C O L L O Q U I U M   M A T H E M A T I C U M

VOL. 148 2017 NO. 1

ON LECH'S LIMIT FORMULA FOR MODULES

BY

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Abstract. Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a standard graded algebra and $M = \bigoplus_{n=0}^{\infty} M_n$ a graded Noetherian R-module. The main objective of this work is to derive a Lech type formula for a sequence of homogeneous elements $a_1, \ldots, a_m$ of degree one which form a g-multiplicity system of R. We also extend to this context the well known Serre Theorem, that is, we prove that for $t \gg 0$ the g-multiplicity symbol $e_t(a_1, \ldots, a_m; R)$, introduced by Kirby (1987), coincides with the Buchsbaum–Rim multiplicity $e_{\mathrm{BR}}(I; R)$ of the R_0-module I generated by $a_1, \ldots, a_m$.

1. Introduction. Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local ring and I be an $\mathfrak{m}$-primary ideal of R. Historically the modern theory of multiplicities began with the study of the asymptotic behavior of powers of ideals. This part of the theory centers around two expressions for the Hilbert–Samuel multiplicity as a limit. One of these limit formulae is due to P. Samuel [S] and the other to C. Lech [L]. P. Samuel has shown that, for n sufficiently large, $\ell(R/I^n)$ is a polynomial in n of degree d (see [S, pp. 24–28], or [N]). P. Samuel defines the multiplicity $e(I; R)$ of I as $d!$ times the leading coefficient of this polynomial. From this definition it can easily be concluded that $e(I; R)$ is a positive integer. On the other hand, it is plain that we can write

$$e(I; R) = \lim_{n \to \infty} \frac{d!\ell(R/I^n)}{n^d}.$$

C. Lech [L] and D. G. Northcott [N] take this expression as the starting point to obtain another limit formula for multiplicity. More specifically, let M be a finitely generated R-module and $\underline{x}$ a sequence $x_1, \ldots, x_s$ of elements of $\mathfrak{m}$ which is a *multiplicity system* of M, that is, $\ell(M/(\underline{x})M)$ is finite (see [N, p. 295]) and so is $\ell(M/\sum_{i=1}^{s} x_i^{n_i} M)$ for all non-negative integers $n_1, \ldots, n_s$ (see [N, p. 296, Proposition 2]).

2010 *Mathematics Subject Classification*: Primary 13H15; Secondary 13H15, 13B22, 13C14, 13C15, 13C40.

Key words and phrases: multiplicity symbol, multiplicity system, Koszul complex.

Received 19 January 2016; revised 7 June 2016.

DOI: 10.4064/cm6870-6-2016

Consider first the case in which $I = (x)$ is an $\mathfrak{m}$-primary ideal of R. Then, because

$$(0 :_M x) \subseteq (0 :_M x^2) \subseteq (0 :_M x^3) \subseteq \cdots$$

is an ascending sequence of submodules of M, there exists an integer m such that $(0 :_M x^n) = (0 :_M x^m)$ for all $n \geq m$. Accordingly, when $n \geq m$,

$$e(x^n; M) = \ell(M/x^n M) - \ell((0 :_M x^n)) = \ell(M/x^n M) - \ell((0 :_M x^m)).$$

Now, by [N, Cor. 1, p. 311], $e(x^n; M) = ne(x; M)$. It follows that

$$\ell(M/x^n M) = ne(x; M) + C$$

for all $n \geq m$, where C is independent of n. In particular we see that

$$\lim_{n\to\infty} \frac{\ell(M/x^n M)}{n} = e(x; M).$$

This is the simplest case of Lech's formula. The general limit formula for multiplicities due to C. Lech [L, Theorem 2] can be generalized as in [N, p. 314] to give

$$\lim_{n_1,\ldots,n_s\to\infty} \frac{\ell(M/(x_1^{n_1}, \ldots, x_s^{n_s})M)}{n_1 \cdots n_s} = e(x_1, \ldots, x_s; M),$$

where $e(x_1, \ldots, x_s; M)$ is the multiplicity symbol of $x_1, \ldots, x_s$ on M (see [N, p. 299]).

Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a standard graded algebra and $M = \bigoplus_{n=0}^{\infty} M_n$ a graded Noetherian R-module. The main aim of this work is to derive a Lech type formula for a sequence $\underline{a}$ of homogeneous elements $a_1, \ldots, a_m$ of degree one which form a g-multiplicity system of R. Precisely, we have

$$\lim_{r_1,\ldots,r_m\to\infty} \frac{\ell\big((M/(a_1^{r_1}, \ldots, a_m^{r_m})M)_{t+\sum(r_i-1)k_i}\big)}{r_1 \cdots r_m} = e_t(a_1, \ldots, a_m; M)$$

for $t \gg 0$, where $e_t(a_1, \ldots, a_m; M)$ is the g-multiplicity symbol of $\underline{a}$.

We also extend to this context the well known Serre Theorem, that is, we prove that for $t \gg 0$ the g-multiplicity symbol $e_t(a_1, \ldots, a_m; R)$ introduced by D. Kirby [Ki] coincides with the Buchsbaum–Rim multiplicity $e_{\mathrm{BR}}(I; R)$ of the R_0-module I generated by $a_1, \ldots, a_m$.

2. Multiplicity system and the Koszul complex.

Throughout the rest of the paper, $R = \bigoplus_{n=0}^{\infty} R_n$ will be a Noetherian graded commutative ring with identity, and $M = \bigoplus_{n=0}^{\infty} M_n$ a graded Noetherian R-module. Our basic tool in this section will be the Koszul complex $K_\bullet(a_1, \ldots, a_m; M)$ with respect to a set of homogeneous elements $a_1, \ldots, a_m$ of R, a_i having degree k_i.

We recall the definition and some basic properties of $K_\bullet(a_1,\dots,a_m;M)$. It is constructed as follows. Let $F=\bigoplus_{i=1}^m R(-k_i)$ be a free R-module with basis $e_1,\dots,e_m$. We consider the homogeneous morphism of graded modules $\phi : F \to R$ defined by $\phi(e_i)=a_i$. Then the Koszul complex $K_\bullet(a_1,\dots,a_m;R)$ is the homological complex such that the nth graded piece is

$$K_n(a_1,\dots,a_m;R)=\bigwedge^n F$$

and the differential $d_n : \bigwedge^n F \to \bigwedge^{n-1} F$ is defined by

$$d_n(x_1\wedge\cdots\wedge x_n)=\sum_{i=1}^n(-1)^{i+1}\phi(x_i)x_1\wedge\cdots\wedge\widehat{x_i}\wedge\cdots\wedge x_n.$$

Notice that each differential is a homogeneous morphism. In fact, considering the basis $\{e_{i_1}\wedge\cdots\wedge e_{i_n}\mid 1\le i_1<\cdots<i_n\le m\}$ of $\bigwedge^n F$, we have

$$\begin{aligned}d_n(e_{i_1}\wedge\cdots\wedge e_{i_n})&=\sum_{j=1}^n(-1)^{j+1}\phi(e_{i_j})e_{i_1}\wedge\cdots\wedge\widehat{e_{i_j}}\wedge\cdots\wedge e_{i_n}\\&=\sum_{j=1}^n(-1)^{j+1}a_{i_j}e_{i_1}\wedge\cdots\wedge\widehat{e_{i_j}}\wedge\cdots\wedge e_{i_n}.\end{aligned}$$

Now, identifying $e_{i_1}\wedge\cdots\wedge e_{i_n}$ with $e_{i_1,\dots,i_n}$ for each element of the basis of $\bigwedge^n F$, we can identify

$$\begin{aligned}K_n(a_1,\dots,a_m;R)&=\bigoplus_{1\le i_1<\cdots<i_n\le m}R(-(k_{i_1}+\cdots+k_{i_n}))\\&=\bigoplus_{1\le i_1<\cdots<i_n\le m}Re_{i_1,\dots,i_n},\end{aligned}$$

where $\deg(e_{i_1,\dots,i_n})=k_{i_1}+\cdots+k_{i_n}$.

From this presentation the graded structure of $K_n(a_1,\dots,a_m;R)$ becomes clear, since $[K_n(a_1,\dots a_m;R)]_t=\bigoplus_{1\le i_1<\cdots<i_n\le m}[Re_{i_1,\dots,i_n}]_t$ and clearly

$$R_k[K_n(a_1,\dots,a_m;R)]_t\subset[K_n(a_1,\dots,a_m;R)]_{k+t}.$$

So $K_n(a_1,\dots,a_m;R)$ is a graded free R-module of rank $\binom{m}{n}$ and with differential

$$d_n(e_{i_1,\dots,i_n})=\sum_{j=1}^n(-1)^{j+1}a_{i_j}e_{i_1,\dots,\widehat{i_j},\dots,i_n}.$$

The Koszul homology modules $H_n(a_1,\dots,a_m;R)$ are defined as the homology modules of the Koszul complex, that is, for $n\ge 0$,

$$H_n(a_1,\dots,a_m;R):=H_n(K_\bullet(a_1,\dots,a_m;R))=\operatorname{Ker}(d_n)/\operatorname{Im}(d_{n+1}).$$

For a graded R-module M, we define the homological Koszul complex $K_\bullet(a_1,\ldots,a_m;M)$ with respect to $a_1,\ldots,a_m$ to be the complex given by

$$K_n(a_1,\ldots,a_m;M)=K_n(a_1,\ldots,a_m;R)\otimes_R M$$

with differentials $d_n\otimes \mathrm{id}_M$. It is clear that $K_n(a_1,\ldots,a_m;M)$ has the structure of a graded R-module in which the component $[K_\bullet(a_1,\ldots,a_m;M)]_t$ of degree t is the complex of R_0-modules

$$0\to M_{t-\sum_{i=1}^m k_i}\to\cdots\to\bigoplus_{i=1}^m M_{t-k_i}\to M_t\to 0,$$

where a_i has degree k_i for $i=1,\ldots,m$. In the same way, define the Koszul homology modules of M as

$$H_n(a_1,\ldots,a_m;M)=\mathrm{Ker}(d_n\otimes\mathrm{id}_M)/\mathrm{Im}(d_{n+1}\otimes\mathrm{id}_M)$$

for $n\geq 0$. The modules $H_\bullet(a_1,\ldots,a_m;M)$ are finitely generated graded R-modules. We denote by $H_iK_t(a_1,\ldots,a_m;M)$ the ith homology module of the component $[K_\bullet(a_1,\ldots,a_m;M)]_t$ of degree t of the Koszul complex $K_\bullet(a_1,\ldots,a_m;M)$.

Recall that a set of homogeneous elements $a_1,\ldots,a_m$ of R is said to form a *g-multiplicity system* for the graded R-module M when the component $(M/\sum_{i=1}^m a_iM)_t$ of degree t in $M/\sum_{i=1}^m a_iM$ has finite R_0-length for all t large (see [Ki]). When $a_1,\ldots,a_m$ is a g-multiplicity system for M, then, for $t\gg 0$, $H_iK_t(a_1,\ldots,a_m;M)$ has finite R_0-length (see [Ki, Proposition 2]) and hence D. Kirby (*loc. cit.*) defined the *multiplicity symbol* $e_t(a_1,\ldots,a_n;M)$ by

$$e_t(a_1,\ldots,a_m;M):=\sum_{i=0}^m(-1)^i\ell_{R_0}(H_iK_t(a_1,\ldots,a_m;M))\qquad\text{for } t\gg 0.$$

REMARK 2.1. D. Kirby [Ki, Proposition 3] proved the following basic properties of the multiplicity symbol $e_t(a_1,\ldots,a_m;M)$.

(i) If $0\to M'\to M\to M''\to 0$ is an exact sequence of graded R-modules and $a_1,\ldots,a_m$ form a g-multiplicity system for M, then

$$e_t(a_1,\ldots,a_m;M)=e_t(a_1,\ldots,a_m;M')+e_t(a_1,\ldots,a_m;M'')$$

for $t\gg 0$.

(ii) If $a_1,\ldots,a_m$ form a g-multiplicity system for M then $a_1,\ldots,a_{m-1}$ form a g-multiplicity system for both $0:_M a_m$ and M/a_mM, and for $t\gg 0$,

$$\begin{aligned}e_t(a_1,\ldots,a_m;M)&=e_t(a_1,\ldots,a_{m-1};M/a_mM)\\&\qquad-e_{t-e}(a_1,\ldots,a_{m-1};0:_M a_m),\end{aligned}$$

where a_m is a form of degree e.

(iii) For elements $a_1, \dots, a_m, a'_m$ where a_m, a'_m have degree e, e' respectively, $a_1, \dots, a_{m-1}, a_m a'_m$ form a g-multiplicity system for M if and only if both $a_1, \dots, a_{m-1}, a_m$ and $a_1, \dots, a_{m-1}, a'_m$ form g-multiplicity systems for M, and then for $t \gg 0$,

$$\begin{aligned} e_t(a_1, \dots, a_{m-1}, a_m a'_m; M) &= e_t(a_1, \dots, a_m; M) + e_{t-e}(a_1, \dots, a'_m; M) \\ &= e_t(a_1, \dots, a'_m; M) + e_{t-e'}(a_1, \dots, a_m; M). \end{aligned}$$

Lemma 2.2. *Let $K_\bullet := K_\bullet(a_1, \dots, a_m; M)$ be the Koszul complex, let $I = (a_1, \dots, a_m)$ be an R_0-submodule of R_1 of finite colength, and for each integer n, let $K_\bullet^{(n)}$ be the subcomplex*

$$K_\bullet^{(n)} : 0 \to I^n K_m \to I^{n+1} K_{m-1} \to \cdots \to I^{n+m} K_0 \to 0$$

of $K_\bullet$. Then

$$H_i(K_\bullet) = H_i(K_\bullet / K_\bullet^{(n)}) \quad \textit{for } n \gg 0.$$

Proof. Since the short exact sequence of graded complexes of R-modules

$$0 \to K_\bullet^{(n)} \to K_\bullet \to K_\bullet / K_\bullet^{(n)} \to 0$$

gives a long exact sequence of Koszul homologies

$$\cdots \to H_{i+1}(K_\bullet / K_\bullet^{(n)}) \to H_i(K_\bullet^{(n)}) \to H_i(K_\bullet) \to H_i(K_\bullet / K_\bullet^{(n)}) \to \cdots,$$

it is enough to prove that for $n \gg 0$ and for every $i \geq 0$ we have

$$H_i(K_\bullet^{(n)}) = 0,$$

i.e. the Koszul complex

$$K_\bullet^{(n)} : 0 \to I^n K_m \to I^{n+1} K_{m-1} \to \cdots \to I^{n+m} K_0 \to 0 \tag{2.1}$$

is exact. Consider the ith cycle module

$$Z_i(K_\bullet^{(n)}) = Z_i(K_\bullet) \cap I^{n+m-i} K_i.$$

By the Artin–Rees Lemma, we have

$$Z_i(K_\bullet) \cap I^{n+m-i} K_i = I(Z_i(K_\bullet) \cap I^{n+m-i-1} K_i) \tag{2.2}$$

for $n \gg 0$. Choose $n_0 > 0$ such that this equality holds for all i and $n \geq n_0$. Now, let $n \geq n_0$ and let $z \in Z_i(K_\bullet^{(n)})$. Then, by (2.2),

$$z = \sum_i a_i z_i \quad \text{with } z_i \in Z_i(K_\bullet) \cap I^{n+m-i-1} K_i \text{ and } a_i \in I.$$

Let $e_1, \dots, e_m$ be a basis of $K_1(\underline{a}; R)$ such that $d_{\underline{a}}(e_i) = a_i$, where $d_{\underline{a}}$ denotes the differential of $K_\bullet(\underline{a}; R)$. Set

$$w := \sum_{i=1}^m e_i z_i \in I^{n+m-i-1} K_{i+1}.$$

Now

$$\begin{aligned} d_{\underline{a}}(w) &= \sum_{i=1}^{m} d_{\underline{a}}(e_i z_i) = \sum_{i=1}^{m} d_{\underline{a}}((e_i) z_i + e_i d_{\underline{a}}(z_i)) \\ &= \sum_{i=1}^{m} d_{\underline{a}}(e_i) z_i \quad \text{(because } z_i \in Z_i(K_\bullet^{(n)})) \\ &= \sum_{i=1}^{m} e_i z_i = z. \end{aligned}$$

This implies that $Z_i(K_\bullet^{(n)}) \subseteq B_i(K_\bullet^{(n)})$ for $i \geq 0$ and $m \geq m_0$. Hence the complex (2.1) is exact. ∎

Lemma 2.3. *Let*

$$F_\bullet : 0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to 0$$

be a complex of finitely generated modules over a Noetherian local ring R such that each F_i is of finite length. Then $\sum_i (-1)^i \ell(H_i(F_\bullet)) = \sum_i (-1)^i \ell(F_i)$.

Proof. Let $\phi_i : F_i \to F_{i-1}$ be the complex maps, for all $i = 0, \ldots, n$. The conclusion follows from the exact sequences $0 \to \operatorname{Ker}(\phi_i) \to F_i \to \operatorname{Im}(\phi_i) \to 0$ and $0 \to \operatorname{Im}(\phi_{i+1}) \to \operatorname{Ker}(\phi_i) \to \operatorname{Ker}(\phi_i)/\operatorname{Im}(\phi_{i+1}) \to 0$. ∎

Remark 2.4. D. Kirby [Ki, Theorem 5] shows that the multiplicity symbol $e_t(a_1, \ldots, a_m; M)$ has a polynomial expression for $t \gg 0$. This also ensures that $e_t(a_1, \ldots, a_m; M)$ is positive, and it is independent of t for $t \gg 0$ if $m = \dim_R(M) - 1$.

Lemma 2.5. *Assume that $\underline{a} = a_1, \ldots, a_m$ is a sequence of homogeneous elements of positive degree which form a g-multiplicity system for M, and $a_m M = 0$. If $m = \dim_R(M) - 1$, then*

$$e_t(\underline{a}; M) = 0.$$

Proof. This follows directly from Remarks 2.1(ii) and 2.4. ∎

Lemma 2.6. *If $\underline{a} = a_1, \ldots, a_m$ is a g-multiplicity system for M, with a_i of degree k_i, then*

$$e_t(\underline{a}; M) \leq \ell((M/(\underline{a})M)_t)$$

for all $t \gg 0$.

Proof. We use induction on m. If $m = 1$ then

$$\begin{aligned} e_t(a_1; M) &= \ell_{R_0}(H_0 K_t(a_1; M)) - \ell_{R_0}(H_1 K_t(a_1, \ldots, a_m; M)) \\ &\leq \ell_{R_0}(H_0 K_t(a_1; M)) = \ell((M/a_1 M)_t). \end{aligned}$$

Hence, we may assume that $m > 1$. Now, by Remark 2.1 we know that $\underline{a'} = a_2, \ldots, a_m$ is a g-multiplicity system for both $(0 :_M a_1)$ and $M/a_1 M$, and for $t \gg 0$ we have

$$\begin{aligned} e_t(\underline{a}; M) &= e_t(\underline{a'}; M/a_1M) - e_{t-k_1}(\underline{a'}; (0 :_M a_1)) \\ &\le e_t(\underline{a'}; M/a_1M) \le \ell((M/(\underline{a})M)_t), \end{aligned}$$

where the last inequality follows by induction hypothesis. ■

Lemma 2.7. *If* $\underline{a} = a_1, \dots, a_m$ *is a g-multiplicity system for* M*, with* a_i *of degree* k_i *and* $m = \dim_R(M) - 1$*, then for* $t \gg 0$*:*

(i) $e_{t+\sum(r_i-1)k_i}(a_1^{r_1}, \dots, a_m^{r_m}; M) = r_1 \cdots r_m e_t(a_1, \dots, a_m; M)$ *for all* $r_i \ge 1$, $i = 1, \dots, m$;

(ii) $0 \le e_t(a_1, \dots, a_m; M) \le \dfrac{\ell\big((M/(a_1^{r_1}, \dots, a_m^{r_m})M)_{t+\sum(r_i-1)k_i}\big)}{r_1 \cdots r_m}$.

Proof. Since $m = \dim_R(M) - 1$, we know from Remark 2.4 that $e_t(a_1, \dots, a_m; M)$ is independent of t for $t \gg 0$. Hence, by Remark 2.1(iii), item (i) follows readily.

We now prove (ii). By (i) we have

$$r_1 \cdots r_m \, e_t(a_1, \dots, a_m; M) = e_{t+\sum(r_i-1)k_i}(a_1^{r_1}, \dots, a_m^{r_m}; M),$$

and by Lemma 2.6,

$$e_{t+\sum(r_i-1)k_i}(a_1^{r_1}, \dots, a_m^{r_m}; M) \le \ell\big((M/(a_1^{r_1}, \dots, a_m^{r_m})M)_{t+\sum(r_i-1)k_i}\big). \ ■$$

3. Lech's formula.

In this section we prove Lech's formula for a g-multiplicity system of a graded R-module M.

Theorem 3.1. *Assume that* $\underline{a} = a_1, \dots, a_m$ *is a sequence of homogeneous elements, with* a_i *of degree* k_i*, which form a g-multiplicity system for* M*, and* $m = \dim_R(M) - 1$*. Then, for* $t \gg 0$*,*

$$\lim_{r_1, \dots, r_m \to \infty} \frac{\ell\big((M/(a_1^{r_1}, \dots, a_m^{r_m})M)_{t+\sum(r_i-1)k_i}\big)}{r_1 \cdots r_m} = e_t(a_1, \dots, a_m; M).$$

Proof. We use induction on m. If $m = 1$ then, by Remark 2.1(ii),

$$e_{t+(r_1-1)k_1}(a_1^{r_1}; M) = \ell((M/a_1^{r_1}M)_{t+(r_1-1)k_1}) - e_{t-k_1}((0 :_M a_1^{r_1})).$$

We have an ascending sequence of submodules of M,

$$(0 :_M a_1) \subseteq (0 :_M a_1^2) \subseteq (0 :_M a_1^3) \subseteq \cdots .$$

Since M is Noetherian, there exists an integer s such that $(0 :_M a_1^{r_1}) = (0 :_M a_1^s)$ for all $r_1 \ge s$. Hence, for $t \gg 0$, $e_{t-k_1}((0 :_M a_1^{r_1}))$ is independent of r_1, and so depends possibly only on t. Thus,

$$\lim_{r_1 \to \infty} \frac{e_{t-k_1}(0 :_M a_1^{r_1})}{r_1} = 0.$$

Now, by Remark 2.1(iii) we have $e_{t+(r_1-1)k_1}(a_1^{r_1};M) = r_1 e_t(a_1;M)$. It follows that

$$\ell\left(\left(\frac{M}{a_1^{r_1}M}\right)_{t+(r_1-1)k_1}\right) = r_1 e_t(a_1;M) + e_{t-k_1}((0:_M a_1^{r_1}))$$

for all $t \gg 0$ and $r_1 \gg 0$. In particular, we see that

$$\lim_{r_1\to\infty} \frac{\ell((M/a_1^{r_1}M)_{t+(r_1-1)k_1})}{r_1} = e_t(a_1;M).$$

Hence, we may assume that $m > 1$. By Lemma 2.7, for all positive integers $r_1,\dots,r_m$ and $t \gg 0$,

$$e_t(a_1,\dots,a_m;M) \leq \lim_{r_1,\dots,r_m\to\infty} \frac{\ell\big((M/(a_1^{r_1},\dots,a_m^{r_m})M)_{t+\sum(r_i-1)k_i}\big)}{r_1\cdots r_m}.$$

Let $I = (a_1^{r_1},\dots,a_m^{r_m}) \subseteq R$. If $0 \to K \to M \to N \to 0$ is a short exact sequence of finitely generated graded R-modules, then $K/IK \to M/IM \to N/IN \to 0$ is another such sequence, and hence $\ell((M/IM)_t) \leq \ell((N/IN)_t) + \ell((K/IK)_t)$. Thus if $0 \subseteq M^0 \subseteq M^1 \subseteq \cdots \subseteq M^k = M$ is a homogeneous prime filtration of M, then $\ell((M/IM)_t) \leq \sum_{i=1}^k \ell((M^i/IM^i + M^{i-1})_t)$. If we know that the conclusion holds for cyclic graded modules R/P where P is a homogeneous prime ideal of R, then by Remark 2.1(i), we have

$$\begin{aligned}\lim_{r_1,\dots,r_m\to\infty} \frac{\ell((M/IM)_{t+\sum(r_i-1)k_i})}{r_1\cdots r_m} &\leq \sum_{i=1}^k e_t(a_1^{r_1},\dots,a_m^{r_m};M^i/M^{i-1})\\ &= e_t(a_1,\dots,a_m;M),\end{aligned}$$

which proves the conclusion for M. Hence we may assume that $M = R$ is a domain. Set $R' = R/a_1R$. By induction hypotheses, we have

$$\lim_{r_2,\dots,r_m\to\infty} \frac{\ell\big((R'/(a_2^{r_2},\dots,a_m^{r_m})R')_{t+\sum_{i\geq 2}(r_i-1)k_i}\big)}{r_2\cdots r_m} = e_t(a_2,\dots,a_m;R').$$

By Remark 2.1(ii), and since R is a domain, we have $e_t(a_2,\dots,a_m;R') = e_t(a_1,\dots,a_m;R)$. From the exact sequence

$$\frac{R}{(a_1,a_2^{r_2},\dots,a_m^{r_m})R} \to \frac{R}{(a_1^{r_1},a_2^{r_2},\dots,a_m^{r_m})R} \to \frac{R}{(a_1^{r_1-1},a_2^{r_2},\dots,a_m^{r_m})R} \to 0,$$

where the leftmost map is multiplication by $a_1^{r_1-1}$, we see that the length of $\big(\frac{R}{(a_1^{r_1},a_2^{r_2},\dots,a_m^{r_m})R}\big)_{t+\sum(r_i-1)k_i}$ is at most $r_1\ell\big(\big(\frac{R}{(a_1,a_2^{r_2},\dots,a_m^{r_m})R}\big)_{t+\sum_{i\geq 2}(r_i-1)k_i}\big)$. Hence

$$
\begin{aligned}
\lim_{r_1,\dots,r_m\to\infty} & \frac{\ell((R/IR)_{t+\sum(r_i-1)k_i})}{r_1\cdots r_m} \\
&\le \lim_{r_1,\dots,r_m\to\infty} \frac{r_1\ell\big((R/(a_1,a_2^{r_2},\dots,a_m^{r_m})R)_{t+\sum_{i\ge 2}(r_i-1)k_i}\big)}{r_1\cdots r_m} \\
&= \lim_{r_2,\dots,r_m\to\infty} \frac{\ell\big((R'/(a_2^{r_2},\dots,a_m^{r_m})R')_{t+\sum_{i\ge 2}(r_i-1)k_i}\big)}{r_2\cdots r_m} \\
&= e_t(a_2,\dots,a_m;R') = e_t(a_1,\dots,a_m;R). \ \blacksquare
\end{aligned}
$$

4. Buchsbaum–Rim multiplicity and multiplicity symbol

Setup. Fix an arbitrary Noetherian local ring $(A,\mathfrak{m})$; fix a graded A-algebra $R=\bigoplus R_n$, generated as an A-algebra by finitely many elements of degree one, and fix a finitely generated graded R-module M. Let I be an A-submodule of R_1 such that $\ell(R_1/I)<\infty$. Let $r:=\dim(\mathrm{Proj}(R))$. The *Hilbert–Samuel function* of n is defined by

$$H_I(n;M):=\ell(M_n/I^nM_0).$$

As a function of n, it is eventually a polynomial, called the *Hilbert–Samuel polynomial* and denoted by $P_I(n;M)$, of total degree $\dim(\mathrm{Supp}(M))$, which is at most r (see [KT1, Theorem 5.7]), and the coefficient of $n^r/r!$ is denoted by $e_{\mathrm{BR}}(I;M)$ and called the *Buchsbaum–Rim multiplicity of I with respect to M.*

Notice that $e_{\mathrm{BR}}(I;M)=0$ if $\dim(\mathrm{Supp}(M))<r$. The notion of Buchsbaum–Rim multiplicity for modules goes back to [BR] and it was considered in the above generality in [Ki], [KR1], [KT1], [KR2], [KT2], [Ka], [CJ] and [SUV].

We now generalize Serre's theorem [AB] relating the multiplicity symbol $e_t(\underline{a};M)$ to the Euler characteristic of the Koszul complex.

THEOREM 4.1. *Assume that $\underline{a}=a_1,\dots,a_m$ is a sequence of homogeneous elements of degree one which form a g-multiplicity system for M, where $m=\dim(M)-1$. Let I be the R_0-submodule of R_1 generated by $\underline{a}$. Then for $t\gg 0$,*

$$e_t(\underline{a};M)=e_{\mathrm{BR}}(I;M).$$

Proof. Notice that

$$
\begin{aligned}
e_t(\underline{a};M) &= \sum_{i\in\mathbb{Z}}(-1)^i\ell_{R_0}(H_i(K_t(\underline{a};M))) \\
&= \sum_{i\in\mathbb{Z}}(-1)^i\ell_{R_0}\big(H_i(K_t(\underline{a};M)/K_t^{(n)}(\underline{a};M))\big) \\
&= \sum_{i\in\mathbb{Z}}(-1)^i\ell_{R_0}\big((K_i(\underline{a};M))_t/(K_i^{(n)}(\underline{a};M))_t\big),
\end{aligned}
$$

where the second equality follows from Lemma 2.2 and the last one from Lemma 2.3, since $K_t(\underline{a};M)/K_t^{(n)}(\underline{a};M)$ is of finite length. Therefore, for $n \gg 0$,

$$\begin{aligned} e_t(\underline{a};M) &= \sum_{i\in\mathbb{Z}}(-1)^i \ell_{R_0}\big((K_i(\underline{a};M))_t/(K_i^{(n)}(\underline{a};M))_t\big) \\ &= \sum_{i\in\mathbb{Z}}(-1)^i \ell_{R_0}\big((K_i(\underline{a};M)/I^{n+m-i}K_i^{(n)}(\underline{a};M))_t\big) \\ &= \sum_{i\in\mathbb{Z}}(-1)^i \binom{m}{i} \ell_{R_0}(M_t/I^{n+m-i}M_{t-m-n+i}), \end{aligned}$$

and setting $t = n + m$ we obtain

$$\begin{aligned} e_t(\underline{a};M) &= \sum_{i\in\mathbb{Z}}(-1)^i \binom{m}{i} \ell_{R_0}(M_{n+m}/I^{n+m-i}M_i) \\ &= \sum_{i\in\mathbb{Z}}(-1)^i \binom{m}{i} P_I(n+m-i;M) \\ &= \triangle^m P_I(n;M) = e_{\mathrm{BR}}(I;M). \ \blacksquare \end{aligned}$$

Acknowledgements. The first author was partially supported by CNPq-Brazil, Grant 308007/2014-0, and the second author was partially supported by CNPq-Brazil, Grant 245872/2012-4, and FAPESP-Brazil, Grant 2012/20304-1.

The authors would like to thank the referee for his/her comments.

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