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Families of periodic Jacobi–Perron algorithms in any dimension with period lengths going to infinity

by

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1. Introduction. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a vector in $\mathbb{R}^n$ $(n \geq 1)$. The *Jacobi–Perron Algorithm* (JPA) expansion [6] of α is given by two sequences:

- $(a^{(\nu)})_{\nu \geq 0}$ in $\mathbb{Z}^n$ where $a^{(\nu)} = (a_1^{(\nu)}, \ldots, a_n^{(\nu)})$,
- $(\alpha^{(\nu)})_{\nu \geq 0}$ in $\mathbb{R}^n$ where $\alpha^{(\nu)} = (\alpha_1^{(\nu)}, \ldots, \alpha_n^{(\nu)})$,

defined by:

$$\begin{cases} \alpha^{(0)} = \alpha, \\ a_i^{(\nu)} = [\alpha_1^{(\nu)}] \quad \text{for } \nu \geq 0 \text{ and } 1 \leq i \leq n, \\ \alpha_n^{(\nu+1)} = \dfrac{1}{\alpha_1^{(\nu)} - a_1^{(\nu)}} \quad \text{if } \alpha_1^{(\nu)} \neq a_1^{(\nu)}, \\ \alpha_i^{(\nu+1)} = \dfrac{\alpha_{i+1}^{(\nu)} - a_{i+1}^{(\nu)}}{\alpha_1^{(\nu)} - a_1^{(\nu)}} \quad \text{for } 1 \leq i < n, \end{cases}$$

where $[x]$ is the integer part of x.

The JPA expansion is *periodic* if there exist integers $k \geq 0$ and $l > 0$ such that $a_i^{(k+\nu)} = a_i^{(k+\nu+l)}$ for all $\nu \geq 0$ and $0 < i \leq n$. If k and l are the least integers satisfying this equality then k is the *preperiod length* and l is the *period length*. If $k = 0$, the expansion is *purely periodic*.

If the JPA expansion of $\alpha = (\alpha_1, \ldots, \alpha_n)$ is purely periodic with period length l, then

$$\epsilon = \prod_{\nu=0}^{l-1} \alpha_n^{(\nu)}$$

is a unit $K = \mathbb{Q}[\alpha_1, \ldots, \alpha_n]$. This is a unit found by Hasse and Bernstein [3].

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2. A family of expansions. For all fixed integers $c \geq 2$, $m, n \geq 1$ and $1 \leq t \leq n$, let

$$f_t(X) = X^{n+1} - c^m X^n - (c-1)X^t - \sum_{i=1}^{t-1}(c^m - 1)X^i - c^m.$$

LEMMA 1. *$f_t(X)$ has only one real root α such that*

$$\begin{cases} c^m + c - 1 < \alpha < c^m + c & \text{for } t = n, \\ c^m < \alpha < c^m + 1 & \text{for } 1 \leq t \leq n-1. \end{cases}$$

Proof. We use the rule of signs of Descartes [5]: the possible number of positive roots of a polynomial is equal to the number of sign changes in the coefficients of the polynomial or less than that number by a multiple of 2.

It follows that $f_t(X)$ has only one positive real root. Moreover,

$$\begin{cases} f_t(c^m + c - 1) < 0 \text{ and } f_t(c^m + c) > 0 & \text{if } t = n, \\ f_t(c^m) < 0 \text{ and } f_t(c^m + 1) > 0 & \text{if } 1 \leq t \leq n-1. \end{cases}$$

So, the result is proved. ∎

Write $f_t(X) = X^{n+1} - \sum_{i=0}^{n} b_t^{(i)} X^i$. We define

$$\begin{cases} \alpha_n = \alpha, \\ \alpha_i = \alpha(\alpha_{i+1} - b_t^{(i+1)}) & \text{for } 1 \leq i \leq n-1. \end{cases}$$

2.1. Main theorem. The main result of this paper is

THEOREM 2. *The JPA expansion of $(\alpha_1, \ldots, \alpha_n)$ is purely periodic with period length $l = (n+t)m + 1$. Moreover $\epsilon = \alpha\big(\frac{\alpha^t}{\alpha - c^m}\big)^m$ is a unit of $\mathbb{Q}(\alpha)$.*

REMARK. If $n = 2$ for $t = 1$ and $t = 2$ we obtain the two families given by C. Levesque and G. Rhin [4]. These results are also explained in my thesis [1].

2.2. Proof of the theorem

(1) *The case $n = t = 1$* (continued fraction). Writing α for the real root of $f(X) = X^2 - (c^m + c - 1)X - c^m$ such that $c^m + c - 1 < \alpha < c^m + c$, we have to prove that the continued fraction expansion of α is purely periodic with period length $l = 2m + 1$ and that $\epsilon = \alpha\big(\frac{\alpha}{\alpha - c^m}\big)^m$ is a unit of $\mathbb{Q}(\alpha)$.

We will use the following formulas:

$$\alpha - (c^m + c - 1) = \frac{c^m}{\alpha}, \quad \alpha - c^m = \frac{c\alpha}{\alpha + 1}.$$

The algorithm starts with

$$\alpha_1^{(0)} = \alpha, \quad a_1^{(0)} = c^m + c - 1.$$

Next,

$$\begin{cases} \alpha_1^{(1)} = \dfrac{1}{\alpha-(c^m+c-1)} = \dfrac{\alpha}{c^m}, & a_1^{(1)} = 1, \\ \alpha_1^{(2)} = \dfrac{1}{\alpha/c^m - 1} = \dfrac{c^m}{\alpha - c^m}, & \\ \qquad = \dfrac{c^m(\alpha+1)}{c\alpha} = c^{m-1} + \dfrac{c^{m-1}}{\alpha}, & a_1^{(2)} = c^{m-1}. \end{cases}$$

Now, by induction, it is easy to prove the following formulas (for $0 \le s \le m-1$):

$$\begin{cases} \alpha_1^{(2s+1)} = \alpha/c^{m-s}, & a_1^{(2s+1)} = c^s, \\ \alpha_1^{(2s+2)} = c^{m-s-1} + c^{m-s-1}/\alpha, & a_1^{(2s+2)} = c^{m-s-1}. \end{cases}$$

So, when $s = m-1$ we have

$$\alpha_1^{(2m)} = 1 + 1/\alpha, \quad a_1^{(2m)} = 1,$$

and finally we obtain

$$\alpha_1^{(2m+1)} = \alpha = \alpha_1^{(0)},$$

which leads to $l = 2m+1$.

The Hasse–Bernstein formula produces in $\mathbb{Q}(\alpha)$ the unit

$$\epsilon = \prod_{\nu=0}^{l-1} \alpha_1^{(\nu)} = \alpha\left(\frac{\alpha}{\alpha-c^m}\right)^m.$$

(2) *The case* $n \neq 1$. In the course of the proof, the equality $1+\alpha+\alpha^2+\cdots+\alpha^{t-1}+\alpha^n = \frac{c\alpha^t}{\alpha-c^m}$ will be useful.

Assume $t \neq 1$. For the first vectors, we easily obtain

$$\begin{cases} \alpha_n^{(0)} = \alpha, \quad a_n^{(0)} = \begin{cases} c^m+c-1 & \text{if } t = n, \\ c^m & \text{if } \le t \le n-1; \end{cases} \\ \text{for } t < i < n, \\ \alpha_i^{(0)} = \alpha^{n-i}(\alpha - c^m) \\ \qquad = (c-1)\dfrac{\alpha^t}{\alpha^i} + \displaystyle\sum_{r=1}^{t-1} \dfrac{(c^m-1)\alpha^{t-r}}{\alpha^i} + \dfrac{c^m}{\alpha^i}, \quad a_i^{(0)} = 0; \\ \alpha_t^{(0)} = \alpha^{n-t}(\alpha - c^m) \\ \qquad = (c-1) + \displaystyle\sum_{r=1}^{t-1} \dfrac{c^m-1}{\alpha^r} + \dfrac{c^m}{\alpha^t}, \qquad a_t^{(0)} = c-1; \\ \text{for } 1 \le i < t, \\ \alpha_i^{(0)} = \displaystyle\sum_{r=0}^{i-1} \dfrac{c^m-1}{\alpha^r} + \dfrac{c^m}{\alpha^i}, \qquad a_i^{(0)} = c^m - 1. \end{cases}$$

Now, we show by induction on s that, for $0 \le s \le m-1$,

$$\tag{2.1} \begin{cases} \text{for } 2 \le i \le n, & \\ \alpha_i^{((n+t)s)} = \alpha_i^{(0)}, & a_i^{((n+t)s)} = a_i^{(0)}; \\ \alpha_1^{((n+t)s)} = (c^{m-s}-1) + c^{m-s}/\alpha, & a_1^{((n+t)s)} = c^{m-s}-1. \end{cases}$$

These formulas hold for $s = 0$. Assume they hold for all $0 \le s \le m-1$. Then

$$\begin{cases} \alpha_n^{((n+t)s+1)} = \alpha/c^{m-s}, & a_n^{((n+t)s+1)} = c^s; \\ \text{for } t \le i < n, & \\ \alpha_i^{((n+t)s+1)} = \alpha^{n-i}(\alpha - c^m)/c^{m-s}, & a_i^{((n+t)s+1)} = 0; \\ \text{for } 1 \le i < t, & \\ \alpha_i^{((n+t)s+1)} = \dfrac{\alpha}{c^{m-s}} \displaystyle\sum_{r=1}^{i} \frac{c^m - 1}{\alpha^r} + \frac{c^s}{\alpha^i} & \\ \qquad = \dfrac{1}{c^{m-s}\alpha^i}\Big(c^m + \displaystyle\sum_{r=1}^{i} \alpha^r (c^m - 1)\Big), & a_i^{((n+t)s+1)} = c^s - 1; \end{cases}$$

and

$$\begin{cases} \alpha_n^{((n+t)s+2)} = \dfrac{\alpha c^{m-s}}{\alpha(c^{m-s}-1) + c^m}, & a_n^{((n+t)s+2)} = 1; \\ \text{for } t-1 \le i < n, & \\ \alpha_i^{((n+t)s+2)} = \dfrac{\alpha^{n-i}(\alpha - c^m)}{\alpha(c^{m-s}-1) + c^m}, & a_i^{((n+t)s+2)} = 0; \\ \text{for } 1 \le i < t-1, & \\ \alpha_i^{((n+t)s+2)} = \dfrac{\alpha^{i+1}(c^{m-s}-1) + c^m + \sum_{r=1}^{i} \alpha^r (c^m - 1)}{\alpha^i(\alpha(c^{m-s}-1) + c^m)}, & a_i^{((n+t)s+2)} = 0. \end{cases}$$

Then it is easy to prove, by induction on ν with $3 \le \nu \le t$, that

$$\begin{cases} \alpha_n^{((n+t)s+\nu)} = \dfrac{\alpha^{\nu-1}(c^{m-s}-1) + c^m\alpha + \sum_{r=2}^{\nu-2} \alpha^r (c^m - 1)}{\alpha^{\nu-1}(c^{m-s}-1) + c^m + \sum_{r=1}^{\nu-2} \alpha^r (c^m - 1)}, & \\ & a_n^{((n+t)s+\nu)} = 1; \\ \text{for } t+1-\nu \le i < n, & \\ \alpha_i^{((n+t)s+\nu)} = \dfrac{\alpha^{n-i}(\alpha - c^m)}{\alpha^{\nu-1}(c^{m-s}-1) + c^m + \sum_{r=1}^{\nu-2} \alpha^r (c^m - 1)}, & \\ & a_i^{((n+t)s+\nu)} = 0; \\ \text{for } 1 \le i < t+1-\nu, & \\ \alpha_i^{((n+t)s+\nu)} = \dfrac{\alpha^{i+\nu-1}(c^{m-s}-1) + c^m + \sum_{r=1}^{i+\nu-2} \alpha^r (c^m - 1)}{\alpha^i\big(\alpha^{\nu-1}(c^{m-s}-1) + c^m + \sum_{r=1}^{\nu-2} \alpha^r (c^m - 1)\big)}, & \\ & a_i^{((n+t)s+\nu)} = 0. \end{cases}$$

As can be verified, all the formulas hold for $\nu = 3$. If we assume that they hold for ν, it is again easy to prove them for $\nu + 1$.

Next,

$$\begin{cases} \alpha_n^{((n+t)s+t+1)} = \dfrac{\alpha^{t-1}(c^{m-s}-1) + c^m + \sum_{r=1}^{t-2} \alpha^r (c^m - 1)}{\alpha^{n-1}(\alpha - c^m)}, \\ \hfill a_n^{((n+t)s+t+1)} = c^{m-s-1}; \\ \text{for } 1 \le i < n, \\ \alpha_i^{((n+t)s+t+1)} = 1/\alpha^i, \qquad a_i^{((n+t)s+t+1)} = 0. \end{cases}$$

Indeed, let us give the details for $a_n^{((n+t)s+t+1)} = c^{m-s-1}$. We have

$$\begin{aligned} \alpha_n^{((n+t)s+t+1)} &= \frac{\alpha^{t-1}c^{m-s} - (\alpha - c^m)\sum_{r=1}^{t-1} \alpha^{r-1}}{\alpha^{n-1}(\alpha - c^m)} \\ &= \frac{c^{m-s-1}}{\alpha^n}\left(\frac{c\alpha^t}{\alpha - c^m}\right) - \sum_{r=1}^{t-1} \frac{1}{\alpha^{n-r}}, \end{aligned}$$

that is,

$$\alpha_n^{((n+t)s+t+1)} = \frac{c^{m-s-1}}{\alpha^n}(1 + \alpha + \alpha^2 + \ldots + \alpha^{t-1} + \alpha^n) - \sum_{r=1}^{t-1} \frac{1}{\alpha^{n-r}}.$$

or

$$\alpha_n^{((n+t)s+t+1)} = c^{m-s-1} + \frac{1}{\alpha^n}\Big((c^{m-s-1} - 1)\sum_{r=1}^{t-1} \alpha^r + c^{m-s-1}\Big),$$

and finally, using the fact that $0 < (c^{m-s-1} - 1)\sum_{r=1}^{t-1} \alpha^r + c^{m-s-1} < \alpha_n$, we obtain $a_n^{((n+t)s+t+1)} = c^{m-s-1}$.

By induction on ν with $2 \le \nu \le n - t + 1$, it is still easy to prove the following formula:

$$\begin{cases} \text{for } n - \nu + 2 \le i \le n, \\ \alpha_i^{((n+t)s+t+\nu)} = \alpha_i^{(0)}, \qquad a_i^{((n+t)s+t+\nu)} = a_i^{(0)}; \\ \alpha_{n-\nu+1}^{((n+t)s+t+\nu)} = \dfrac{c^{m-s-1} - 1}{\alpha^{n-\nu+1}}\Big(\displaystyle\sum_{r=1}^{t-1} \alpha^r\Big) + \dfrac{c^{m-s-1}}{\alpha^{n-\nu+1}}, \quad a_{n-\nu+1}^{((n+t)s+t+\nu)} = 0; \\ \text{for } 1 \le i \le n - \nu, \\ \alpha_i^{((n+t)s+t+\nu)} = 1/\alpha^i, \qquad a_i^{((n+t)s+t+\nu)} = 0. \end{cases}$$

All the formulas hold for $\nu = 2$ and if we assume they hold for ν, it is again easy to prove them for $\nu + 1$.

Then, always by induction on ν with $2 \le \nu \le t$ we have

$$\begin{cases} \text{for } t-\nu+2 \le i \le n, & \\ \alpha_i^{((n+t)s+n+\nu)} = \alpha_i^{(0)}, & a_i^{((n+t)s+n+\nu)} = a_i^{(0)}; \\ \alpha_{t-\nu+1}^{((n+t)s+n+\nu)} = \dfrac{c^{m-s-1}-1}{\alpha^{t-\nu+1}}\Big(\displaystyle\sum_{r=1}^{t-\nu+1}\alpha^r\Big) + \dfrac{c^{m-s-1}}{\alpha^{t-\nu+1}}, & \\ & a_{t-\nu+1}^{((n+t)s+n+\nu)} = c^{m-s-1}-1; \\ \text{for } 1 \le i \le t-\nu, & \\ \alpha_i^{((n+t)s+n+\nu)} = 1/\alpha^i, & a_i^{((n+t)s+n+\nu)} = 0. \end{cases}$$

In the special case of $\nu = t$,

$$\begin{cases} \text{for } 2 \le i \le n, & \\ \alpha_i^{((n+t)s+n+t)} = \alpha_i^{(0)}, & a_i^{((n+t)s+n+t)} = a_i^{(0)}; \\ \alpha_1^{((n+t)s+n+t)} = (c^{m-s-1}-1) + c^{m-s-1}/\alpha, & a_1^{((n+t)s+n+t)} = c^{m-s-1}-1. \end{cases}$$

Therefore, (2.1) holds for $s+1$, and so all the previous formulas follow by induction.

For $t = 1$, it is again easy to prove that, for $0 \le s \le m$,

$$\begin{cases} \alpha_i^{((n+t)s)} = \alpha_i^{(0)} \quad \text{for } 2 \le i \le n, \\ \alpha_1^{((n+t)s)} = \begin{cases} c^{m-s}/\alpha & \text{if } 1 \le s \le m, \\ c-1+c^m/\alpha & \text{if } s = 0. \end{cases} \end{cases}$$

So, for all t such that $1 \le t \le n$,

$$\begin{cases} \text{for } 2 \le i \le n, & \\ \alpha_i^{((n+t)m)} = \alpha_i^{(0)}, & a_i^{((n+t)m)} = a_i^{(0)}; \\ \alpha_1^{((n+t)m)} = 1/\alpha, & a_1^{((n+t)m)} = 0. \end{cases}$$

Finally, we obtain

$$\alpha_i^{((n+t)m+1)} = \alpha_i^{(0)}, \qquad a_i^{((n+t)m+1)} = a_i^{(0)} \qquad \text{for } 1 \le i \le n,$$

leading to $l = (n+t)m+1$.

We deduce that the Hasse–Bernstein formula provides the unit

$$\epsilon = \prod_{\nu=0}^{l-1} \alpha_n^{(\nu)} = \alpha\Big(\frac{\alpha^t}{\alpha - c^m}\Big)^m. \ \blacksquare$$

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