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The first moment of central values of symmetric square L-functions of cusp forms

by

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1. Introduction. Let f be a primitive holomorphic cusp form of even weight k, level 1 and trivial central character with Fourier expansion $f = \sum_{n=1}^{\infty} n^{(k-1)/2}\lambda_f(n)e(nz)$ for $z \in \mathbb{H}$. Define, for $\Re(s) > 1$,

$$L(\mathrm{sym}^2 f, s) = \zeta(2s)\sum_{n\ge 1}^{\infty}\frac{\lambda_f(n^2)}{n^s}.$$

We also define

$$\gamma_\infty(\mathrm{sym}^2, s) = \pi^{-3s/2}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right).$$

Then we have the functional equation, by Shimura [20],

$$\begin{aligned}\Lambda(\mathrm{sym}^2 f, s) &:= \gamma_\infty(\mathrm{sym}^2, s)L(\mathrm{sym}^2 f, s)\\ &= \gamma_\infty(\mathrm{sym}^2, 1-s)L(\mathrm{sym}^2 f, 1-s)\\ &= \Lambda(\mathrm{sym}^2 f, 1-s).\end{aligned}$$

Note that $L(\mathrm{sym}^2 f, s)$ is holomorphic over $\mathbb{C}$. Define the *harmonic weight* of f by

$$\omega_f = \frac{\Gamma(k-1)}{(4\pi)^{k-1}\|f\|^2} = \frac{2\pi^2}{k-1}L(\mathrm{sym}^2 f, 1)^{-1}.$$

In view of [5] and [4], we have

$$k^{-\epsilon} \ll L(\mathrm{sym}^2 f, 1) \ll k^{\epsilon}.$$

The trivial bound of ω_f is $\ll k^{\epsilon-1}$.

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The moments of $L(\mathrm{sym}^2 f, s)$ in the level aspect have been extensively studied in [2, 6, 22, 19]. Let H_k be the set of primitive forms of weight k and level 1. In 2002, Lau [13, (3.13)] proved that

$$\sum_{f\in H_k} \omega_f L(\mathrm{sym}^2 f, 1/2) = \log k + C + O(k^{-0.008}).$$

But he did not try to make the error term as small as possible. Then, in 2007, Khan [7] gave a detailed study of the first moment of $L(\mathrm{sym}^2 f, 1/2)$ in the weight aspect and he got the asymptotic formula

$$\sum_{f\in H_k} \omega_f L(\mathrm{sym}^2 f, 1/2) = \log k + C + O(k^{-1/20+\epsilon})$$

where C is an absolute constant. Later in 2012 Sun [21] sharpened the error term to $O(k^{-1/2})$. However, there was a gap in the proof. To fix the gap, we shall establish Theorem 1.1 by following the idea of Lau and Tsang in [15]. It is worth noting that Khan [8] also got the first mollified harmonic moment of $L(\mathrm{sym}^2 f, 1/2)$ over the average of k. For the natural moment (i.e. without harmonic weight), Kohnen and Sengupta [10] obtained an upper bound on the first moment under the condition that $L(\mathrm{sym}^2 f, 1/2) \geq 0$ for all $f \in H_k$. Their approach was to use Zagier's kernel (see [24]). Inspired by their method, in 2012 Luo [18] gave an upper bound for the second moment, while recently Lam [12] shortened the length of the extra average of k. Lam's approach used Petersson's trace formula. The formula also appeared in Lau's paper [14] which gave a mean square estimate in the weight aspect for symmetric square L-functions on the critical line.

The following result generalizes the results of Khan and Sun.

THEOREM 1.1. *For any $A \geq 1$ and $l \ll k^A$, we have*

$$\sum_{f\in H_k} \omega_f \lambda_f(l^2) L(\mathrm{sym}^2 f, 1/2) = \frac{2}{\sqrt{l}}\left(-\frac{\log l}{2} + \gamma + \frac{1}{2}\frac{\gamma_\infty'(\mathrm{sym}^2, 1/2)}{\gamma_\infty(\mathrm{sym}^2, 1/2)}\right) + O_{A,\epsilon}(k^{-1/2+\epsilon} l).$$

REMARK 1.2. Since

$$\frac{\gamma_\infty'(\mathrm{sym}^2, 1/2)}{\gamma_\infty(\mathrm{sym}^2, 1/2)} = -\frac{3}{2}\log\pi + \frac{1}{2}\frac{\Gamma'(3/4)}{\Gamma(3/4)} + \frac{1}{2}\frac{\Gamma'\left(\frac{2k-1}{4}\right)}{\Gamma\left(\frac{2k-1}{4}\right)} + \frac{1}{2}\frac{\Gamma'\left(\frac{2k+1}{4}\right)}{\Gamma\left(\frac{2k+1}{4}\right)} \sim \log k,$$

the above result can be restated as: for $l \ll k^A$,

$$\sum_{f\in H_k} \omega_f \lambda_f(l^2) L(\mathrm{sym}^2 f, 1/2) = \frac{2}{\sqrt{l}}\left(-\frac{\log l}{2} + \gamma - \frac{3}{4}\log\pi + \frac{1}{4}\frac{\Gamma'(3/4)}{\Gamma(3/4)} + \frac{1}{4}\frac{\Gamma'\left(\frac{2k-1}{4}\right)}{\Gamma\left(\frac{2k-1}{4}\right)} + \frac{1}{4}\frac{\Gamma'\left(\frac{2k+1}{4}\right)}{\Gamma\left(\frac{2k+1}{4}\right)}\right) + O_\epsilon(k^{-1/2+\epsilon} l).$$

REMARK 1.3. The above main terms are dominating when $l \ll k^{1/3-\epsilon}$.

REMARK 1.4. When $l = 1$, Theorem 1.1 recovers Sun's result [21].

REMARK 1.5. The factor $k^{-1/2+\epsilon}$ in the error term of Theorem 1.1 seems to be the limit of the method. It would be interesting to obtain an omega result. Also notice that if an extra average on k is introduced, the error term could be sharpened (see [7]).

REMARK 1.6. Very recently Theorem 1.1 was improved in [17] and [1]. An asymptotic formula with an arbitrary power saving error term for the case $l = 1$ was established in [17] and [1] independently, while the error bound was improved to $l^{5/6+\epsilon} k^{1/2+\epsilon}$ for $l > 1$.

Using the same method as in the proof of Theorem 1.1, we obtain the remark below.

REMARK 1.7. For any $A \geq 1$ and for all non-square natural numbers l with $l \ll k^A$, we have

$$\sum_{f\in H_k} \omega_f \lambda_f(l) L(\mathrm{sym}^2 f, 1/2) \ll_\epsilon k^{-1/2+\epsilon} l^{1/2}.$$

Note that, in the above remark, there is no main term since the diagonal term (in Petersson's trace formula) δ_{l,n^2} is zero.

In the literature, there are three useful tools to obtain the moment: Petersson's trace formula, Selberg's trace formula and Zagier's kernel. Inspired by [11], our key tool in computing the first moment without harmonic weight is Petersson's trace formula. Its advantage over Selberg's trace formula and Zagier's kernel is that we can make use of two classical tools: Perron's formula and the large sieve inequality. Our method here is based on the application of Theorem 1.1. To remove the harmonic weight, we use Perron's formula to give an approximation with a Dirichlet polynomial. With the help of the large sieve inequality, we get an asymptotic formula for the first moment without harmonic weight.

Corollary 1.8. *Suppose that* $L(\operatorname{sym}^2 f, 1/2) \geq 0$ *for all* $f \in H_k$ *and all* k. *Then*

$$\sum_{f\in H_k} L(\operatorname{sym}^2 f, 1/2) = \frac{k-1}{12}\left(\zeta'\left(\frac{3}{2}\right) + 2\gamma\zeta\left(\frac{3}{2}\right) + \frac{\gamma_\infty'(\operatorname{sym}^2, 1/2)}{\gamma_\infty(\operatorname{sym}^2, 1/2)}\zeta\left(\frac{3}{2}\right)\right) + O_\epsilon\big(k^{\frac{159}{160}+\epsilon}\big).$$

Note that $\frac{\gamma_\infty'(\operatorname{sym}^2,1/2)}{\gamma_\infty(\operatorname{sym}^2,1/2)} \sim \log k$. The corollary improves the result of [10], in which only the upper bound is obtained.

Remark 1.9. The condition $L(\operatorname{sym}^2 f, 1/2) \geq 0$ for all $f \in H_k$ follows from the Generalized Riemann Hypothesis.

Unconditional results similar to the above corollary hold for the first and second moments of Hecke L-functions:

Theorem 1.10. *For* $k \equiv 0 \pmod 4$, *we have*

$$\sum_{f\in H_k} L(f, 1/2) = \frac{k-1}{12}\zeta(2) + O_\epsilon(k^{59/60+\epsilon}).$$

Theorem 1.11. *For* $k \equiv 0 \pmod 4$, *we have*

$$\sum_{f\in H_k} L(f, 1/2)^2 = \frac{k-1}{3}\left(\frac{\zeta^3(2)}{\zeta(4)}\left(\frac{\Gamma'(k/2)}{\Gamma(k/2)} + \gamma - \log(2\pi)\right) + \frac{3\zeta^2(2)\zeta'(2) - 2\zeta^3(2)\zeta'(4)}{\zeta^2(4)}\right) + O_\epsilon\big(k^{\frac{179}{180}+\epsilon}\big).$$

We omit the proofs of Theorems 1.10 and 1.11 as they are similar to that of Corollary 1.8.

Remark 1.12. Unlike Corollary 1.8, the above two theorems are unconditional due to the fact that $L(f, 1/2) \geq 0$. This fact was proved by using Waldspurger's formula, and in [9] a good account of Waldspurger's formula can be found.

Throughout this paper, ϵ will always denote an arbitrarily small positive constant, not necessarily the same one at each occurrence.

2. Preparation. In this section, we prepare several lemmas for the proof of our theorem. The first lemma applies Perron's formula to give an approximation of the harmonic weight by a Dirichlet polynomial. Then we give some technical lemmas.

LEMMA 2.1. *Let $f \in H_k$ and $y \ge 1$. For any fixed $\epsilon > 0$,*

$$L(\mathrm{sym}^2 f, 1) = \zeta(2) \sum_{n \le y} \frac{\lambda_f(n^2)}{n} + O_\epsilon\big(k^\epsilon y^\epsilon (y^{-2/7} + k^{3/4} y^{-1/2})\big).$$

Proof. Employing Perron's formula [23, Corollary II.2.1] with $B(x) = x^\epsilon$, $\alpha = 3$ and $T = y^{2/7}$, we deduce that

$$\sum_{n \le y} \frac{\lambda_f(n^2)}{n} = \frac{1}{2\pi i} \int_{1/\log y - iy^{2/7}}^{1/\log y + iy^{2/7}} \frac{L(\mathrm{sym}^2 f, 1+s)}{\zeta(2+2s)} \frac{y^s}{s}\, ds + O_\epsilon\big(k^\epsilon y^\epsilon (y^{-2/7} + y^{-1})\big).$$

Moving the line segment of integration to $\Re s = -1/2 + \epsilon$ and using the classical convexity bound $L(\mathrm{sym}^2 f, s) \ll_\epsilon (k + |\Im s|)^{\frac{3}{2}\max\{0, 1 - \Re s\} + \epsilon}$ yields the result. ∎

The next lemma controls the "tail" part of the Dirichlet polynomial. We define

$$\omega_f(x, y) = \sum_{x < n \le y} \frac{\lambda_f(n^2)}{n}.$$

LEMMA 2.2 ([16, Lemma 2.5]). *For any $A, \epsilon > 0$ and every natural number j, we have*

$$\sum_{f \in H_k} \omega_f(x, y)^{2j} \ll_{A,\epsilon,j} k^\epsilon$$

uniformly for $2 \mid k$ and $k^5 \le x^j < y^j \le k^A$.

The lemma below transforms an integral involving the gamma function and the exponential function into an integral involving the Bessel functions and the exponential function, which is easier to control.

LEMMA 2.3. *Let $y > 0$ and $0 < \Re s < k/2 - 1$. Then*

$$\frac{1}{2\pi i} \int_{(1/2)} \frac{\Gamma\big(\frac{k+s-w}{2}\big)}{\Gamma\big(\frac{k-s+w}{2}\big)} \Gamma(w) e(\pm w/4) y^{-w}\, dw = 2^{-s} \int_0^\infty J_{k-1}(x) \exp(\pm iyx/2) x^s\, dx.$$

Proof. The two integrals are holomorphic in s for $0 < \Re s < k/2 - 1$. Using the relation $\Gamma\big(\frac{k+s}{2}\big)/\Gamma\big(\frac{k-s}{2}\big) = \int_0^\infty J_{k-1}(x)(x/2)^s\, dx$ and moving the line of integration to $\Re w = -\infty$ yields the result. ∎

The following lemma estimates an integral involving Bessel functions and the exponential function. The proof follows [15, proof of Lemma 2.4] with minor modifications.

LEMMA 2.4. *For $\Re u \ge 1/2$ and any $A_1, A_2 \ge 1$, we have:*

(a) *if $a > A_1$, then*

$$\int_0^\infty J_{k-1}(x)\exp(\pm iax)x^{-u-1/2}\,dx \ll \exp\Big(\frac{\pi}{2}|\Im u|\Big)a^{\Re u-k+1/2}(1-a^{-2})^{-1},$$

(b) *if $a \le A_2$, then*

$$\int_0^\infty J_{k-1}(x)\exp(\pm iax)x^{-u-1/2}\,dx \ll (|\Im u|+1)k^{-\Re u-1/2}(\log k)^2.$$

All the implied constants are absolute.

Proof. (a) For $a > 2$, by [3, Formula 1 in 6.621], we have

$$\int_0^\infty J_{k-1}(x)\exp(\pm iax)x^{-u-1/2}\,dx$$
$$= \frac{1}{2^{k-1}(\pm ia)^{k-u-1/2}}\frac{\Gamma(k-u+1/2)}{\Gamma(k)}F\Big(\frac{k-u+1/2}{2},\frac{k-u+3/2}{2};k,a^{-2}\Big),$$

where F is the hypergeometric function

$$F(\alpha,\beta;\gamma,z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{r\ge 0}\frac{\Gamma(\alpha+r)\Gamma(\beta+r)}{\Gamma(\gamma+r)}\,\frac{z^r}{r!}.$$

Since $\Gamma(z)\Gamma(z+1/2) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z)$, the integral is

$$\ll \frac{\exp(\frac{\pi}{2}|\Im u|)}{2^u a^{k-u-1/2}}\sum_{r\ge 0}\frac{\big|\Gamma(\frac{k-u+1/2}{2}+r)\Gamma(\frac{k-u+3/2}{2}+r)\big|}{\Gamma(k+r)}\,\frac{a^{-2r}}{r!}.$$

From $\Gamma(z)\Gamma(z+1/2) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z)$ again, we have

$$\Gamma\Big(\frac{k-u+1/2}{2}+r\Big)\Gamma\Big(\frac{k-u+3/2}{2}+r\Big)$$
$$= \sqrt{\pi}\,2^{u-2r-k+1/2}\Gamma(k-u+1/2+2r),$$

and Stirling's estimate gives

$$\frac{\Gamma(k-u+1/2+2r)}{\Gamma(k+r)\Gamma(r+1)} \ll \frac{\Gamma(k+2r)}{\Gamma(k+r)\Gamma(r+1)}$$
$$\ll \frac{(k+2r)^{k+2r-1/2}}{(k+r)^{k+r-1/2}(r+1)^{r+1-1/2}}$$
$$\ll \Big(1-\frac{k}{2k+2r}\Big)^{k+r}\Big(1+\frac{k-2}{2r+2}\Big)^{r+1}2^{k+2r+1}$$
$$\ll \exp\Big(-\frac{k}{2}+\frac{k}{2}\Big)2^{k+2r+1} = 2^{k+2r+1}.$$

Hence the integral is

$$\begin{aligned} &\ll \frac{\exp\left(\frac{\pi}{2}|\Im u|\right)}{2^u a^{k-u-1/2}} \sum_{r\geq 0} 2^{u-2r-k+1/2} 2^{k+2r+1} a^{-2r} \\ &\ll \frac{\exp\left(\frac{\pi}{2}|\Im u|\right)}{a^{k-u-1/2}} \sum_{r\geq 0} a^{-2r} \\ &\ll \frac{\exp\left(\frac{\pi}{2}|\Im u|\right)}{a^{k-u-1/2}} (1-a^{-2})^{-1}. \end{aligned}$$

(b) This follows from [15, proof of Lemma 2.4]. ∎

We need the following lemma to bound the average of Kloosterman sums.

LEMMA 2.5.

$$\sum_{b \,(\mathrm{mod}\, c)} |S(b^2, l^2; c)| \ll c^{3/2+\epsilon}.$$

Proof. Weil's bound for the Kloosterman sum yields

$$|S(n,m;c)| \leq d(c) c^{1/2} (n,m,c)^{1/2}.$$

Then the following estimate follows readily:

$$\begin{aligned} \sum_{b \,(\mathrm{mod}\, c)} S(b^2, l^2; c) &\ll c^{1/2+\epsilon} \sum_{b \,(\mathrm{mod}\, c)} (b^2, c)^{1/2} \\ &\ll c^{1/2+\epsilon} \sum_{d|c} \sum_{d|b^2,\, b\leq c} d^{1/2} \ll c^{3/2+\epsilon}. \end{aligned}$$ ∎

Note that Lemma 2.5 is not sufficient to control the non-diagonal terms. However, its proof is much simpler than that of the next lemma. The latter, which is [2, Corollary 4], applies to the character sum and hence gives a better estimate.

LEMMA 2.6. *Consider the unique factorization*

$$c = 2^r c_1 c_2^2 c_3^2$$

where c_1, c_2, c_3 *are odd,* $\mu^2(c_1) = 1$, $c_2 \,|\, c_1^\infty$, $(c_1, c_3) = 1$. *If* $n = \pm 2l$, *then*

$$\sum_{b \,(\mathrm{mod}\, c)} S(b^2, l^2; c) e\left(\pm \frac{b}{c} n\right)$$

vanishes unless c *is of the form* t^2 *for some integer* t, *in which case*

$$\sum_{b \,(\mathrm{mod}\, c)} S(b^2, l^2; c) e\left(\pm \frac{b}{c}(2l)\right) \ll \sqrt{c}\, \phi(c).$$

In any case,

$$\sum_{b\,(\mathrm{mod}\,c)} S(b^2,l^2;c)e\Big(\pm\frac{b}{c}n\Big) \ll \sqrt{cc_1}(c/c_1,4n^2-l^2).$$

The next lemma is the starting point of the proof of the main theorem.

LEMMA 2.7.

$$L(\mathrm{sym}^2 f,1/2) = 2\sum_{n\ge 1}\frac{\lambda_f(n^2)}{\sqrt{n}}V(n)$$

where

$$V(n) = \frac{1}{2\pi i}\int_{(1+\epsilon)} \frac{\gamma_\infty(\mathrm{sym}^2,1/2+u)}{\gamma_\infty(\mathrm{sym}^2,1/2)}\zeta(1+2u)\frac{G(u)}{n^u u}\,du$$

and $G(z)=\exp(z^2)$.

Proof. We start with

$$\begin{aligned} I(\mathrm{sym}^2 f,s) &:= \frac{1}{2\pi i}\int_{\Re(u)=1+\epsilon} \Lambda(\mathrm{sym}^2 f,s+u)\frac{G(u)\,du}{u} \\ &= \Lambda(\mathrm{sym}^2 f,s) + \frac{1}{2\pi i}\int_{\Re(u)=-1-\epsilon} \Lambda(\mathrm{sym}^2 f,s+u)\frac{G(u)\,du}{u} \\ &= \Lambda(\mathrm{sym}^2 f,s) + \frac{1}{2\pi i}\int_{\Re(u)=-1-\epsilon} \Lambda(\mathrm{sym}^2 f,1-s-u)\frac{G(u)\,du}{u} \\ &= \Lambda(\mathrm{sym}^2 f,s) - \frac{1}{2\pi i}\int_{\Re(u)=1+\epsilon} \Lambda(\mathrm{sym}^2 f,1-s+u)\frac{G(u)\,du}{u}, \end{aligned}$$

by the functional equation. Then we are done by rearrangement. ∎

3. Proof of the twisted first moment. We start with

$$L(\mathrm{sym}^2 f,1/2) = 2\sum_{n\ge 1}\frac{\lambda_f(n^2)}{\sqrt{n}}V(n)$$

from Lemma 2.7, where

$$V(n) = \frac{1}{2\pi i}\int_{(1+\epsilon)} \frac{\gamma_\infty(\mathrm{sym}^2,1/2+u)}{\gamma_\infty(\mathrm{sym}^2,1/2)}\zeta(1+2u)\frac{G(u)}{n^u u}\,du.$$

By Petersson's trace formula

$$\sum_{f\in H_k}\omega_f\lambda_f(m)\lambda_f(n) = \delta_{m,n} + 2\pi i^{-k}\sum_{c>0}\frac{S(m,n;c)}{c}J_{k-1}\Big(\frac{4\pi\sqrt{mn}}{c}\Big),$$

we have

$$\begin{aligned}\sum_{f\in H_k}\omega_f\lambda_f(l^2)L(\mathrm{sym}^2 f,1/2)\\ &= 2\sum_{n>0}\frac{V(n)}{\sqrt{n}}\sum_{f\in H_k}\omega_f\lambda_f(l^2)\lambda_f(n^2)\\ &= 2\sum_{n>0}\frac{V(n)}{\sqrt{n}}\left(\delta_{l^2,n^2}+2\pi i^{-k}\sum_{c>0}\frac{S(l^2,n^2;c)}{c}J_{k-1}\left(\frac{4\pi ln}{c}\right)\right)\\ &=: D+2\pi i^{-k}ND,\qquad \text{say.}\end{aligned}$$

3.1. Main terms. The main term here comes from the diagonal term D:

$$D=2\frac{V(l)}{\sqrt{l}}=2\frac{1}{\sqrt{l}}\frac{1}{2\pi i}\int_{(1+\epsilon)}\frac{\gamma_\infty(\mathrm{sym}^2,1/2+u)}{\gamma_\infty(\mathrm{sym}^2,1/2)}\zeta(1+2u)\frac{G(u)}{l^u u}\,du.$$

Moving the line of integration to $\Re u=-1+\epsilon$, we obtain

$$\begin{aligned}D&=2\frac{1}{\sqrt{l}}\mathrm{Res}_{u=0}\left(\frac{\gamma_\infty(\mathrm{sym}^2,1/2+u)}{\gamma_\infty(\mathrm{sym}^2,1/2)}\zeta(1+2u)\frac{G(u)}{l^u u}\right)\\ &\quad+2\frac{1}{\sqrt{l}}\frac{1}{2\pi i}\int_{(-1+\epsilon)}\frac{\gamma_\infty(\mathrm{sym}^2,1/2+u)}{\gamma_\infty(\mathrm{sym}^2,1/2)}\zeta(1+2u)\frac{G(u)}{l^u u}\,du.\end{aligned}$$

The residue here is

$$\frac{2}{\sqrt{l}}\left(-\frac{\log l}{2}+\gamma+1/2\,\frac{\gamma_\infty'(\mathrm{sym}^2,1/2)}{\gamma_\infty(\mathrm{sym}^2,1/2)}\right),$$

by computing the following Laurent expansions at $u=0$:

$$\begin{aligned}\zeta(1+2u)&=\frac{1}{2u}+\gamma+\cdots,\\ \frac{G(u)}{l^u u}&=\frac{1}{u}-\log l+\cdots,\\ \frac{\gamma_\infty(\mathrm{sym}^2,1/2+u)}{\gamma_\infty(\mathrm{sym}^2,1/2)}&=1+\frac{\gamma_\infty'(\mathrm{sym}^2,1/2+u)}{\gamma_\infty(\mathrm{sym}^2,1/2)}u+\cdots.\end{aligned}$$

The last integral is $\ll k^{-1}l^{1/2}$, because by Stirling's estimates,

$$\Gamma(k/2+s)\ll k^{\Re s}\Gamma(k/2),$$

and the integral converges as the auxiliary function $G(u)$ decays much faster than the other terms as $|\Im u|\to\infty$.

3.2. Error terms. In this section we shall bound the non-diagonal term. Using the Mellin transform formula for the J-Bessel function

$$J_\nu=\frac{1}{4\pi i}\int_{(-1)}\frac{\Gamma\left(\frac{\nu+s}{2}\right)}{\Gamma\left(1+\frac{\nu-s}{2}\right)}\left(\frac{x}{2}\right)^{-s}ds$$

and interchanging the sum and integral, we obtain

$$ND = \frac{1}{2(2\pi i)^2} \int\limits_{(2)} \int\limits_{(-1)} \frac{\gamma_\infty(\mathrm{sym}^2, 1/2+u)}{\gamma_\infty(\mathrm{sym}^2, 1/2)} \zeta(1+2u) \frac{G(u)}{u} \frac{\Gamma\left(\frac{k-1+s}{2}\right)}{\Gamma\left(\frac{k+1-s}{2}\right)} \times \sum_n \sum_c \frac{S(n^2, l^2; c)}{n^{1/2+s+u} c^{1-s}} \frac{ds\, du}{(2\pi l)^s}.$$

Next we note that

$$\begin{aligned}\sum_{n\geq 1}\sum_{c\geq 1} \frac{S(n^2, l^2; c)}{n^{1/2+s+u} c^{1-s}} &= \sum_{c\geq 1} c^{s-1} \sum_n \frac{S(n^2, l^2; c)}{n^{1/2+s+u}} \\ &= \sum_{c\geq 1} c^{s-1} \sum_{\substack{1\leq b\leq c \\ b\,(\mathrm{mod}\, c)}} S(b^2, l^2; c) \sum_{n\equiv b\,(\mathrm{mod}\, c)} n^{-1/2-s-u} \\ &= \sum_{c\geq 1} c^{-u-3/2} \sum_{\substack{1\leq b\leq c \\ b\,(\mathrm{mod}\, c)}} S(b^2, l^2; c) \sum_{h\geq 0} (h + b/c)^{-1/2-s-u} \\ &= \sum_{c\geq 1} c^{-u-3/2} \sum_{\substack{1\leq b\leq c \\ b\,(\mathrm{mod}\, c)}} S(b^2, l^2; c) \zeta(1/2+s+u, b/c),\end{aligned}$$

where $\zeta(s,\alpha)$ for $\Re s > 1$ and $0 < \alpha \leq 1$ is the Hurwitz zeta function defined as

$$\zeta(s,\alpha) = \sum_{n\geq 0} (n+\alpha)^{-s}.$$

The Hurwitz zeta function $\zeta(s,\alpha)$ has only a simple pole at $s = 1$ with residue 1. If we move the line of integration to $\Re s = -5$, it passes a simple pole at $s = 1/2 - u$ with residue 1. Then we have

$$ND = ND_1 + ND_2,$$

where

$$ND_1 = \frac{1}{4\pi i} \int\limits_{(2)} \frac{\gamma_\infty(\mathrm{sym}^2, 1/2+u)}{\gamma_\infty(\mathrm{sym}^2, 1/2)} \zeta(1+2u) \frac{G(u)}{u} \frac{\Gamma\left(\frac{k-1/2-u}{2}\right)}{\Gamma\left(\frac{k+1/2+u}{2}\right)} \times \sum_{c\geq 1} c^{-u-3/2} \sum_{\substack{1\leq b\leq c \\ b\,(\mathrm{mod}\, c)}} S(b^2, l^2; c) \frac{du}{(2\pi l)^{1/2-u}},$$

$$ND_2 = \frac{1}{2(2\pi i)^2} \int\limits_{(2)} \int\limits_{(-5)} \frac{\gamma_\infty(\mathrm{sym}^2, 1/2+u)}{\gamma_\infty(\mathrm{sym}^2, 1/2)} \zeta(1+2u) \frac{G(u)}{u} \frac{\Gamma\left(\frac{k-1+s}{2}\right)}{\Gamma\left(\frac{k+1-s}{2}\right)} \times \sum_{c\geq 1} c^{-u-3/2} \sum_{\substack{1\leq b\leq c \\ b\,(\mathrm{mod}\, c)}} S(b^2, l^2; c) \zeta(1/2+s+u, b/c) \frac{ds\, du}{(2\pi l)^s}.$$

3.2.1. *Treatment of ND_1.* From Lemma 2.5, we know that

$$\sum_{c\ge 1} c^{-u-3/2} \sum_{\substack{1\le b\le c\\ b\,(\mathrm{mod}\,c)}} S(b^2,l^2;c)$$

is absolutely convergent. Using Stirling's estimates, we obtain

$$ND_1 \ll_\epsilon k^{-1/2+\epsilon} l^{1/2+\epsilon}.$$

3.2.2. *Treatment of ND_2.* Letting $w = 1/2 - s - u$ and using the functional equation of the Hurwitz zeta function as follows:

$$\begin{aligned}\zeta(s+u+1/2, b/c) &= \zeta(1-w, b/c)\\ &= \frac{\Gamma(w)}{(2\pi)^w}(e(-w/4)F(b/c,w) + e(w/4)F(-b/c,w)),\end{aligned}$$

where

$$F(\alpha,s) = \sum_{n\ge 1} e(\alpha n) n^{-s},$$

we arrive at

$$\begin{aligned}ND_2 = \frac{1}{2(2\pi i)^2}\int_{(2)}\int_{(7/2)} &\frac{\gamma_\infty(\mathrm{sym}^2, 1/2+u)}{\gamma_\infty(\mathrm{sym}^2,1/2)}\zeta(1+2u)\frac{G(u)}{u}\,\frac{\Gamma\big(\frac{k-1/2-u-w}{2}\big)}{\Gamma\big(\frac{k+1/2+u+w}{2}\big)}\\ &\times \sum_{c\ge 1} c^{-u-3/2}\sum_{\substack{1\le b\le c\\ b\,(\mathrm{mod}\,c)}} S(b^2,l^2;c)\frac{\Gamma(w)}{(2\pi)^w}\big(e(-w/4)F(b/c,w)\\ &+ e(w/4)F(-b/c,w)\big)\frac{dw\,du}{(2\pi l)^{1/2-w-u}}\\ &=: ND_2^- + ND_2^+, \qquad \text{say.}\end{aligned}$$

It suffices to bound ND_2^+ since the treatment of ND_2^- is similar. With the help of Lemma 2.3, we have

$$\begin{aligned}ND_2^+ = \frac{1}{2(2\pi i)^2}\int_{(2)} &\frac{\gamma_\infty(\mathrm{sym}^2,1/2+u)}{\gamma_\infty(\mathrm{sym}^2,1/2)}\zeta(1+2u)\frac{G(u)}{u}\\ &\times \sum_{c\ge 1} c^{-u-3/2}\sum_{\substack{1\le b\le c\\ b\,(\mathrm{mod}\,c)}} S(b^2,l^2;c)\sum_{n\ge 1} e\Big(-\frac{b}{c}n\Big)2^{1/2+u}\\ &\times \int_0^\infty J_{k-1}(x)\exp(inx/2l)x^{-1/2-u}\,dx\,\frac{du}{(2\pi l)^{1/2-u}}.\end{aligned}$$

To bound ND_2^+, we split it into two parts, $ND_2^+ = ND_2^{+,1} + ND_2^{+,2}$, according to $n \le 4l$ and $n > 4l$. It follows that, by Lemma 2.4(a) (taking $A_1 = 2$) and Lemma 2.6,

$$\begin{aligned} ND_2^{+,2} &\ll \int\limits_{(2)} \left| \frac{\gamma_\infty(\mathrm{sym}^2, 1/2+u)}{\gamma_\infty(\mathrm{sym}^2,1/2)} \zeta(1+2u) \frac{G(u)}{u} \right| \\ &\quad \times \sum_{n \geq 4l+1} \exp\left(\frac{\pi}{2}|\Im u|\right) \left(\frac{n}{2l}\right)^{\Re u - k + 1/2} dx \, \frac{|2^{1/2+u}| \, du}{|(2\pi l)^{1/2-u}|} \\ &\quad \times \sum_{c \geq 1} \left| c^{-u-3/2} \sum_{b \,(\mathrm{mod}\, c)} S(b^2, l^2; c) e\left(\frac{b}{c} n\right) \right| \\ &\ll_\epsilon k^{1/2+\epsilon} l^\epsilon (4l)^{-k+2+\epsilon} (2l)^{k-1-\epsilon} \\ &\ll_{A,\epsilon} k^{-B} \quad \text{for any large } B. \end{aligned}$$

The bound here is negligible since $l \ll k^A$.

Remark 3.1. This is the only place where we use the condition $l \ll k^A$.

We now turn to $ND_2^{+,1}$. By Lemma 2.4(b) (taking $A_2 = 2$), Lemma 2.6 and Stirling's estimates, we infer that

$$\begin{aligned} ND_2^{+,1} &\ll \int\limits_{(2)} \left| \frac{\gamma_\infty(\mathrm{sym}^2, 1/2+u)}{\gamma_\infty(\mathrm{sym}^2,1/2)} \zeta(1+2u) \frac{G(u)}{u} \right| \\ &\quad \times \sum_{n \leq 4l} (|\Im u| + 1) k^{-\Re u - 1/2 + \epsilon} \, dx \, \frac{|2^{1/2+u}| \, du}{|(2\pi l)^{1/2-u}|} \\ &\quad \times \sum_{c \geq 1} \left| c^{-u-3/2} \sum_{\substack{1 \leq b \leq c \\ b \,(\mathrm{mod}\, c)}} S(b^2, l^2; c) e\left(\frac{b}{c} n\right) \right| \\ &\ll_\epsilon k^{-1/2+\epsilon} l^{1+\epsilon}. \end{aligned}$$

By collecting all this information, the following asymptotic formula is established: for any $A \geq 1$ and $l \ll k^A$, we have

$$\begin{aligned} &\sum_{f \in H_k} \omega_f \lambda_f(l^2) L(\mathrm{sym}^2 f, 1/2) \\ &\qquad = \frac{2}{\sqrt{l}} \left(-\frac{\log l}{2} + \gamma + \frac{1}{2} \frac{\gamma_\infty'(\mathrm{sym}^2, 1/2)}{\gamma_\infty(\mathrm{sym}^2, 1/2)} \right) + O_\epsilon(k^{-1/2+\epsilon} l^{1+\epsilon}). \end{aligned}$$

Remark 3.2. By combining the above results, the following bound is proved:

$$2\pi i^{-k} \sum_n \frac{V(n)}{\sqrt{n}} \sum_{c>0} \frac{S(l^2, n^2; c)}{c} J_{k-1}\left(\frac{4\pi l n}{c}\right) \ll k^{-1/2+\epsilon} l^{1+\epsilon}.$$

Following the same argument, one gets

$$2\pi i^{-k} \sum_n \frac{V(n)}{\sqrt{n}} \sum_{c>0} \frac{S(l, n^2; c)}{c} J_{k-1}\left(\frac{4\pi \sqrt{l}\, n}{c}\right) \ll k^{-1/2+\epsilon} l^{1/2+\epsilon}.$$

4. Proof of Corollary 1.8. We begin with

$$\begin{aligned}\sum_{f\in H_k} L(\mathrm{sym}^2 f,1/2) &= \sum_{f\in H_k} \omega_f\omega_f^{-1}L(\mathrm{sym}^2 f,1/2)\\ &= \frac{k-1}{2\pi^2}\sum_{f\in H_k}\omega_f L(\mathrm{sym}^2 f,1/2)L(\mathrm{sym}^2 f,1)\\ &= \frac{k-1}{12\zeta(2)}\sum_{f\in H_k}\omega_f L(\mathrm{sym}^2 f,1/2)\Big(\zeta(2)\sum_{l\le x}\frac{\lambda_f(l^2)}{l}\\ &\quad+\zeta(2)\sum_{x<l\le y}\frac{\lambda_f(l^2)}{l}+O_\epsilon\big(k^\epsilon y^\epsilon\big(y^{-2/7}+k^{3/4}y^{-1/2}\big)\big)\Big)\\ &=: M_1+M_2+M_3,\qquad \text{say;}\end{aligned}$$

by Lemma 2.1. M_1 contributes the main terms, the other two constitute the error terms.

4.1. Main terms. By using the key Lemma 1.1, we have

$$\begin{aligned}M_1 &= \frac{k-1}{12\zeta(2)}\sum_{f\in H_k}\omega_f L(\mathrm{sym}^2 f,1/2)\times\zeta(2)\sum_{l\le x}\frac{\lambda_f(l^2)}{l}\\ &= \frac{k-1}{12}\sum_{l\le x}\frac{1}{l}\sum_{f\in H_k}\omega_f\lambda_f(l^2)L(\mathrm{sym}^2 f,1/2)\\ &= \frac{k-1}{12}\sum_{l\le x}\frac{1}{l}\bigg(\frac{2}{\sqrt{l}}\Big(-\frac{\log l}{2}+\gamma+1/2\,\frac{\gamma_\infty'(\mathrm{sym}^2,1/2)}{\gamma_\infty(\mathrm{sym}^2,1/2)}\Big)\\ &\qquad\qquad\qquad\qquad+O_\epsilon(k^{-1/2+\epsilon}l^{3/2+\epsilon})\bigg)\\ &= \frac{k-1}{12}\sum_{l}\frac{2}{l^{3/2}}\Big(-\frac{\log l}{2}+\gamma+\frac{1}{2}\frac{\gamma_\infty'(\mathrm{sym}^2,1/2)}{\gamma_\infty(\mathrm{sym}^2,1/2)}\Big)\\ &\quad-\frac{k-1}{12}\sum_{l>x}\frac{2}{l^{3/2}}\Big(-\frac{\log l}{2}+\gamma+\frac{1}{2}\frac{\gamma_\infty'(\mathrm{sym}^2,1/2)}{\gamma_\infty(\mathrm{sym}^2,1/2)}\Big)+O_\epsilon(k^{1/2+\epsilon}x^{3/2+\epsilon})\\ &= \frac{k-1}{12}\Big(\zeta'\Big(\frac{3}{2}\Big)+2\gamma\zeta\Big(\frac{3}{2}\Big)+\frac{\gamma_\infty'(\mathrm{sym}^2,1/2)}{\gamma_\infty(\mathrm{sym}^2,1/2)}\zeta'\Big(\frac{3}{2}\Big)\Big)\\ &\quad+O_\epsilon(k^{1+\epsilon}x^{-1/2+\epsilon}+k^{1/2+\epsilon}x^{3/2+\epsilon}).\end{aligned}$$

Choosing $x=k^{1/4}$, we get

$$M_1=\frac{k-1}{12}\Big(\zeta'\Big(\frac{3}{2}\Big)+2\gamma\zeta\Big(\frac{3}{2}\Big)+\frac{\gamma_\infty'(\mathrm{sym}^2,1/2)}{\gamma_\infty(\mathrm{sym}^2,1/2)}\zeta'\Big(\frac{3}{2}\Big)\Big)+O_\epsilon(k^{7/8+\epsilon}).$$

4.2. Error terms. Before estimating the error terms, we take y to be k^{100}. Then we bound M_3 first:

$$\begin{aligned} M_3 &= \frac{k-1}{12\zeta(2)} \sum_{f\in H_k} \omega_f L(\operatorname{sym}^2 f, 1/2) \times O_\epsilon(k^{-200/7+\epsilon}) \\ &= O_\epsilon\Big(\sum_{f\in H_k} \omega_f L(\operatorname{sym}^2 f, 1/2) \times k^{-193/7+\epsilon}\Big) = O_\epsilon(k^{-193/7+\epsilon}), \end{aligned}$$

where we have used the assumption that $L(\operatorname{sym}^2 f, 1/2) \geq 0$ and the key Lemma 1.1 (with $l = 1$).

To bound M_2, we apply Hölder's inequality, the condition $L(\operatorname{sym}^2 f, 1/2) \geq 0$ and Lemma 2.2, to obtain

$$\begin{aligned} M_2 &= \frac{k-1}{12\zeta(2)} \sum_{f\in H_k} \omega_f L(\operatorname{sym}^2 f, 1/2) \times \zeta(2) \sum_{x<l\leq y} \frac{\lambda_f(l^2)}{l} \\ &\ll k\Big(\sum_{f\in H_k} \omega_f (k^{1/4}, k^{100})^{40}\Big)^{1/40} \Big(\sum_{f\in H_k} (\omega_f L(\operatorname{sym}^2 f, 1/2))^{40/39}\Big)^{39/40} \\ &\ll_\epsilon k^{1+\epsilon}\Big(\sum_{f\in H_k} \omega_f L(\operatorname{sym}^2 f, 1/2)(\omega_f L(\operatorname{sym}^2 f, 1/2))^{1/39}\Big)^{39/40}. \end{aligned}$$

Applying the trivial bound of $\omega_f \ll k^{\epsilon-1}$ and the convexity estimate of $L(\operatorname{sym}^2 f, 1/2) \ll k^{3/4+\epsilon}$, we have

$$M_2 \ll_\epsilon k^{1+\epsilon}\Big(\sum_{f\in H_k} \omega_f L(\operatorname{sym}^2 f, 1/2)(k^{-1/4+\epsilon})^{1/39}\Big)^{39/40} \ll_\epsilon k^{159/160+\epsilon}.$$

The last estimate is again by Theorem 1.1 (with $l = 1$). Putting everything together completes the proof.

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