, Th. 2.12, p. 146]). This special property can be used to define the weak Mellin transform as a vector valued integral.

Consider the vector space $L^n$, where $L$ is any locally compact field, denote by $e_1$ the vector $(1, 0, 0, \ldots)$ and by $K_{e_1}$ the stabilizer of $e_1$ in $K$, i.e. the subgroup of elements $k$ of $K$ such that $ke_1 = e_1$. We say that an irreductible representation $(\pi, V_\pi)$ of $K$ is *spherical* if it has, up to a scalar factor, a unique vector fixed under the action of $K_{e_1}$.

The following proposition can be considered as a generalization of Proposition 5.4:

Proposition 5.16. *Let $(\pi, V_\pi)$ be a spherical representation of $K$, and let $s \in \mathbb{C}$. Then there exists a unique, up to a scalar factor, function $\nu_{s,\pi}$ defined on $L^n \setminus \{0\}$ with values in $V_\pi$ and satisfying:*

- $\nu_{s,\pi}(\mu x) = |\mu|^s \nu_{s,\pi}(x)$ *for all* $\mu \in \mathbb{R}_+^*$ *if* $L = \mathbb{R}$ *or* $L = \mathbb{C}$, *and* $\mu = \varpi$ *if $L$ is local,*
- $\nu_{s,\pi}(kx) = \pi_{s,\pi}(k)\nu_{s,\pi}(x)$ *for all $k$ in $K$.*

*Proof.* Let us first prove unicity. Denote by $v_0$ a vector of $V_\pi$ invariant under the action of $K_{e_1}$ by the representation $\pi$. Since the action of $K$ on the unit sphere $\|x\| = 1$ is transitive, it is immediate that the restriction of $\nu_{\pi,s}$ to the sphere $\|x\| = 1$ is uniquely defined by $f(w)$, where $w$ is any vector in the sphere, for example $w = e_1 = (1, 0, 0, \ldots)$, by the formula

$$f(ke_1) = \pi(k) f(e_1).$$

$e_1$ is invariant under the action of $K_{e_1}$. As a consequence, $f(e_1)$ should then be a vector in $V_\pi$ invariant under the action of $K_{e_1}$ by the representation $\pi$. We have supposed, however, that there exists, up to a scalar factor, exactly one vector with this property. Hence $\nu_{\pi,s}(e_1) = \mu v_0$ for some scalar $\mu$, and the restriction of $\nu_{\pi,s}$ to the sphere can be given by the formula

$$\nu_{\pi,s}(ke_1) = \mu\pi(k)v_0.$$

Conversely, this equation gives a well defined function $\nu_{\pi,s}$ on the sphere which satisfies the conditions of the proposition. The extension of $\nu_{\pi,s}$ to $L^n \setminus 0$ using dilations is immediate. ∎

If $\phi$ is any smooth function with compact support from $\mathrm{GL}_n(L)$ to $\mathrm{GL}(V_\pi)$ and $f$ any continuous function defined on $L^n$ with values in $V_\pi$

or $\mathbb{C}$, we can again define the function $\lambda(\phi)f$ by the formula

$$\lambda(\phi)f(x) = \int\limits_{\mathrm{GL}_n(L)} \phi(g)f(g^{-1}x)\,d^\times g.$$

If $f$ has values in $\mathbb{C}$, $\lambda(\phi)f$ has values in $\mathrm{GL}(V_\pi)$. If $f$ has values in $V_\pi$, $\lambda(\phi)f$ has values in $V_\pi$.

For example, if $f$ is a non-degenerate second degree character defined on $L^n$ with values in $\mathbb{C}$, then the function $\lambda(\phi)f$ has values in $\mathrm{GL}(V_\pi)$ and is a Schwartz function (because each of the matrix coefficients is a Schwartz function).

We also have to replace the spherical Hecke algebra $\mathcal{H}(\mathrm{GL}_n)$ by the $\pi$-spherical Hecke algebra $\mathcal{H}(\mathrm{GL}_n,\pi)$ associated to the representation $\pi$, i.e. the set of smooth functions from $\mathrm{GL}_n(L)$ to $\mathrm{GL}(V_\pi)$ having compact support and satisfying $\phi(k_1gk_2) = \pi(k_1)\phi(g)\pi(k_2)$ for all $k_1, k_2$ in $K$. With these definitions, the generalization of Proposition 5.5 is immediate:

PROPOSITION 5.17. *Let $\phi \in \mathcal{H}(\mathrm{GL}_n,\pi)$. Then for all $s \in \mathbb{C}$, there exists a scalar $\xi_{s,\pi}(\phi)$ such that*

$$\lambda(\phi)\nu_{s,\pi} = \xi_{s,\pi}(\phi)\nu_{s,\pi}.$$

*Proof.* It is immediate that $\lambda(\phi)\nu_{s,\pi}$ satisfies the conditions of the previous proposition:

$$\begin{aligned}\lambda(\phi)\nu_{\pi,s}(\mu kx) &= \int\limits_{\mathrm{GL}_n(L)} \phi(g)\nu_{\pi,s}(g^{-1}\mu kx)\,d^\times g\\ &= |\mu|^s \int\limits_{\mathrm{GL}_n(L)} \phi(g)\nu_{\pi,s}(g^{-1}kx)\,d^\times g.\end{aligned}$$

Writing $g^{-1}k = g'^{-1}$, we get

$$\begin{aligned}\lambda(\phi)\nu_{\pi,s}(\mu kx) &= |\mu|^s \int\limits_{\mathrm{GL}_n(L)} \phi(kg')\nu_{\pi,s}((g')^{-1}x)\,d^\times g'\\ &= |\mu|^s\pi(k) \int\limits_{\mathrm{GL}_n(L)} \phi(g')\nu_{\pi,s}((g')^{-1}x)\,d^\times g'.\end{aligned}$$

This function is then equal to $\nu_{\pi,s}$ on $L^n \setminus \{0\}$ up to a scalar factor. ∎

By using this proposition, the method used in the previous sections to define the weak Mellin transform can be easily extended, by replacing the function $\|x\|^s$ with the functions $\nu_{\pi,s}$: We first define the ramified Mellin transform as a vector valued integral

$$M(f,s,\pi) = \int\limits_{L^n} f(x)\nu_{s,\pi}(x)\,\frac{dx}{\|x\|^n} \in V_\pi.$$

Note that this formula still makes sense (the integral is absolutely convergent for $\Re(s) > 0$) if $f$ has values in $\mathrm{GL}(V_\pi)$, since this linear map acts on the vector $\nu_{\pi,s}(x)$ so that if $f$ is a second degree character and $\phi$ any smooth function with compact support in $\mathrm{GL}_n(L)$ with values in $\mathrm{GL}(V_\pi)$, then $M(\lambda(\phi)f, \pi, s)$ is a well defined vector in $V_\pi$.

Using this definition of $M(f, s, \pi)$, the definition of the weak Mellin transform of a second degree character as an element of $V_\pi$ can be done using exactly the same method as for $n = 1$.

Proposition 5.18. *For all Schwartz functions $f$ defined on $L^n$ with values in $\mathbb{C}$ and $\phi \in \mathcal{H}(\mathrm{GL}_n, \pi)$, we have*

$$M(\lambda(\phi)f, \pi, s) = \xi_{s-n,\pi}(\phi^*) M(f, \pi, s).$$

Remark: $f$ has values in $\mathbb{C}$, but $\lambda(\phi)f$ has values in $\mathrm{GL}(V_\pi)$.

*Proof of Proposition 5.18.* We have

$$\begin{aligned} M(\lambda(\phi)f, \pi, s) &= \int_{L^n} (\lambda(\phi)f)(x)\nu_{s,\pi}(x)\, \frac{dx}{\|x\|^n} \\ &= \int_{x\in L^n} \int_{g\in \mathrm{GL}_n(L)} \phi(g) f(g^{-1}x)\nu_{s-n,\pi}(x)\, d^\times g\, dx. \end{aligned}$$

Writing $y = g^{-1}x$ and replacing $g$ with $g^{-1}$ yields

$$\begin{aligned} M(\lambda(\phi)f, \pi, s) &= \int_{y\in L^n} \int_{g\in \mathrm{GL}_n(L)} f(y)\phi(g^{-1})\nu_{s-n,\pi}(g^{-1}y)\, \frac{1}{|\det g|}\, d^\times g\, dy \\ &= \int_{y\in L^n} f(y)\lambda(\phi^*)\nu_{s-n,\pi}(y)\, dy \\ &= \xi_{s-n,\pi}(\phi^*) \int_{y\in L^n} f(y)\nu_{s-n,\pi}(y)\, dy. \ \blacksquare \end{aligned}$$

Using this formula, we can define in a reasonable way the weak Mellin transform of a second degree character.

Definition 5.19. Let $f$ be a non-degenerate second degree character defined on $L^n$. Choose $\phi \in \mathcal{H}(\mathrm{GL}_n, \pi)$ so that $\xi_{s-n,\pi}(\phi^*) \neq 0$. We define the weak Mellin transform $M(f, s, \pi)$ of $f$ by the formula

$$M(\lambda(\phi)f, s, \pi) = \xi_{s-n,\pi}(\phi^*) M(f, s, \pi).$$

This quantity does not depend on the choice of $\phi$.

*Proof.* We cannot use the same proof as in the unramified case because the algebra $\mathcal{H}(\mathrm{GL}_n, \pi)$ is not commutative. First suppose that $L$ is a local field and let $\phi \in \mathcal{H}(\mathrm{GL}_n, \pi)$. It is immediate that $\mathbf{1}_K(x)\pi(x)$ is a unit of the algebra $\mathcal{H}(\mathrm{GL}_n, \pi)$ (we assume that the Haar measure on $\mathrm{GL}_n(L)$ is normalized so that the measure of $K$ is 1), so that

$$M(\lambda(\phi)f) = M(\lambda(\phi)\lambda(\mathbf{1}_K\pi)f).$$

Applying Proposition 5.18, we get

$$M(\lambda(\phi)f) = \xi_{s-n}(\phi^*)M(\lambda(\mathbf{1}_K\pi)f, s),$$

and we observe that $M(\lambda(\mathbf{1}_K\pi)f, s)$ does not depend on $\phi$. Now suppose that $L = \mathbb{R}$ or $\mathbb{C}$. Choose $\phi_1$ so that $\xi_{s-n}(\phi_1^*) \neq 0$, consider the family of functions $f_b(x) = f(x)\psi(b.x)$ and assume that the functions $M_1(f_b, s)$ are defined for all $b$ in $L^n$ by the formula

$$M(\lambda(\phi_1)f_b, s) = \xi_{s-n}(\phi_1^*)M(f_b, s).$$

The computations in Proposition 3.9 can be generalized to vector spaces, showing that $\zeta_{f_b}(s)$, considered as a function of $b$ (or, more precisely, as a distribution in the variable $b$), is the weak Fourier transform of the distribution $f(x)\nu_{s,\pi}(x)\|x\|^{-n}$, which proves unicity because $\zeta_{f_b}(s)$ is a continuous function of $b$. ∎

This weak Mellin transform has the same kind of scaling properties as the usual Mellin transform:

PROPOSITION 5.20. *Suppose that the weak Mellin transform $M(f, s, \pi)$ of a function $f$ on $L^n$ with values in $\mathbb{C}$ is well defined for some $s$ and $\pi$, and let $k \in K$ and $\mu \in \mathbb{R}_+^*$ if $L$ is $\mathbb{R}$ or $\mathbb{C}$ (or $\mu = \varpi^k$ for some $k$ in $\mathbb{Z}$ if $L$ is local). Then*

$$M(f(k\mu x), \pi, s) = |\mu|^{-s}\pi(k)^{-1}M(f, \pi, s).$$

*Proof.* The definition of the weak Mellin transform of $f$ is that for all $\phi$ in $\mathcal{H}(\mathrm{GL}_n, \pi)$,

$$M(\lambda(\phi)f, s, \pi) = \xi_{s-n,\pi}(\phi^*)M(f, s, \pi).$$

Denote by $f_\mu$ the function $f(\mu\,\cdot)$ and by $f_k$ the function $f(k\,\cdot)$. It is then enough to prove that

$$M(\lambda(\phi)f_\mu, s, \pi) = \mu^{-s}M(\lambda(\phi)f, s, \pi),$$
$$M(\lambda(\phi)f_k, s, \pi) = \pi(k)^{-1}M(\lambda(\phi)f, s, \pi)$$

The first identity is immediate. The second is a consequence of the following computation:

$$M(\lambda(\phi)f_k, s, \pi) = \int_{L^n} \Big(\int_G \phi(g)f(kg^{-1}x)\,d^\times g\Big)\nu_{s,\pi}(x)\,\frac{dx}{\|x\|^n}.$$

Writing $h^{-1} = kg^{-1}$, we get

$$M(\lambda(\phi)f_k, s, \pi) = \int_{L^n} \Big(\int_G \phi(hk)f(h^{-1}x)\,d^\times g\Big)\nu_{s,\pi}(x)\,\frac{dx}{\|x\|^n}.$$

Using the right equivariance of $\phi$ we obtain

$$M(\lambda(\phi)f_k,s,\pi)=\int\limits_{L^n}\Big(\int\limits_{G}\phi(h)f(h^{-1}x)\,d^{\times}g\Big)\pi(k)\nu_{s,\pi}(x)\,\frac{dx}{\|x\|^n}.$$

The functional property of $\nu_{s,\pi}$ yields

$$M(\lambda(\phi)f_k,s,\pi)=\int\limits_{L^n}\Big(\int\limits_{G}\phi(h)f(h^{-1}x)\,d^{\times}g\Big)\nu_{s,\pi}(kx)\,\frac{dx}{\|x\|^n}.$$

Writing $y=kx$, then $h^{-1}k^{-1}=g^{-1}$, we get

$$M(\lambda(\phi)f_k,s,\pi)=\int\limits_{L^n}\Big(\int\limits_{G}\phi(k^{-1}g)f(g^{-1}y)\,d^{\times}g\Big)\nu_{s,\pi}(y)\,\frac{dy}{\|y\|^n}.$$

Finally, by the left equivariance of $\phi$,

$$M(\lambda(\phi)f_k,s,\pi)=\pi(k)^{-1}M(\lambda(\phi)f,s,\pi). \ \blacksquare$$

We now specialize to the case $L=\mathbb{R}$.

Proposition 5.21. *Assume that $L=\mathbb{R}$ and $(\pi,V_\pi)$ is a spherical representation of $K_L=O(n)$ and $0<\Re(s)<n$. Then the weak Fourier transform of $\nu_{\pi,s}(x)\|x\|^{-n}$ is equal to $\nu_{\pi,n-s}(x)\|x\|^{-n}$ up to a scalar factor.*

*Proof.* Let $\Delta$ denote the Fourier transform of $\nu_{\pi,s}(x)\|x\|^{-n}$, which is a tempered distribution because $\Re(s)>0$. We know (cf. [6, p. 130]) that if $\nu$ is a $C^\infty$ function on $\mathbb{R}^n\setminus\{0\}$ that is homogeneous, then the restriction of its Fourier transform $\mathfrak{F}(\nu)$ to $\mathbb{R}^n\setminus\{0\}$ is also a $C^\infty$ function on $\mathbb{R}^\infty\setminus\{0\}$. It is immediate, using the commutation relation of the Fourier transform, that the restriction of $\Delta$ to $\mathbb{R}^n\setminus\{0\}$ satisfies $\Delta(\mu x)=\mu^{-s}\Delta(x)$ and $\Delta(kx)=\pi((k^{-1})^t)\Delta(x)=\pi(k)\Delta(x)$. We then see that $\Delta$ is equal to some multiple $\lambda_{\pi,s}$ of $\nu_{\pi,n-s}\|x\|^{-n}$ plus a tempered distribution $\sigma_{\pi,s}$ supported at zero, i.e. a finite sum of derivatives of the Dirac mass $\delta_0$ at zero.

$$\mathfrak{F}(\nu_{\pi,s}\|x\|^{-n})=\lambda_{\pi,s}\nu_{\pi,n-s}\|x\|^{-n}+\sigma_{\pi,s}.$$

In order to prove that $\sigma_{\pi,s}=0$, we use the Fourier inversion formula and the fact that for $\Re(n-s)>0$,

$$\mathfrak{F}(\mathfrak{F}(\nu_{\pi,s}\|x\|^{-n}))=\mathfrak{F}(\lambda_{\pi,s}\nu_{\pi,n-s}\|x\|^{-n})+\mathfrak{F}(\sigma_{\pi,s}),$$
$$\nu_{\pi,s}(-x)\|x\|^{-n}=\lambda_{\pi,s}\lambda_{\pi,n-s}\nu_{\pi,s}(x)\|x\|^{-n}+\lambda_{\pi,s}\sigma_{\pi,n-s}+\mathfrak{F}(\sigma_{\pi,s}),$$

which shows that $\mathfrak{F}(\sigma_{\pi,s})$ is supported at zero, which is possible if and only if $\sigma_{\pi,s}=0$. $\blacksquare$

Using this proposition, it is not difficult to see that the weak Mellin transform of a second degree character of the form $\psi\big(\frac{a}{2}x.x+b.x\big)$ satisfies a functional equation. Let us now consider the location of the zeroes of this weak Mellin transform. We first remark that on $\mathbb{R}^n$ the function $\zeta_f(s,\pi)$, considered as a function of $s$, is vector valued, but behaves like a scalar. Let us for example consider the second degree character $f_{a,e_1} = \psi\big(\frac{a}{2}x.x + e_1.x\big)$. We have

Proposition 5.22. *For any value of $s$, $\zeta_{f_{a,e_1}}(s,\pi)$ is equal, up to a scalar factor, to the unique vector $v_0$ in $V_\pi$ fixed under the action of $K_{e_1}$.*

*Proof.* The function $f_{a,e_1}(x)$ remains unchanged if we replace $x$ with $kx$ with $k \in K_{e_1}$. As a consequence of Proposition 5.20, the function $\zeta_{f_{a,e_1}}(s,\pi)$ is a vector of $V_\pi$ invariant under the action of $K_{e_1}$. Hence it is equal to $v_0$ up to a scalar factor. ∎

One can also prove that the zeroes of $\zeta_f$ lie on the line $\Re(s) = n/2$ if the morphism associated to $f$ is a scalar:

Theorem 5.23. *Consider on $\mathbb{R}^n$ a second degree character of the form $f_{a,b}(x) = \psi\big(\frac{1}{2}ax.x + b.x\big)$ with $a \in \mathbb{R}^*$ and $b \in \mathbb{R}^n$, $(\pi, V_\pi)$ an irreductible spherical representation of $K = O(n)$, and denote by $\zeta_{a,b}(s,\pi)$ the weak Mellin transform of $f_{a,b}$. Then*

- *if $b = 0$ and $\pi$ is not trivial, then $\zeta_{a,b}(s,\pi) = 0$ for all values of $s$,*
- *if $b \neq 0$, all the zeroes of $\zeta_{a,b}$ lie on the line $\Re(s) = n/2$.*

*Proof.* The case $b = 0$ is clear: the function $\psi\big(\frac{1}{2}ax.x\big)$ is invariant under the action of $K = O(n)$. As a consequence of Proposition 5.20, $\zeta_{a,b}(s)$ is a vector in $V_\pi$ invariant under the action of $\pi(k)$ for all $k$ in $K$. Since the representation $\pi$ is assumed to be irreductible, the only possible value of $\zeta_{a,b}(s,\pi)$ is zero.

Now suppose that $b \neq 0$. If $\zeta_{a,b_0}(s,\pi) = 0$ for some $b_0$, we also have $\zeta_{a,kb_0}(s,\pi) = 0$ for $k \in O(n)$ as a consequence of Proposition 5.20, so that the function vanishes on the whole sphere $\|b\| = \|b_0\|$. The idea is then to use Sturm–Liouville theory in $\mathbb{R}^n$ with boundary conditions on the sphere $\|b\| = \|b_0\|$. Let us first find the partial differential equation satisfied by $\zeta_{a,b}(s,\pi)$ considered as a function of $b$.

We observe that the function $f_{a,b} = \psi\big(\frac{1}{2}ax.x + b.x\big)$ satisfies the formula

$$\Delta_b f_{a,b} = (-4\pi i)\frac{\partial}{\partial a} f_{a,b},$$

where $\Delta_b$ is the laplacian of $f_{a,b}$ considered as a function of the vector variable $b$.

As a consequence, the function $\zeta_{a,b}(s,\pi)$ satisfies the differential equation

$$\Delta_b \zeta_{a,b}(s,\pi) = (-4\pi i)\frac{\partial \zeta_{a,b}(s,\pi)}{\partial a}. \tag{5.3}$$

Using Proposition 5.20 we also have, for any $\lambda > 0$,

$$\zeta_{\lambda^2 u,\lambda v}(s,\pi) = \lambda^{-s}\zeta_{u,v}(s,\pi),$$

which we can also write as

$$\frac{\partial}{\partial \lambda}(\lambda^s \zeta_{\lambda^2 u, \lambda v}(s,\pi)) = 0.$$

Let us develop this equation, writing $\frac{\partial}{\partial a}$ for the derivative of $\zeta_{a,b}(s,\pi)$ with respect to $a$, and $\nabla_b$ for the gradient of $\zeta_{a,b}(s,\pi)$ with respect to the vector variable $b$:

$$s\lambda^{s-1}\zeta_{\lambda^2 u,\lambda v}(s,\pi) + \lambda^s(2\lambda u)\frac{\partial}{\partial a}\zeta_{\lambda^2 u,\lambda v}(s,\pi) + \lambda^s \nabla_b \zeta_{\lambda^2 u,\lambda v}(s,\pi).v = 0.$$

Writing $\lambda^2 u = a$, $\lambda v = b$ and using the partial differential equation (5.3), we get

$$s\zeta_{a,b}(s,\pi) - \frac{a}{2\pi i}\Delta_b \zeta_{a,b}(s,\pi) + \nabla_b \zeta_{a,b}(s,\pi).b = 0. \tag{5.4}$$

In order to apply Sturm–Liouville theory, we have to get rid of the first order term. Hence for $\pi$ fixed we introduce the vector valued function $\phi_{a,b}(s)$ defined by

$$\zeta_{a,b}(s,\pi) = \phi_{a,b}(s)e^{\frac{\pi i b.b}{2a}}.$$

Elementary calculations show that equation (5.4) becomes

$$\Delta_b \phi_{a,b}(s) + \left(\frac{\pi i}{a}(n-2s) + \|b\|^2\left(\frac{\pi}{a}\right)^2\right)\phi_{a,b}(s) = 0.$$

We multiply this equation with the vector $\bar{\phi}_{a,b}(s)$ (whose coordinates are the complex conjugates of those of $\phi_{a,b}(s)$) and integrate on the ball $B$ defined by $\|b\| \le \|b_0\|$:

$$0 = \int_{b\in B}\left(\Delta_b \phi_{a,b}(s) + \left(\frac{\pi i}{a}(n-2s) + \|b\|^2\left(\frac{\pi}{a}\right)^2\right)\phi_{a,b}(s)\right).\bar{\phi}_{a,b}(s)\,db.$$

Using the boundary conditions, we get

$$0 = -\int_{b\in B}\|\nabla_b \phi_{a,b}(s)\|^2\,db + \int_{b\in B}\left(\frac{\pi i}{a}(n-2s) + \|b\|^2\left(\frac{\pi}{a}\right)^2\right)\|\phi_{a,b}(s))\|^2\,db,$$

so that

$$\frac{\pi i}{a}(n-2s)\int_{b\in B}\|\phi_{a,b}(s))\|^2\,db = \int_{b\in B}\|\nabla_b \phi_{a,b}(s)\|^2\,db - \int_{b\in B}\|b\|^2\left(\frac{\pi}{a}\right)^2\|\phi_{a,b}(s))\|^2\,db,$$

and this last expression is real. It is clear that $\phi_{a,b}(s)$ cannot vanish on the whole ball $B$ (cf. Proposition 3.9), so that $n-2s$ has to be imaginary, i.e. $\Re(s) = n/2$. ∎

**References**

[1] R. Beals and R. Wong, *Special Functions*, Cambridge Stud. Adv. Math. 126, Cambridge Univ. Press, 2010.
[2] D. Bump and E. K.-S. Ng, *On Riemann's zeta function*, Math. Z. 192 (1986), 195–204.
[3] D. Bump, K.-K. Choi, P. Kurlberg and J. Vaaler, *A local Riemann hypothesis I*, Math. Z. 233 (1996), 1–19.
[4] P. Cartier, *Über einige Integralformeln in der Theorie der quadratischen Formen*, Math. Z. 84 (1964), 93–100.
[5] D. Goldfeld and J. Hundley, *Automorphic Representations and L-functions for the General Linear Group*, Vol. I, Cambridge Stud. Adv. Math. 129, Cambridge Univ. Press, 2011.
[6] L. Grafakos, *Classical Fourier Analysis*, 2nd ed., Grad. Texts in Math. 249, 2008.
[7] R. Howe, *On the role of the Heisenberg group in harmonic analysis*, Bull. Amer. Math. Soc. 3 (1980), 821–843.
[8] P. Kurlberg, *A local Riemann Hypothesis II*, Math. Z. 233 (2000), 21–37.
[9] M. Morimoto, *Analytic Functionals on the Sphere*, Transl. Math. Monogr. 178, Amer. Math. Soc., 1998.
[10] P. D. Nelson, A. Pitale and A.Saha, *Bounds for Rankin–Selberg integrals and quantum unique ergodicity for powerful levels*, J. Amer. Math. Soc. 27 (2014), 147–191.
[11] R. Olofsson, *Local Riemann hypothesis for complex numbers*, Int. J. Number Theory 5 (2006), 1221–1230.
[12] P. Perrin, *Représentations de Schrödinger, indice de Maslov et groupe métaplectique*, in: Lecture Notes in Math. 880, Springer, 1981, 370–407.
[13] K. Soundararajan and M. P. Young, *The prime geodesic theorem*, J. Reine Angew. Math. 676 (2013), 105–120.
[14] J. Tate, *Fourier analysis in number fields and Hecke's zeta-functions*, Ph.D. Thesis, Princeton Univ., Princeton, NJ, 1950.
[15] H. Weber, *Ueber die Integration der partiellen Differentialgleichung:* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0$, Math. Ann. 1 (1869), 1–36.
[16] A. Weil, *Sur certains groupes d'opérateurs unitaires*, Acta Math. 111 (1964), 143–211.
[17] A. Weil, *Fonction zêta et distributions*, Séminaire Bourbaki, exp. 312, 1966, 523–531.

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