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Small prime solutions to linear equations in three variables

by

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1. Introduction. In this paper, we investigate linear equations of the form

$$a_1p_1 + a_2p_2 + a_3p_3 = b \tag{1.1}$$

where a_1, a_2, a_3, b are fixed integer constants and p_1, p_2, p_3 are variables which are prime numbers. When $a_1 = a_2 = a_3 = 1$, this is the weak Goldbach problem, which has been shown to be solvable for all sufficiently large odd integers b by Vinogradov [13] in 1937, and Helfgott [6] has recently proved that this holds for all odd integers $b > 5$. Vinogradov's argument works for the general case (1.1) where a_1, a_2, a_3 are nonzero integers such that $(a_1, a_2, a_3) = 1$ and b satisfies

$$b \equiv a_1 + a_2 + a_3 \pmod 2 \quad \text{and} \quad (b, a_i, a_j) = 1 \quad \text{for } 1 \le i < j \le 3. \tag{1.2}$$

In connection with his work on a Diophantine approximation problem, Baker [1] considered a special case of (1.1) and obtained an explicit bound for small prime solutions. This then leads to the interesting question on the size of small prime solutions of (1.1) in terms of the coefficients a_1, a_2, a_3 and b. An account on the development in this area can be found in [11].

In this paper, our main result is

Theorem 1. *Suppose a_1, a_2, a_3 are nonzero integers with $(a_1, a_2, a_3) = 1$ and conditions (1.2) hold. Let $A = \max_{1\le j\le 3} |a_j|$.*

(i) *If not all of a_1, a_2, a_3 have the same sign, then (1.1) has a prime solution satisfying*

$$\max_{1\le j\le 3} |a_j|p_j \ll |b| + A^{25}.$$

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(ii) *If* $a_1, a_2, a_3 > 0$, *then* (1.1) *is solvable whenever*

$$b \gg A^{25}.$$

Liu and Tsang [11] proved Theorem 1 with the exponent 25 replaced by some unspecified constant n_0. Choi [2] gave the first explicit numerical value $n_0 = 4190$. This was reduced to $n_0 = 45$ in [12] and then to $n_0 = 38$ in [9]. In [10], by assuming an extra condition $(a_i, a_j) = (a_i, b) = 1$ for $1 \le i < j \le 3$, Liu and Tsang obtained the much better bound where A^{n_0} is replaced by $A|a_1a_2a_3|^{5/2}\log^{26} A$. The exponent 5/2 was further reduced to 20/9 by Choi and Kumchev [3].

The proof of Theorem 1 is based on the circle method and we use the key ideas of [11] and [12]. The explicit value of n_0 depends, among other things, on estimations related to the zero density of Dirichlet's L-functions.

The improved value 25 of the exponent is a result of two new ingredients. The first concerns estimation of the generating function $S_j(x)$ (defined in (3.4)) for x in the major arc via Lemma 3.1. In order that the contribution of the error term in (3.11) to the main term is negligible, we have to impose a lower bound on N (where N has the same size as $|b|$) in terms of T and Q, which in turn gives a relation between N and A. In this paper, we find that those zeros of Dirichlet's L-function with large imaginary parts can be handled separately with a negligible contribution. As a result, a smaller N suffices for the estimation of the error term in (3.11).

Our second improvement is in the estimation of the important sum $Z(q; \chi_1, \chi_2, \chi_3)$ (defined in (4.1)) for $q \le Q$. In Section 6, we give a detailed study of the contribution arising from that sum. Our estimation reveals that the contribution is admissible unless the modulus q has a special kind of factorization depending on the coefficients a_1, a_2, a_3 and b. In particular, we only need zero density estimates for $q \le Q^{2/3+\epsilon}$ in all cases (as compared with $q \le Q$ in [12]). For some special cases, we can further restrict the range to $q \le Q^{1/3+\epsilon}$. Therefore, a smaller N is sufficient to establish the corresponding zero density estimates in Lemma 6.1.

2. Preliminaries. One of the major components in the proof is to estimate the zero density of Dirichlet's L-function. In this section, we first cite a few related results. As usual, $\chi \pmod q$ refers to a Dirichlet character modulo q, and χ_0 is the principal character, whose modulus should be clear from the context. Let $L(s, \chi)$ be Dirichlet's L-function of χ and let

$$\prod(s) = \prod_{q \le Q} \prod_{\chi \,(\mathrm{mod}\, q)}^{*} L(s, \chi) \tag{2.1}$$

where Q will be taken to be sufficiently large and the product with an asterisk runs over all primitive characters modulo q.

LEMMA 2.1 ([12, Proposition 2.3]). *For fixed $C>0$, there exists $K(C)>0$ depending only on C such that whenever $Q \geq K(C)$, the function $\prod(s)$ defined by* (2.1) *has at most one zero in the region*

$$\sigma \geq 1 - \frac{0.364}{\log Q}, \quad |t| \leq C,$$

where $s = \sigma + it$. If such an exceptional zero $\widetilde{\beta}$ exists, then it is real and simple, corresponding to a nonprincipal, real, primitive character $\widetilde{\chi}$ (mod $\widetilde{r}$) *where $\widetilde{r} \leq Q$.*

The possible exceptional zero is called the *Siegel zero.*

LEMMA 2.2 ([9, Lemma 3]). *If the exceptional zero $\widetilde{\beta}$ in Lemma* 2.1 *exists, then for any constant c with $0 < c < 1$ and any small $\epsilon > 0$ there exists $K(c,\epsilon)>0$ depending only on c and ϵ such that any zero $\rho=\beta+i\gamma\neq\widetilde{\beta}$ corresponding to χ* (mod q) *of the function $\prod(s)$ defined by* (2.1) *satisfies*

$$\beta \leq 1 - \min\left\{\frac{c}{6}, \frac{(1-c)(8/9-\epsilon)}{\log([\widetilde{r},q]|\gamma|)} \log\left(\frac{(1-c)(8/9-\epsilon)}{(1-\widetilde{\beta})\log([\widetilde{r},q]|\gamma|)}\right)\right\} \tag{2.2}$$

if $[\widetilde{r},q]|\gamma| > K(c,\epsilon)$.

LEMMA 2.3 ([7, (1.1)] and [8, Theorem 1]). *Define*

$$N(\alpha,x,y) = \sum_{q\leq x} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \sum_{\substack{|\gamma|\leq y \\ \beta\geq\alpha}} 1$$

where the innermost sum is over all nontrivial zeros $\rho = \beta + i\gamma$ of $L(s,\chi)$ satisfying $|\gamma| \leq y$ and $\beta \geq \alpha$. Then for any $x \geq 1$ and $y \geq 2$, we have

$$N(\alpha,x,y) \ll_\epsilon (x^2y)^{\frac{12}{5}(1-\alpha)+\epsilon} \quad \text{for } 1/2 \leq \alpha \leq 4/5, \tag{2.3}$$

$$N(\alpha,x,y) \ll_\epsilon (x^2y)^{(2+\epsilon)(1-\alpha)} \quad \text{for } 4/5 \leq \alpha \leq 1. \tag{2.4}$$

3. Circle method. Our proof is based on the circle method as in [12]. In the following, we suppose C is a sufficiently large fixed constant, while ϵ_1 and $\epsilon_2 := \epsilon_2(\epsilon_1)$ are fixed small positive constants. The symbol ϵ always refers to a sufficiently small positive number, which can vary from place to place. The variable p, with or without subscripts, always denotes a prime number.

Let N be a sufficiently large positive number depending on $C, \epsilon_1, \epsilon_2$, and let

$$Q = (NA^{-1})^{\frac{1}{8-100\epsilon_1}} \tag{3.1}$$

where $A := \max\{|a_1|, |a_2|, |a_3|\}$. We assume further that

$$Q^{1/3-\epsilon_1} \gg A. \tag{3.2}$$

From (3.1) and (3.2), this is the same as

$$N \gg A^{25-\frac{228\epsilon_1}{1-3\epsilon_1}}. \tag{3.3}$$

Define $N_j = N/|a_j|$, $N_j' = N/(4|a_j|)$ and $\mathcal{L} = \log N$. As usual, $\Lambda(n)$ is the von Mangoldt function. For any real x, $e(x) := \exp(2\pi i x)$. For $j = 1, 2, 3$ define

$$S_j(x) = \sum_{N_j' < n \le N_j} \Lambda(n) e(a_j n x), \tag{3.4}$$

and set

$$G(n, \chi) = \sum_{l=1}^{q} \chi(l) e(nl/q). \tag{3.5}$$

Our proof starts by investigating the sum

$$\sum_{\substack{N_j' < n_j \le N_j \\ a_1 n_1 + a_2 n_2 + a_3 n_3 = b}} \Lambda(n_1)\Lambda(n_2)\Lambda(n_3). \tag{3.6}$$

It can be shown that the contribution from powers of primes higher than 1 is negligible. Hence, if (3.6) is positive, we get a prime solution where each $a_j p_j$ has the size of N. In view of (3.3), it follows that there is a prime solution whenever $N \gg A^{25}$. By taking N to have the same size as $|b|$, as in [11], we establish Theorem 1.

From [12], the sum (3.6) can be written as

$$\Big(\int_{\mathcal{M}} + \int_{\overline{\mathcal{M}}}\Big) e(-bx) S_1(x) S_2(x) S_3(x)\, dx =: I_1 + I_2 \tag{3.7}$$

where I_1 is the contribution from the major arc $\mathcal{M}$, which is the disjoint union of the intervals $[(h-\tau)/q, (h+\tau)/q]$ with $1 \le h \le q \le Q$, $(h, q) = 1$ and

$$\tau := Q^{1+\epsilon_1} N^{-1}, \tag{3.8}$$

and I_2 is the contribution from the minor arc $\overline{\mathcal{M}}$, the complement of $\mathcal{M}$ in $[\tau, 1+\tau]$.

The integral I_2 over the minor arc can be estimated in exactly the same way as in [12], which gives

$$I_2 \ll N^2 Q^{-1/2} |a_1 a_2 a_3|^{-1/2} \mathcal{L}^5. \tag{3.9}$$

This estimation is sufficiently good in view of the assumption (3.2).

Our main task is to estimate I_1 over the major arc. Each $x \in \mathcal{M}$ can be written uniquely as $x = h/q + \eta$ where $|\eta| \le \tau/q$. For any complex number

s and any real number η and $j = 1, 2, 3$, define

$$I_j(s,\eta) = \int\limits_{N_j'}^{N_j} x^{s-1} e(a_j x \eta)\, dx. \tag{3.10}$$

As in [11] and [12], we have the following

LEMMA 3.1. *Under the definitions* (3.4), (3.5) *and* (3.10), *we have*

$$\begin{aligned} S_j\left(\frac{h}{q}+\eta\right) = \frac{1}{\varphi(q)}\Big(&G(a_jh,\chi_0)I_j(1,\eta) - \widetilde{\delta}_q G(a_jh,\widetilde{\chi}\chi_0)I_j(\widetilde{\beta},\eta) \\ &- \sum_{\chi_j \ (\mathrm{mod}\, q)} \sideset{}{'}\sum_{|\gamma|\le T} G(a_jh,\overline{\chi}_j)I_j(\rho,\eta) \\ &- \sum_{\chi_j \ (\mathrm{mod}\, q)} G(a_jh,\overline{\chi}_j) \times O((1+N|\eta|)N_jT^{-1}\mathcal{L}^2)\Big) \end{aligned} \tag{3.11}$$

where

$$T := Q^{4/3+3\epsilon_1} \tag{3.12}$$

and $\widetilde{\delta}_q$ *is* 1 *if* $\widetilde{r}$ *divides* q, *and is* 0 *otherwise. In the inner sum of the third term over the nontrivial zeros* $\rho = \beta + i\gamma$, *we add a prime to indicate that the possible Siegel zero* $\widetilde{\beta}$ *is excluded, since it is isolated as the second term for separate treatment.*

Later when we apply zero density estimates, we have to impose a lower bound on N in terms of Q and T. This is one of the main constraints that determine the lower bound for the exponent 25 in Theorem 1 using this method. One will see that it is better to have T being a smaller power of Q. At the same time, T cannot be too small as the error term in (3.11) should be admissible when we apply Lemma 3.1. In [12], T is taken to be Q^3 as the authors simply use $|G(a_jh,\overline{\chi}_j)| \le q^{1/2}$ to bound the Gauss sum. Indeed, by using exactly the same argument, the best that one can do is to take T to be $Q^{2.5+\epsilon}$, which is one of the amendments made in [9] to achieve a better result.

In this paper, we use the more detailed expression given in (3.11) to estimate the errors. It turns out that we can take $T = Q^{4/3+3\epsilon_1}$.

LEMMA 3.2 ([12, Lemma 4.3] up to a change of variable). *Under the definition* (3.10), *for any* $\rho = \beta + i\gamma$ *with* $1/2 \le \beta \le 1$ *and any real* η, *we have*

$$I_j(\rho,\eta) \ll \begin{cases} N_j^{\beta}|\gamma|^{-1} & \text{if } |\eta| \le |\gamma|/(4\pi N),\ \gamma \ne 0, \\ N_j^{\beta}|\gamma|^{-1/2} & \text{if } |\gamma|/(4\pi N) < |\eta| \le 4|\gamma|/(\pi N),\ \gamma \ne 0, \\ N_j^{\beta}N^{-1}|\eta|^{-1} & \text{if } |\eta| > 4|\gamma|/(\pi N), \end{cases} \tag{3.13}$$

and

$$I_j(\rho,\eta) \ll N_j^{\beta}. \tag{3.14}$$

LEMMA 3.3. *Let*

$$T' = 4\pi Q^{1+\epsilon_1}. \tag{3.15}$$

Then

$$\Sigma_1 := \sum_{r\le Q} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, r)} \sideset{}{'}\sum_{T'<|\gamma|\le T} |\gamma|^{-1}\left(\frac{N}{A}\right)^{\beta-1} \ll Q^{-1},$$

$$\Sigma_2 := \sum_{r\le Q} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, r)} \sideset{}{'}\sum_{|\gamma|\le T'} \left(\frac{N}{A}\right)^{\beta-1} \ll 1.$$

Proof. The proof is similar to the corresponding one in [12]. Firstly,

$$\Sigma_1 \ll \mathcal{L} \max_{T'<y\le T} y^{-1} \sum_{r\le Q} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, r)} \sideset{}{'}\sum_{y<|\gamma|\le 2y} \left(\frac{N}{A}\right)^{\beta-1}$$

where by Lemma 2.3,

$$\begin{aligned}\sum_{r\le Q} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, r)} \sideset{}{'}\sum_{y<|\gamma|\le 2y} \left(\frac{N}{A}\right)^{\beta-1} &\ll -\int_{1/2}^{1} \left(\frac{N}{A}\right)^{\alpha-1} dN(\alpha,Q,y)\\ &\ll (Q^2y)^{6/5+\epsilon}N^{-1/2}A^{1/2} + N^{\epsilon}\int_{1/2}^{4/5} ((Q^2y)^{12/5}N^{-1}A)^{1-\alpha}\,d\alpha\\ &\quad + \mathcal{L}\int_{4/5}^{1} ((Q^2y)^{2+\epsilon}N^{-1}A)^{1-\alpha}\,d\alpha\\ &\ll N^{\epsilon}(Q^2y)^{6/5}N^{-1/2}A^{1/2} + N^{\epsilon}(Q^2y)^{12/25}N^{-1/5}A^{1/5} + 1.\end{aligned}$$

Hence,

$$\begin{aligned}\Sigma_1 &\ll N^{\epsilon}Q^{12/5}T^{1/5}N^{-1/2}A^{1/2}\\ &\quad + N^{\epsilon}Q^{24/25}Q^{-13/25-13\epsilon_1/25}N^{-1/5}A^{1/5} + \mathcal{L}Q^{-1-\epsilon_1} \ll Q^{-1}.\end{aligned}$$

This gives the first assertion. Similarly,

$$\begin{aligned}\Sigma_2 &\ll (Q^2Q^{1+\epsilon_1})^{6/5+\epsilon}N^{-1/2}A^{1/2} + N^{\epsilon}\int_{1/2}^{4/5} ((Q^2Q^{1+\epsilon_1})^{12/5}N^{-1}A)^{1-\alpha}\,d\alpha\\ &\quad + \mathcal{L}\int_{4/5}^{1} ((Q^2Q^{1+\epsilon_1})^{2+\epsilon}N^{-1}A)^{1-\alpha}\,d\alpha \ll 1. \ \blacksquare\end{aligned}$$

Now, we use (3.11) to replace $S_1(x)S_2(x)S_3(x)$ in I_1 defined by (3.7) and expand the three brackets. In the remainder of this section, we shall deal with the error terms, which consist of the following ten types of expressions:

$$I_i(1,\eta)I_j(1,\eta)O_k, \quad I_i(1,\eta)I_j(\widetilde{\beta},\eta)O_k, \quad I_i(\widetilde{\beta},\eta)I_j(\widetilde{\beta},\eta)O_k,$$
$$I_i(1,\eta)I_j(\rho_j,\eta)O_k, \quad I_i(\widetilde{\beta},\eta)I_j(\rho_j,\eta)O_k, \quad I_i(\rho_i,\eta)I_j(\rho_j,\eta)O_k,$$
$$I_i(1,\eta)O_jO_k, \quad I_i(\widetilde{\beta},\eta)O_jO_k, \quad I_i(\rho_i,\eta)O_jO_k, \quad O_iO_jO_k.$$

We use $E_1,\ldots,E_{10}$ to denote a typical term of each of these types.

For the estimation of the Gauss sum, we use the well-known expression given in [5, p. 450]. It says that for any character χ (mod q) induced by the primitive character χ^* (mod q^*), if we let $q=q_1q_2$ be such that $(q_2,q^*)=1$ and any prime factor of q_1 divides q^*, then

$$\begin{aligned}(3.16)\qquad & G(n,\chi)\\ &= \overline{\chi^*}\Big(\frac{n}{(n,q)}\Big)\chi^*\Big(\frac{q}{q^*(n,q)}\Big)\mu\Big(\frac{q}{q^*(n,q)}\Big)\varphi(q)\varphi^{-1}\Big(\frac{q}{(n,q)}\Big)G(1,\chi^*)\end{aligned}$$

if $q^*=q_1/(n,q_1)$, and $G(n,\chi)=0$ otherwise. It is also well-known that $|G(1,\chi^*)|=(q^*)^{1/2}$. For our purpose, this gives

$$|G(n,\chi)|\le (n,q)(q^*)^{1/2}. \tag{3.17}$$

For $G(n,\chi)$ to be nonzero, we must have

$$(n,q)q^*\mid q. \tag{3.18}$$

The method of estimating each error term E_j is similar; we shall demonstrate the estimation for the typical term E_6.

Firstly, we take absolute values and sum over the primitive characters which induce the characters. This gives

$$\begin{aligned}E_6 \ll N_3T^{-1}\mathcal{L}^2 \sum_{r_1,r_2,r_3\le Q}\ \sideset{}{^*}\sum_{\chi_j\,(\mathrm{mod}\,r_j)}\ \sideset{}{'}\sum_{|\gamma_1|,|\gamma_2|\le T}\ \sum_{\substack{q\le Q\\ r_j\mid q}}\ \sideset{}{'}\sum_{h=1}^{q}\frac{1}{\varphi(q)^3}\Big|\prod_{j=1}^{3}G(a_jh,\overline{\chi_j\chi_0})\Big|\\ \times\int_{-\tau/q}^{\tau/q}|I_1(\rho_1,\eta)I_2(\rho_2,\eta)|(1+N|\eta|)\,d\eta.\end{aligned}$$

The sum over h is restricted to those relatively prime to q. Each sum over $|\gamma_j|\le T$ can be divided into the two ranges $T'<|\gamma_j|\le T$ and $|\gamma_j|\le T'$ where T' is defined in (3.15). For the first range, from (3.8), we have $|\gamma_j|>4\pi Q^{1+\epsilon_1}\ge 4\pi N\tau/q\ge 4\pi N|\eta|$, so Lemma 3.2 provides a better bound to estimate the integrand than the general case, as we have the additional factor $|\gamma_j|^{-1}$. We shall study each case in detail.

When $T' < |\gamma_1|, |\gamma_2| \le T$, we use the first range of (3.13), which gives

$$\int_{-\tau/q}^{\tau/q} |I_1(\rho_1,\eta)I_2(\rho_2,\eta)|(1+N|\eta|)\,d\eta \ll N_1^{\beta_1}N_2^{\beta_2}|\gamma_1|^{-1}|\gamma_2|^{-1}\int_{-\tau/q}^{\tau/q}(1+N|\eta|)\,d\eta \ll N^{-1}Q^{2+2\epsilon_1}q^{-2}N_1^{\beta_1}N_2^{\beta_2}|\gamma_1|^{-1}|\gamma_2|^{-1}.$$

When $|\gamma_1| \le T' < |\gamma_2| \le T$, we divide the integral into four ranges

$$\int_{-\tau/q}^{\tau/q} = \int_{|\eta|\le 1/N} + \int_{1/N<|\eta|\le|\gamma_1|/(4\pi N)} + \int_{|\gamma_1|/(4\pi N)<|\eta|\le 4|\gamma_1|/(\pi N)} + \int_{4|\gamma_1|/(\pi N)<|\eta|\le\tau/q}.$$

For $|\eta| \le 1/N$, we apply the first range of (3.13) for $I_2(\rho_2,\eta)$ and the trivial bound (3.14) for $I_1(\rho_1,\eta)$, yielding $\int_{|\eta|\le 1/N} \ll N^{-1}N_1^{\beta_1}N_2^{\beta_2}|\gamma_2|^{-1}$. For the other ranges of η, we apply (3.13). In all cases, we find that the integral over η is $\ll N^{-1}Q^{3/2+3\epsilon_1/2}q^{-3/2}N_1^{\beta_1}N_2^{\beta_2}|\gamma_2|^{-1}$.

When $|\gamma_1|, |\gamma_2| \le T'$, we can similarly divide the integral into several ranges according to Lemma 3.2. The integral is $\ll N^{-1}Q^{1+\epsilon_1}q^{-1}N_1^{\beta_1}N_2^{\beta_2}$ for all ranges.

From these cases, we find that E_6 is

$$\ll N^{2+\epsilon}|a_1a_2a_3|^{-1}T^{-1}\sum_{r_1,r_2,r_3\le Q}\ \sideset{}{^*}\sum_{\substack{\chi_1 \,(\mathrm{mod}\, r_1)\\ \chi_2\,(\mathrm{mod}\, r_2)}}\ \max_{\chi_3\,(\mathrm{mod}\, r_3)}$$

$$\Bigg(Q^{2+2\epsilon_1}\sideset{}{'}\sum_{T'<|\gamma_1|,|\gamma_2|\le T}\ \sum_{\substack{q\le Q\\ r_j|q}}\ \max_{\substack{1\le h\le q\\ (h,q)=1}}\frac{r_3}{q^2\varphi(q)^2}\Big|\prod_{j=1}^{3}G(a_jh,\overline{\chi_j\chi_0})\Big|\prod_{j=1}^{2}|\gamma_j|^{-1}N_j^{\beta_j-1}$$

$$+\,Q^{3/2+3\epsilon_1/2}\sideset{}{'}\sum_{\substack{|\gamma_1|\le T'\\ T'<|\gamma_2|\le T}}\ \sum_{\substack{q\le Q\\ r_j|q}}\ \max_{\substack{1\le h\le q\\ (h,q)=1}}\frac{r_3}{q^{3/2}\varphi(q)^2}\Big|\prod_{j=1}^{3}G(a_jh,\overline{\chi_j\chi_0})\Big||\gamma_2|^{-1}\prod_{j=1}^{2}N_j^{\beta_j-1}$$

$$+\,Q^{1+\epsilon_1}\sideset{}{'}\sum_{|\gamma_1|,|\gamma_2|\le T'}\ \sum_{\substack{q\le Q\\ r_j|q}}\ \max_{\substack{1\le h\le q\\ (h,q)=1}}\frac{r_3}{q\varphi(q)^2}\Big|\prod_{j=1}^{3}G(a_jh,\overline{\chi_j\chi_0})\Big|\prod_{j=1}^{2}N_j^{\beta_j-1}\Bigg).$$

For each sum over r_3, we use (3.17) to estimate the inner sum over q, subject to the constraint (3.18). Firstly, let $d_j = (a_j, q)$. We have

$$\sum_{r_3\le Q}\sum_{\substack{q\le Q\\ r_j|q}}\max_{\chi_3\,(\mathrm{mod}\,r_3)}\max_{\substack{1\le h\le q\\ (h,q)=1}}\frac{r_3}{q^2\varphi(q)^2}\Big|\prod_{j=1}^{3}G(a_jh,\overline{\chi_j\chi_0})\Big| \tag{3.19}$$

$$\ll N^\epsilon\sum_{d_j|a_j}\sum_{r_3\le Q}\sum_{\substack{q\le Q\\ [d_1r_1,d_2r_2,d_3r_3]|q}}\frac{d_1d_2d_3r_1^{1/2}r_2^{1/2}r_3^{3/2}}{q^4}$$

$$\ll N^\epsilon\sum_{d_j|a_j}\sum_{r_3\le Q}\frac{d_1d_2d_3r_1^{1/2}r_2^{1/2}r_3^{3/2}}{[d_1r_1,d_2r_2,d_3r_3]^4}.$$

The fraction equals $d_1d_2d_3^{-3}r_1^{1/2}r_2^{1/2}r_3^{-5/2}[d_1r_1,d_2r_2]^{-4}([d_1r_1,d_2r_2],d_3r_3)^4$. We define $u_1=([d_1r_1,d_2r_2],d_3r_3)$ and $u_2=(u_1,d_3)$. The sum (3.19) is

$$\ll N^\epsilon\sum_{d_j|a_j}\sum_{\substack{u_2|d_3\\ u_2|u_1|[d_1r_1,d_2r_2]}}\sum_{\substack{r_3\le Q\\ \frac{u_1}{u_2}|r_3}}d_1d_2d_3^{-3}r_1^{1/2}r_2^{1/2}r_3^{-5/2}[d_1r_1,d_2r_2]^{-4}u_1^4$$

$$\ll N^\epsilon\sum_{d_j|a_j}d_1d_2r_1^{1/2}r_2^{1/2}[d_1r_1,d_2r_2]^{-5/2}\ll N^\epsilon r_1^{-1/2}r_2^{-1}.$$

We handle the other two sums over r_3 similarly. This gives

$$\begin{aligned}E_6 &\ll N^{2+\epsilon}|a_1a_2a_3|^{-1}T^{-1}\\ &\times\Bigg(Q^{2+2\epsilon_1}\sum_{r_1,r_2\le Q}\sideset{}{^*}\sum_{\substack{\chi_1\,(\mathrm{mod}\,r_1)\\ \chi_2\,(\mathrm{mod}\,r_2)}}\sideset{}{'}\sum_{T'<|\gamma_1|,|\gamma_2|\le T}r_1^{-1/2}r_2^{-1}\prod_{j=1}^{2}|\gamma_j|^{-1}N_j^{\beta_j-1}\\ &\quad+Q^{3/2+3\epsilon_1/2}\sum_{r_1,r_2\le Q}\sideset{}{^*}\sum_{\substack{\chi_1\,(\mathrm{mod}\,r_1)\\ \chi_2\,(\mathrm{mod}\,r_2)}}\sideset{}{'}\sum_{\substack{|\gamma_1|\le T'\\ T'<|\gamma_2|\le T}}r_1^{-1/2}r_2^{-1/2}|\gamma_2|^{-1}\prod_{j=1}^{2}N_j^{\beta_j-1}\\ &\quad+Q^{1+\epsilon_1}A^{1/2}\sum_{r_1,r_2\le Q}\sideset{}{^*}\sum_{\substack{\chi_1\,(\mathrm{mod}\,r_1)\\ \chi_2\,(\mathrm{mod}\,r_2)}}\sideset{}{'}\sum_{|\gamma_1|,|\gamma_2|\le T'}r_1^{-1/2}\prod_{j=1}^{2}N_j^{\beta_j-1}\Bigg)\\ &\ll N^2|a_1a_2a_3|^{-1}Q^{-1/6},\end{aligned}$$

by Lemma 3.3 and (3.12).

After analyzing all terms E_j, we find that the errors are all

$$\ll N^2|a_1a_2a_3|^{-1}Q^{-\epsilon_1}.$$

Finally, we conclude from Lemma 3.1 that

$$(3.20)\qquad I_1 = \sum_{q\le Q}\sideset{}{'}\sum_{h=1}^{q} \frac{1}{\varphi(q)^3} e\Big(-\frac{b}{q}h\Big) \int_{-\tau/q}^{\tau/q} e(-b\eta) \prod_{j=1}^{3} \Big(G(a_jh,\chi_0)I_j(1,\eta) - \widetilde{\delta}_q G(a_jh,\widetilde{\chi}\chi_0)I_j(\widetilde{\beta},\eta) - \sum_{\chi_j \,(\mathrm{mod}\, q)} \sideset{}{'}\sum_{|\gamma|\le T} G(a_jh,\overline{\chi}_j)I_j(\rho,\eta)\Big)\, d\eta + O(N^2|a_1a_2a_3|^{-1}Q^{-\epsilon_1}).$$

4. An expression for $Z(r;\chi_1,\chi_2,\chi_3)$. When we expand the product of the three terms in the integrand on the right hand side of (3.20), we need to handle the term

$$(4.1)\qquad Z(q;\chi_1,\chi_2,\chi_3) := \sideset{}{'}\sum_{h=1}^{q} e\Big(-\frac{b}{q}h\Big) G(a_1h,\chi_1\chi_0)G(a_2h,\chi_2\chi_0)G(a_3h,\chi_3\chi_0).$$

Here, if r_j is the modulus of the primitive character χ_j, then we always assume $r := [r_1,r_2,r_3]$ divides q. Also, χ_0 refers to the principal character modulo q. We will give the necessary estimations involving $Z(q;\chi_1,\chi_2,\chi_3)$ in this section. In particular, we need to estimate

$$(4.2)\qquad \sum_{\substack{q\le Q\\ r|q}} \frac{1}{\varphi(q)^3} Z(q;\chi_1,\chi_2,\chi_3).$$

Define

$$(4.3)\qquad A(q) = \frac{1}{\varphi(q)^3} Z(q;\chi_0,\chi_0,\chi_0),\qquad s(p) = 1 + A(p).$$

From [11], we have the following

Lemma 4.1. *Under the definitions* (4.3), *we have*

$$\sum_{\substack{q\le Q\\ r|q}} \frac{1}{\varphi(q)^3} Z(q;\chi_1,\chi_2,\chi_3) = \frac{1}{\varphi(r)^3} Z(r;\chi_1,\chi_2,\chi_3) \sum_{\substack{q\le Q/r\\ (q,r)=1}} A(q)$$

where

$$\sum_{\substack{q\le Q/r\\ (q,r)=1}} A(q) \ll \prod_{p\nmid r} s(p) \ll \prod_{p} s(p).$$

Liu and Wang [12] use this to derive a rather loose bound where (4.2) is bounded by $\prod_p s(p)$. But we can do a lot better in many cases by estimating the term $Z(r;\chi_1,\chi_2,\chi_3)$ in detail. Li [9] also obtains an improvement by using a better estimation of this expression. He finds that better bounds can be used when one or two of the characters χ_j are principal. For example, when $\chi_1 = \chi_0$, he proves that

$$\sum_{\substack{q \le Q \\ r \mid q}} \frac{1}{\varphi(q)^3} Z(q; \chi_0, \chi_2, \chi_3) \ll Q^{\epsilon} \frac{(a_1, r)}{\sqrt{r}} \prod_p s(p).$$

As $(a_1, r) \le A < Q^{1/3}$, this shows the error is negligible when $r > Q^{2/3+\epsilon}$. Therefore, when we apply the zero density estimates, we may restrict the moduli to the range $r \le Q^{2/3+\epsilon}$ instead of $r \le Q$ as in [12].

We can do better here by using the explicit expression for $Z(r; \chi_1, \chi_2, \chi_3)$. When $(h, q) = 1$, each Gauss sum $G(a_j h, \chi_j \chi_0)$ equals $\overline{\chi}_j(h) G(a_j, \chi_j \chi_0)$ so that

$$Z(q; \chi_1, \chi_2, \chi_3) = G(a_1, \chi_1 \chi_0) G(a_2, \chi_2 \chi_0) G(a_3, \chi_3 \chi_0) G(-b, \overline{\chi_1 \chi_2 \chi_3}).$$

Then we apply (3.16) to get the following result.

Lemma 4.2. *Let* $\chi_4 \pmod{r_4}$ *be the primitive character inducing* $\overline{\chi_1 \chi_2 \chi_3}$. *For convenience, we write* $a_4 = b$ *occasionally. For* $1 \le j \le 4$, *write* $r = r_1^{(j)} r_2^{(j)}$ *such that* $(r_2^{(j)}, r_j) = 1$ *and any prime factor of* $r_1^{(j)}$ *divides* r_j. *Then*

$$Z(r; \chi_1, \chi_2, \chi_3) = \prod_{j=1}^{4} \overline{\chi}_j \Big(\frac{a_j}{(a_j, r)} \Big) \chi_j \Big(\frac{r}{r_j (a_j, r)} \Big) \mu \Big(\frac{r}{r_j (a_j, r)} \Big) \varphi(r) \varphi^{-1} \Big(\frac{r}{(a_j, r)} \Big) G(1, \chi_j)$$

if

$$r_j = \frac{r_1^{(j)}}{(a_j, r_1^{(j)})} \tag{4.4}$$

for each $1 \le j \le 4$, *and* $Z(r; \chi_1, \chi_2, \chi_3) = 0$ *otherwise. In particular, for* $Z(r; \chi_1, \chi_2, \chi_3)$ *to be nonzero, we must have* $(a_j, r) r_j \mid r$; *moreover,*

$$|Z(r; \chi_1, \chi_2, \chi_3)| \ll \prod_{j=1}^{4} (a_j, r) r_j^{1/2}. \tag{4.5}$$

Lemma 4.2 gives a sharp bound of $Z(r; \chi_1, \chi_2, \chi_3)$ for our estimation. Regarding the sum (4.2), we quote the following result.

Lemma 4.3 ([12, Lemma 5.2]). *Under the definitions* (4.3), *we have*

$$\prod_p s(p) \gg 1, \tag{4.6}$$

$$\Big| \sum_{\substack{q \le Q \\ r \mid q}} \frac{1}{\varphi(q)^3} Z(q; \chi_1, \chi_2, \chi_3) \Big| \le 2.140782 \prod_p s(p). \tag{4.7}$$

5. The terms M_1 and M_3. We now proceed to estimate the main term of I_1 in (3.20). When we expand the three brackets on the right side, we get 27 terms which are of the following three types:

- Type 1: the term $\prod_{j=1}^{3} G(a_j h, \chi_0) I_j(1, \eta)$;
- Type 2: the 19 terms, each of which has at least one sum over the zeros of $L(s, \chi_j)$ for some $\chi_j \pmod q$;
- Type 3: the seven remaining terms.

For $j = 1, 2, 3$, define

(5.1)
$$M_i = \sum_{q \le Q} \sideset{}{'}\sum_{h=1}^{q} \frac{1}{\varphi(q)^3} e\Big(-\frac{b}{q}h\Big) \int_{-\tau/q}^{\tau/q} e(-b\eta)(\text{sum of terms of type } j)\, d\eta.$$

In this section, we first consider M_1 and M_3. For any x_1, x_2, define

$$x_3 = \frac{1}{a_3}\bigg(\frac{b}{N} - a_1 x_1 - a_2 x_2\bigg).$$

Let

$$M_0 = N^2 |a_3|^{-1} \prod_p s(p) \int_{\mathcal{D}} dx_1\, dx_2 \tag{5.2}$$

where

$$\mathcal{D} := \bigg\{(x_1, x_2) : \frac{1}{4|a_j|} \le x_j \le \frac{1}{|a_j|} \text{ for } j = 1, 2, 3\bigg\}.$$

In the estimation of M_1, M_2 and M_3, we need the following result.

Lemma 5.1 ([12, Lemma 5.1]).

$$\int_{-\infty}^{\infty} e(-b\eta) \prod_{j=1}^{3} I_j(\rho_j, \eta)\, d\eta = N^2 |a_3|^{-1} \int_{\mathcal{D}} \prod_{j=1}^{3} (N x_j)^{\rho_j - 1}\, dx_1\, dx_2.$$

Now, we estimate M_1 in the same way as in [11] and [12]. We obtain

Lemma 5.2.

$$M_1 = M_0 + O\Big(N^2 |a_1 a_2 a_3|^{-1} Q^{-2\epsilon_1} \prod_p s(p)\Big).$$

Here, the error term arises from the extension of the range of integration to $(-\infty, \infty)$.

Next, suppose $\widetilde{\beta}$ exists and we shall estimate M_3. For $\widetilde{r} > Q^{1/3+\epsilon_1}$, we estimate each term as we have done for the terms E_j. There are three types of terms in M_3. The first type is

$$\sum_{\substack{q\le Q\\ \widetilde{r}\mid q}} \frac{1}{\varphi(q)^3} Z(q;\chi_0,\chi_0,\widetilde{\chi}) \int_{-\tau/q}^{\tau/q} e(-b\eta) I_1(1,\eta) I_2(1,\eta) I_3(\widetilde{\beta},\eta)\, d\eta$$

$$\ll N^2|a_1a_2a_3|^{-1} \sum_{\substack{q\le Q\\ \widetilde{r}\mid q}} \frac{1}{\varphi(q)^3} |Z(q;\chi_0,\chi_0,\widetilde{\chi})|$$

$$\ll N^{2+\epsilon}|a_1a_2a_3|^{-1} \prod_p s(p) \frac{(\widetilde{r},a_1)(\widetilde{r},a_2)}{\widetilde{r}^2}$$

$$\ll N^{2+\epsilon}|a_1a_2a_3|^{-1} A\widetilde{r}^{-1} \prod_p s(p).$$

The other two types can be handled in the same way. On noting (3.2), we get

$$M_3 \ll N^{2+\epsilon}|a_1a_2a_3|^{-1} A\widetilde{r}^{-1} \prod_p s(p) \ll N^2|a_1a_2a_3|^{-1} Q^{-\epsilon_1} \prod_p s(p).$$

If $\widetilde{r} \le Q^{1/3+\epsilon_1}$, we apply the method of [12, Lemma 5.5] to get

Lemma 5.3. *Let*

$$\omega = (1-\widetilde{\beta})\log Q. \tag{5.3}$$

If $\widetilde{r} \le Q^{1/3+\epsilon_1}$, *then*

$$M_1 + M_3 \ge M_0 \times \begin{cases} 511\omega^3 & \text{if } \omega \le 10^{-5},\\ 496\omega^3 & \text{if } 10^{-5} < \omega \le 0.0025,\\ 240\omega^3 & \text{if } 0.0025 < \omega \le 0.066,\\ 63\omega^3 & \text{if } 0.066 < \omega \le 0.2,\\ 26\omega^3 & \text{if } 0.2 < \omega \le 0.306,\\ 17\omega^3 & \text{if } 0.306 < \omega \le 0.364 \end{cases} + O\Big(N^2|a_1a_2a_3|^{-1} Q^{-2\epsilon_1} \prod_p s(p)\Big). \tag{5.4}$$

The proof of this lemma is the same as that of [12, Lemma 5.5]. The only difference is that we investigate various ranges of ω to get a lower bound of the main term in each case, instead of using a single bound for all cases.

6. Estimation of M_2. In this section, we shall estimate M_2 defined in (5.1). We first consider the case where the exceptional zero $\widetilde{\beta}$ does not exist or $\widetilde{\beta}$ exists with $\omega > \epsilon_2$, where ω is defined in (5.3) and ϵ_2 is a sufficiently small positive constant. When we follow the steps in the estimation of M_1

and M_3 to handle M_2, we will end up with a main term involving sums over zeros of Dirichlet's L-function, and it has the same order of magnitude as M_0. Therefore, we need explicit zero density estimates to compare the contributions of the M_i's. Indeed, we have the following

LEMMA 6.1. *If $\widetilde{\beta}$ does not exist, then*

$$\Sigma_3 := \sum_{r \le Q^{2/3+\epsilon_1}} \sum_{\chi \,(\mathrm{mod}\, r)}^{*} \sum_{|\gamma| \le C}{}' \left(\frac{N}{4A}\right)^{\beta-1} \le 0.416, \tag{6.1}$$

$$\Sigma_4 := \sum_{r \le Q^{1/3+\epsilon_1}} \sum_{\chi \,(\mathrm{mod}\, r)}^{*} \sum_{|\gamma| \le C}{}' \left(\frac{N}{4A}\right)^{\beta-1} \le 8.87 \times 10^{-4}. \tag{6.2}$$

Under the definition (5.3), *if $\widetilde{\beta}$ exists with $\omega > \epsilon_2$ and $\widetilde{r} \le Q^{2/3+\epsilon_1}$, then*

$$\Sigma_3 \le \begin{cases} 0.0109\omega^3 & \text{if } \omega \le 10^{-5}, \\ 0.135\omega^3 & \text{if } 10^{-5} < \omega \le 0.0025, \\ 120\omega^3 & \text{if } 0.0025 < \omega \le 0.066, \\ 11.3\omega^3 & \text{if } 0.066 < \omega \le 0.2, \\ 4.99\omega^3 & \text{if } 0.2 < \omega \le 0.306, \\ 3.70\omega^3 & \text{if } 0.306 < \omega \le 0.364 \end{cases} \tag{6.3}$$

and

$$\Sigma_4 \le \begin{cases} 3.60 \times 10^{-6}\omega^3 & \text{if } \omega \le 10^{-5}, \\ 2.26 \times 10^{-25}\omega^3 & \text{if } 10^{-5} < \omega \le 0.0025, \\ 3.89 \times 10^{-9}\omega^3 & \text{if } 0.0025 < \omega \le 0.066, \\ 8.53 \times 10^{-6}\omega^3 & \text{if } 0.066 < \omega \le 0.2, \\ 1.25 \times 10^{-4}\omega^3 & \text{if } 0.2 < \omega \le 0.306, \\ 3.74 \times 10^{-4}\omega^3 & \text{if } 0.306 < \omega \le 0.364. \end{cases} \tag{6.4}$$

The proof of these results is similar to that of [12, Lemma 6.2]. Note that when we apply [12, Lemma 3.1], we replace Q by $Q' := Q^{2/3+\epsilon_1}$ or $Q^{1/3+\epsilon_1}$ in the respective cases. For the case where $\widetilde{\beta}$ exists with $\omega > \epsilon_2$, we use our (2.2) to replace [12, Lemma 2.6].

We now estimate M_2 defined in (5.1) when $\widetilde{\beta}$ does not exist or $\widetilde{\beta}$ exists with $\omega > \epsilon_2$. As in [12], the 19 terms can be further classified into six types:

- Type 2.1: three terms of the form $I_1(1,\eta)I_2(1,\eta)I_3(\rho_3,\eta)$;
- Type 2.2: six terms of the form $I_1(1,\eta)I_2(\widetilde{\beta},\eta)I_3(\rho_3,\eta)$;
- Type 2.3: three terms of the form $I_1(\widetilde{\beta},\eta)I_2(\widetilde{\beta},\eta)I_3(\rho_3,\eta)$;
- Type 2.4: three terms of the form $I_1(1,\eta)I_2(\rho_2,\eta)I_3(\rho_3,\eta)$;

- Type 2.5: three terms of the form $I_1(\widetilde{\beta},\eta)I_2(\rho_2,\eta)I_3(\rho_3,\eta)$;
- Type 2.6: the remaining term of the form $I_1(\rho_1,\eta)I_2(\rho_2,\eta)I_3(\rho_3,\eta)$.

Let $M_{21}, M_{22},\dots,M_{26}$ be the contribution from type $2.1, 2.2,\dots,2.6$ respectively. Owing to similarity, we only demonstrate the estimations for M_{23} and M_{26}. For M_{23}, a typical term has the form

$$(6.5)\qquad \sum_{r_3\le Q}\sideset{}{^*}\sum_{\chi_3\,(\mathrm{mod}\,r_3)}\sideset{}{'}\sum_{|\gamma_3|\le T}\sum_{\substack{q\le Q\\ [\widetilde{r},r_3]\mid q}}\frac{Z(q;\widetilde{\chi},\widetilde{\chi},\overline{\chi}_3)}{\varphi(q)^3} \times\int_{-\tau/q}^{\tau/q} e(-b\eta)I_1(\widetilde{\beta},\eta)I_2(\widetilde{\beta},\eta)I_3(\rho_3,\eta)\,d\eta.$$

We divide the sum over the nontrivial zeros into the three ranges $T'<|\gamma_3|\le T$, $C<|\gamma_3|\le T'$ and $|\gamma_3|\le C$, and denote the corresponding subsums of (6.5) by H_1, H_2, H_3 respectively. For H_1, H_2, we use Lemmas 3.2, 4.3 and 3.3 as in the estimation for the errors E_j. Firstly, H_1 is

$$\begin{aligned}
&\ll N^2|a_1a_2a_3|^{-1}\sum_{r_3\le Q}\sideset{}{^*}\sum_{\chi_3\,(\mathrm{mod}\,r_3)}\sideset{}{'}\sum_{T'<|\gamma_3|\le T}\sum_{\substack{q\le Q\\ [\widetilde{r},r_3]\mid q}}\frac{|Z(q;\widetilde{\chi},\widetilde{\chi},\overline{\chi}_3)|}{\varphi(q)^3}|\gamma_3|^{-1}N_3^{\beta_3-1}\\
&\ll N^2|a_1a_2a_3|^{-1}\prod_p s(p)\sum_{r_3\le Q}\sideset{}{^*}\sum_{\chi_3\,(\mathrm{mod}\,r_3)}\sideset{}{'}\sum_{T'<|\gamma_3|\le T}|\gamma_3|^{-1}N_3^{\beta_3-1}\\
&\ll N^2|a_1a_2a_3|^{-1}Q^{-1}\prod_p s(p).
\end{aligned}$$

Secondly, we have

$$\begin{aligned}
H_2 &\ll N^2|a_1a_2a_3|^{-1}C^{-1}\prod_p s(p)\sum_{r_3\le Q}\sideset{}{^*}\sum_{\chi_3\,(\mathrm{mod}\,r_3)}\sideset{}{'}\sum_{C<|\gamma_3|\le T'}N_3^{\beta_3-1}\\
&\ll N^2|a_1a_2a_3|^{-1}C^{-1}\prod_p s(p).
\end{aligned}$$

Note that the implied constant of the Vinogradov symbol here does not depend on C.

For H_3, we extend the range of integration to $(-\infty,\infty)$ for convenience. For $|\eta|\ge\tau/q$, we have $|\eta|>4C/(\pi N)\ge 4|\gamma_3|/(\pi N)$ and we use the third bound in Lemma 3.2. The error is thus

$$\ll\sum_{r_3\le Q}\sideset{}{^*}\sum_{\chi_3\,(\mathrm{mod}\,r_3)}\sideset{}{'}\sum_{|\gamma_3|\le C}\sum_{\substack{q\le Q\\ [\widetilde{r},r_3]\mid q}}\frac{|Z(q;\widetilde{\chi},\widetilde{\chi},\overline{\chi}_3)|}{\varphi(q)^3}\int_{\tau/q}^{\infty}\Big|\Big(\prod_{j=1}^{2}I_j(\widetilde{\beta},\eta)\Big)I_3(\rho_3,\eta)\Big|\,d\eta$$

$$\ll N^2|a_1a_2a_3|^{-1}Q^{-2-2\epsilon_1}$$
$$\times \sum_{r_3\le Q}\ \sideset{}{^*}\sum_{\chi_3\,(\mathrm{mod}\,r_3)}\ \sideset{}{'}\sum_{|\gamma_3|\le C}\ \sum_{\substack{q\le Q\\ [\widetilde{r},r_3]|q}} \frac{q^2}{\varphi(q)^3}|Z(q;\widetilde{\chi},\widetilde{\chi},\overline{\chi}_3)|N_3^{\beta_3-1}$$
$$\ll N^2|a_1a_2a_3|^{-1}Q^{-2\epsilon_1}\prod_p s(p),$$

which is admissible.

On applying Lemma 5.1, (6.5) equals

$$(6.6)\qquad N^2|a_3|^{-1}\sum_{r_3\le Q}\ \sideset{}{^*}\sum_{\chi_3\,(\mathrm{mod}\,r_3)}\ \sideset{}{'}\sum_{|\gamma_3|\le C}\ \sum_{\substack{q\le Q\\ [\widetilde{r},r_3]|q}} \frac{1}{\varphi(q)^3}Z(q;\widetilde{\chi},\widetilde{\chi},\overline{\chi}_3)$$
$$\times\int_{\mathcal{D}}(Nx_1)^{\widetilde{\beta}-1}(Nx_2)^{\widetilde{\beta}-1}(Nx_3)^{\rho_3-1}\,dx_1\,dx_2+O\Big(N^2|a_1a_2a_3|^{-1}C^{-1}\prod_p s(p)\Big).$$

We first use Lemma 4.1 and (4.5) to estimate the inner sum over q in (6.6). It equals zero unless (4.4) is satisfied, in which case it is bounded above by

$$(6.7)\qquad \mathcal{L}^3\prod_p s(p)\frac{(a_1,r)(a_2,r)(a_3,r)(b,r)\widetilde{r}r_3}{r^3}$$

where $r:=[\widetilde{r},r_3]$. As the fraction in (6.7) is multiplicative in r, we may consider the contribution from each prime p. We add a superscript $^{(p)}$ to a variable to mean its component corresponding to the prime p, possibly equal to 1. In particular, we see that $r^{(p)}=\max\{\widetilde{r}^{(p)},r_3^{(p)}\}$. Denote by $F^{(p)}$ the p-component of $(a_1,r)(a_2,r)(a_3,r)(b,r)\widetilde{r}r_3/r^3$ in (6.7). We shall consider each case in detail.

(i) For $r_3^{(p)}>1$ and $\widetilde{r}^{(p)}=1$, the conditions (4.4) for $j=3,4$ yield $(a_3^{(p)},r^{(p)})=(b^{(p)},r^{(p)})=1$. Hence,
$$F^{(p)}=(a_1^{(p)},r^{(p)})(a_2^{(p)},r^{(p)})(r^{(p)})^{-2}\le(a_1^{(p)},a_2^{(p)},r^{(p)})(r^{(p)})^{-1}$$
since this can be rewritten as $[(a_1^{(p)},r^{(p)}),(a_2^{(p)},r^{(p)})]\le r^{(p)}$, which is true on noting that each term on the left-hand side divides $r^{(p)}$.

(ii) For $r_3^{(p)}\ge\widetilde{r}^{(p)}>1$, again we have $(a_3^{(p)},r^{(p)})=(b^{(p)},r^{(p)})=1$. Also, the conditions (4.4) for $j=1,2$ yield $(a_1^{(p)},r^{(p)})=(a_2^{(p)},r^{(p)})=r^{(p)}(\widetilde{r}^{(p)})^{-1}$. Thus we get $F^{(p)}=(\widetilde{r}^{(p)})^{-1}=(a_1^{(p)},a_2^{(p)},r^{(p)})(r^{(p)})^{-1}$ in this case.

(iii) The case $\widetilde{r}^{(p)}>r_3^{(p)}>1$ is very similar to (ii), with the only difference that (4.4) implies $a_1^{(p)}=a_2^{(p)}=1$ and $(a_3^{(p)},r^{(p)})=(b^{(p)},r^{(p)})=\widetilde{r}^{(p)}(r_3^{(p)})^{-1}$. We again obtain $F^{(p)}=(r_3^{(p)})^{-1}=(a_1^{(p)},a_2^{(p)},r_3^{(p)})(r_3^{(p)})^{-1}$. Indeed, one may notice that since $\widetilde{r}^{(p)}\ge 2$, p must be 2 and it is known that $\widetilde{r}^{(p)}$ can only be 4 or 8 (for example, see [4, p. 40]). So this case occurs rarely and $F^{(p)} \ll 1$ suffices for our use.

(iv) For $\widetilde{r}^{(p)} > r_3^{(p)} = 1$, as in case (i), we have
$$F^{(p)} = (a_3^{(p)}, \widetilde{r}^{(p)})(b^{(p)}, \widetilde{r}^{(p)})(\widetilde{r}^{(p)})^{-2} \le (a_3^{(p)}, b^{(p)}, \widetilde{r}^{(p)})(\widetilde{r}^{(p)})^{-1}.$$

From these cases, if we write $r_3 = r_3' r_3''$ so that $r_3' = (a_1, a_2, r_3)$, the fraction in (6.7) is bounded above by $(r_3'')^{-1}$. Consider those $r_3 \le Q$ for which $r_3'' > Q^{\epsilon_1}$. We see that their total contribution to the main term of (6.6) is
$$\ll Q^{-\epsilon_1} \mathcal{L}^3 M_0 \sum_{r_3 \le Q} \sideset{}{^*}\sum_{\chi_3 \,(\mathrm{mod}\, r_3)} \sideset{}{'}\sum_{|\gamma_3| \le C} \left(\frac{N}{4A}\right)^{\beta_3 - 1} \ll Q^{-\epsilon_1} \mathcal{L}^3 M_0,$$
which is again admissible. Hence, we may restrict the sum over r_3 in (6.6) to those which can be represented as $r_3' r_3''$ where $r_3' \mid (a_1, a_2)$ and $r_3'' \le Q^{\epsilon_1}$. A similar reasoning allows us to impose the constraint $\widetilde{r} \le (a_3, b) Q^{\epsilon_1}$, as otherwise the main term in (6.6) is negligible. Next, we use (4.7). Then the absolute value of (6.6) is
$$\le 2.140782 M_0 \sum_{r_3} \sideset{}{^*}\sum_{\chi_3 \,(\mathrm{mod}\, r_3)} \sideset{}{'}\sum_{|\gamma_3| \le C} \left(\frac{N}{4A}\right)^{\beta_3 - 1} + O\Big(N^2 |a_1 a_2 a_3|^{-1} C^{-1} \prod_p s(p)\Big).$$

Now, there are two other similar terms of the same type, corresponding to sums over r_1 and r_2. Note that r_1 equals a factor of (a_2, a_3) multiplied by a small term $\le Q^{\epsilon_1}$, while r_2 equals a factor of (a_1, a_3) multiplied by a small term $\le Q^{\epsilon_1}$. Therefore, r_1, r_2, r_3 are almost distinct, except possibly when they are $\le Q^{2\epsilon_1}$. The contribution to M_{23} from these overlapping cases can be estimated as in Lemma 6.1, by replacing the upper bound of r by $Q^{2\epsilon_1}$. The corresponding sum, like Σ_3 and Σ_4, is bounded above by $\epsilon_2 \omega^3$, which is negligible. Hence, we can combine the sums over r_1, r_2, r_3 into a single one. Note that in any case, each r_j and $\widetilde{r}$ is at most $A Q^{\epsilon_1} \le Q^{1/3 + \epsilon_1}$. Since $\omega > \epsilon_2$, by taking C sufficiently large depending on ϵ_2, we conclude that
$$M_{23} \le (2.140782 + \epsilon_2) M_0 \Sigma_4 \tag{6.8}$$
where Σ_4 is defined in (6.2).

Next, for the term M_{26}, there is only a single term
$$\sum_{r_1, r_2, r_3 \le Q} \sideset{}{^*}\sum_{\chi_j \,(\mathrm{mod}\, r_j)} \sideset{}{'}\sum_{|\gamma_j| \le T} \sum_{\substack{q \le Q \\ r_j \mid q}} \frac{Z(q; \overline{\chi}_1, \overline{\chi}_2, \overline{\chi}_3)}{\varphi(q)^3} \int_{-\tau/q}^{\tau/q} e(-b\eta) \prod_{j=1}^{3} I_j(\rho_j, \eta)\, d\eta.$$

As in the case of M_{23}, we easily find that the contribution when at least one of $|\gamma_j|$ is larger than C is admissible. By similar arguments, we need to consider

$$G := \frac{(a_1,r)(a_2,r)(a_3,r)(b,r)r_1^{1/2}r_2^{1/2}r_3^{1/2}r_4^{1/2}}{r^3}$$

where $r := [r_1, r_2, r_3]$. Note that for each prime p, the largest two among $r_1^{(p)}, r_2^{(p)}, r_3^{(p)}, r_4^{(p)}$ are equal, since r_4 is the conductor of $\chi_1\chi_2\chi_3$. Again we have a few cases to consider.

(i) For $r_1^{(p)} = r_2^{(p)} = r_3^{(p)} > 1$, we have $a_1^{(p)} = a_2^{(p)} = a_3^{(p)} = 1$ and hence $G^{(p)} = (b^{(p)}, r^{(p)})(r_4^{(p)})^{1/2}(r^{(p)})^{-3/2}$. Since $(b^{(p)}, r^{(p)})(r_4^{(p)}) \le r^{(p)}$, $G^{(p)}$ is bounded above by $(r^{(p)})^{-1/2}$.

(ii) For $r_1^{(p)} = r_2^{(p)} > r_3^{(p)}$ (and in two other similar cases), we have $a_1^{(p)} = a_2^{(p)} = 1$ and

$$\begin{aligned} G^{(p)} &= (a_3^{(p)}, r^{(p)})(b^{(p)}, r^{(p)})(r_3^{(p)})^{1/2}(r_4^{(p)})^{1/2}(r^{(p)})^{-2} \\ &\le (a_3^{(p)}, r^{(p)})^{1/2}(b^{(p)}, r^{(p)})^{1/2}(r^{(p)})^{-1} \\ &\le (a_3^{(p)}, b^{(p)}, r^{(p)})^{1/2}(r^{(p)})^{-1/2}. \end{aligned}$$

(iii) For $r_1^{(p)} > r_2^{(p)}, r_3^{(p)}$ (and in two other similar cases), we have $r_4^{(p)} = r_1^{(p)}$ and $a_1^{(p)} = b^{(p)} = 1$. In this case

$$\begin{aligned} G^{(p)} &= (a_2^{(p)}, r^{(p)})(a_3^{(p)}, r^{(p)})(r_2^{(p)})^{1/2}(r_3^{(p)})^{1/2}(r^{(p)})^{-2} \\ &\le (a_2^{(p)}, r^{(p)})^{1/2}(a_3^{(p)}, r^{(p)})^{1/2}(r^{(p)})^{-1} \\ &\le (a_2^{(p)}, a_3^{(p)}, r^{(p)})^{1/2}(r^{(p)})^{-1/2}. \end{aligned}$$

If we write $r_1 = r_1' r_1''$ where r_1' consists of the components corresponding to primes dividing $(a_2, a_3)(a_2, b)(a_3, b)$, then we easily find that the contribution to M_{26} from those r_1 with $r_1'' > Q^{\epsilon_1}$ is negligible, as in the case of M_{23}. Similar results hold for r_2 and r_3. Hence, we restrict r_i to be a factor of $(a_j, a_k)(a_j, b)(a_k, b)$ multiplied by a small term $\le Q^{\epsilon_1}$ for each $1 \le i \le 3$, where $\{i, j, k\} = \{1, 2, 3\}$. This time each r_i is bounded above by $A^2 Q^{\epsilon_1} \le Q^{2/3+\epsilon_1}$ instead of $Q^{1/3+\epsilon_1}$. Therefore, we have

$$M_{26} \le (2.140782 + \epsilon_2) M_0 \Sigma_3^3 \tag{6.9}$$

where Σ_3 is defined in (6.1).

Now we summarize the contributions in the remaining cases. Note that for M_{22}, we can only combine the six terms into two groups of 3. Thus,

$$\begin{aligned} &M_{21} \le (2.140782 + \epsilon_2) M_0 \Sigma_4, \qquad M_{22} \le (2.140782 + \epsilon_2) M_0 \times 2\Sigma_4, \\ &M_{24} \le (2.140782 + \epsilon_2) M_0 \Sigma_4^2, \qquad M_{25} \le (2.140782 + \epsilon_2) M_0 \Sigma_3^2. \end{aligned} \tag{6.10}$$

From (6.8)–(6.10), we have

$$M_2 \le (2.140782 + \epsilon_2) M_0 (4\Sigma_4 + \Sigma_3^2 + \Sigma_4^2 + \Sigma_3^3)$$

if $\widetilde{\beta}$ exists. Using (6.3) and (6.4), and comparing the results to (5.4), we always have

$$M_1 + M_2 + M_3 \gg \omega^3 M_0. \tag{6.11}$$

For the case where $\widetilde{\beta}$ exists with $\omega \le \epsilon_2$, similar estimations to the case $\omega > \epsilon_2$ apply. Firstly, we can restrict the nontrivial zeros to the range $|\gamma_j| \le Q^{\epsilon_1}$. Secondly, the contribution to M_2 when one of the moduli r_j is $> Q^{2/3+\epsilon_1}$ or $\widetilde{r} > Q^{2/3+\epsilon_1}$ is negligible, as in the estimation for M_{26}. In view of these and the arguments in [12], the remaining key step is to establish

LEMMA 6.2. *Let*

$$\Sigma_5 = \sum_{r \le Q^{2/3+\epsilon_1}} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, r)} \sideset{}{'}\sum_{|\gamma| \le Q^{\epsilon_1}} \left(\frac{N}{4A}\right)^{\beta-1}.$$

Under the definition (5.3), *if* $\omega \le \epsilon_2$ *and* Q *is sufficiently large, then*

$$\Sigma_5 \ll \epsilon_2^{1/2} \omega^3.$$

The proof is similar to that of [12, Lemma 6.1], except that we use our (2.2).

From this, the triple sum in Lemma 6.2 is negligible compared to the coefficient in (5.4). Hence, (6.11) still applies in the case $\omega \le \epsilon_2$.

If $\widetilde{\beta}$ does not exist, the terms M_{22}, M_{23}, M_{25} vanish and hence

$$M_2 \le (2.140782 + \epsilon_2) M_0 (\Sigma_4 + \Sigma_4^2 + \Sigma_3^3).$$

Note that M_3 vanishes in this case. From Lemma 5.2, (6.1) and (6.2), we get $M_1 + M_2 \gg M_0$. Here, the error terms are absorbed in view of the definition of M_0 in (5.2) and noting (4.6).

Combining (3.2), (3.9), (3.20) and (5.1), we find that the sum (3.6) is $\gg \omega^3 M_0$ or M_0 in the respective cases. This completes the proof of Theorem 1.

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