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Triads of integers with equal sums of squares and equal products and a related multigrade chain

by

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1. Introduction. The problem of finding two triads of integers with equal sums of squares and equal products was first posed by Bini [1] in 1908. Two-parameter solutions of the problem were presented separately by Dubouis and by Mathieu in 1909 [9]. Subsequently, Gloden [10, pp. 36–37] and Choudhry [3] gave different six-parameter solutions of the problem. The problem was included by Bremner and Guy in their list of "dozen difficult diophantine dilemmas" [2] and it was also mentioned by Guy in his book "Unsolved Problems in Number Theory" [11, p. 214]. Two complete solutions were later obtained independently by Kelly [12] and by Choudhry [4].

This paper is concerned with a natural extension of the problem, namely, whether for any given integer $n > 2$, there exists a set of n triads of positive integers with equal sums of squares and equal products, that is, whether there exists a solution in positive integers of the simultaneous diophantine chains

$$\begin{aligned} x_1^2+y_1^2+z_1^2 &= x_2^2+y_2^2+z_2^2 = \cdots = x_n^2+y_n^2+z_n^2,\\ x_1y_1z_1 &= x_2y_2z_2 = \cdots = x_ny_nz_n. \end{aligned} \tag{1.1}$$

An extensive computer search was carried out to find integer solutions of (1.1). While numerous solutions were found for $n = 3$, only a limited number of solutions were obtained when $n = 4$ or $n = 5$.

An analysis of the numerical solutions yielded two parametric solutions when $n = 3$. Two more such solutions were obtained by directly solving (1.1) for $n = 3$. In fact, infinitely many parametric solutions of the diophantine chains (1.1) may be obtained by the methods given in this paper.

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In Section 2 we present the findings of the computer search. In Section 3 we derive the four parametric solutions mentioned above and in Section 4 we use the solutions already found to obtain multigrade chains of the type

$$(1.2)\quad \begin{aligned} a_1+b_1+c_1+d_1 &= a_2+b_2+c_2+d_2=\cdots=a_n+b_n+c_n+d_n,\\ a_1^2+b_1^2+c_1^2+d_1^2 &= a_2^2+b_2^2+c_2^2+d_2^2=\cdots=a_n^2+b_n^2+c_n^2+d_n^2,\\ a_1^3+b_1^3+c_1^3+d_1^3 &= a_2^3+b_2^3+c_2^3+d_2^3=\cdots=a_n^3+b_n^3+c_n^3+d_n^3,\\ a_1^5+b_1^5+c_1^5+d_1^5 &= a_2^5+b_2^5+c_2^5+d_2^5=\cdots=a_n^5+b_n^5+c_n^5+d_n^5. \end{aligned}$$

In Section 5 we discuss an open problem concerning the length of the diophantine chains (1.1).

We note that all the diophantine equations considered in this paper are homogeneous, and hence any solution in rational numbers may be multiplied through by a suitable constant to yield a solution in integers. It follows that in the parametric solutions we obtain, the parameters may be assigned arbitrary rational values and solutions in integers may be obtained by appropriate scaling.

2. Numerical results. We carried out an exhaustive search to determine all positive integer solutions of (1.1) with terms below 30000. We obtained 4611 solutions with $n=3$ and thirteen solutions with $n=4$. We did not find any solutions with $n=5$.

In Table 1 we present the 32 solutions of (1.1) with terms less than 700. Observe that the solutions at No. 27 and No. 31 have the same sums of squares and also some common terms. The thirteen solutions with $n=4$ and terms less than 30000 are presented in Table 2.

A selective search to a higher range produced 127 solutions of (1.1) with $n=4$ and two solutions with $n=5$. Among the solutions with $n=4$, we found a pair of solutions with the same sum of squares. This pair of solutions is as follows:

$$\begin{aligned} &(3720960, 2644180, 2427685), \quad (3674125, 2800512, 2321384),\\ &(3571360, 3002285, 2227680), \quad (3536000, 3056872, 2209779); \end{aligned}$$

and

$$\begin{aligned} &(5056480, 937040, 533715), \quad (4919460, 1555840, 330395),\\ &(4559060, 2427685, 228480), \quad (4489485, 2554760, 220480). \end{aligned}$$

The two solutions of (1.1) that we found for $n=5$ are as follows:

$$\begin{aligned} &(250582040, 82469040, 70402865), \quad (244317489, 108953000, 54655952),\\ &(222583465, 152187360, 42949660), \quad (218290540, 158526615, 42043040),\\ &(203908744, 177048625, 40299792), \end{aligned}$$

and

$$(13728417456, 1175013242, 918393125),$$
$$(13430700400, 3192246485, 345538770),$$
$$(13377826150, 3409075280, 324840165),$$
$$(13210166710, 4013347845, 279432400),$$
$$(9836636875, 9690715632, 155413726).$$

Table 1. Three triads with equal sums of squares and equal products

No.	x_1, y_1, z_1	x_2, y_2, z_2	x_3, y_3, z_3
1	143, 40, 34	136, 65, 22	110, 104, 17
2	196, 39, 18	182, 84, 9	156, 126, 7
3	250, 28, 21	245, 60, 10	210, 140, 5
4	252, 130, 51	238, 156, 45	221, 180, 42
5	260, 45, 30	234, 125, 12	195, 180, 10
6	260, 110, 95	250, 143, 76	220, 190, 65
7	261, 70, 28	245, 116, 18	203, 180, 14
8	323, 65, 39	285, 169, 17	247, 221, 15
9	357, 130, 100	340, 182, 75	325, 210, 68
10	378, 91, 60	315, 234, 28	294, 260, 27
11	440, 185, 30	407, 250, 24	375, 296, 22
12	455, 148, 66	444, 182, 55	370, 308, 39
13	464, 78, 35	435, 182, 16	406, 240, 13
14	494, 75, 20	475, 156, 10	380, 325, 6
15	495, 120, 80	480, 180, 55	375, 352, 36
16	513, 220, 126	495, 266, 108	462, 324, 95
17	516, 112, 85	476, 240, 43	408, 344, 35
18	517, 66, 40	440, 282, 11	376, 363, 10
19	520, 63, 29	504, 145, 13	455, 261, 8
20	576, 144, 85	540, 256, 51	459, 384, 40
21	588, 85, 34	578, 140, 21	476, 357, 10
22	588, 154, 65	572, 210, 49	490, 364, 33
23	611, 185, 63	585, 259, 47	481, 423, 35
24	612, 209, 54	561, 324, 38	513, 396, 34
25	625, 143, 99	605, 225, 65	585, 275, 55
26	629, 95, 93	589, 255, 37	555, 323, 31
27	650, 240, 95	625, 304, 78	570, 400, 65
28	656, 190, 87	615, 304, 58	551, 410, 48
29	665, 192, 104	640, 273, 76	560, 416, 57
30	666, 180, 53	636, 270, 37	530, 444, 27
31	680, 150, 65	650, 255, 40	625, 312, 34
32	688, 375, 266	665, 430, 240	570, 560, 215

Table 2. Four triads with equal sums of squares and equal products

No.	x_1, y_1, z_1 x_2, y_2, z_2	x_3, y_3, z_3 x_4, y_4, z_4
1	4455, 1596, 902 4428, 1694, 855	3895, 2772, 594 3630, 3116, 567
2	6545, 1544, 1178 6479, 1930, 952	5983, 3230, 616 5320, 4246, 527
3	8788, 1665, 954 8658, 2340, 689	8268, 3510, 481 6890, 5772, 351
4	11063, 4576, 4290 10985, 5106, 3872	10582, 6240, 3289 9568, 7865, 2886
5	11648, 1239, 500 11505, 2240, 280	11375, 2832, 224 8320, 8260, 105
6	12025, 1045, 885 11875, 2301, 407	10915, 5225, 195 10725, 5605, 185
7	14625, 1404, 1378 13780, 5265, 390	13182, 6625, 324 10530, 10335, 260
8	14945, 2405, 935 14875, 2849, 793	14245, 5185, 455 12155, 9065, 305
9	20770, 3140, 975 20410, 5025, 620	19375, 8164, 402 17420, 11775, 310
10	21824, 4028, 1105 21736, 4505, 992	21359, 6080, 748 18073, 12920, 416
11	24820, 7475, 5590 23725, 11180, 3910	21250, 15548, 3139 18980, 18275, 2990
12	28405, 11102, 9776 28106, 12688, 8645	27664, 14053, 7930 22448, 21970, 6251
13	29393, 8874, 7500 28900, 11310, 5985	26100, 17290, 4335 24225, 19890, 4060

3. Three triads of integers with equal sums of squares and equal products. In this section we find parametric solutions of the simultaneous diophantine chains

(3.1)
$$\begin{aligned} x_1^2+y_1^2+z_1^2 &= x_2^2+y_2^2+z_2^2 = x_3^2+y_3^2+z_3^2,\\ x_1y_1z_1 &= \quad x_2y_2z_2 \quad = x_3y_3z_3. \end{aligned}$$

In Section 3.1 we find a parametric solution of (3.1) by analysing the numerical solutions $(x_1, y_1, z_1;\ x_2, y_2, z_2;\ x_3, y_3, z_3)$ obtained by computer search. The solutions thus obtained are somewhat special as they are part of a larger diophantine system. Accordingly, in Sections 3.2 and 3.3 we directly solve (3.1) and obtain two solutions in terms of one rational parameter and three rational parameters respectively.

3.1. We detected several pairs of solutions sharing common sums of squares and having some common terms. The smallest such example is

$$\begin{aligned}&(\mathbf{650},\ -240,\ 95;\quad \mathbf{625},\ 304,\ -78;\quad \mathbf{65},\ -570,\ 400),\\ &(\mathbf{650},\ -40,\ -255;\ \mathbf{625},\ -34,\ -312;\ \mathbf{65},\ 680,\ 150),\end{aligned} \tag{3.2}$$

where the order of terms and their signs are chosen to satisfy the additional equations specified below.

A closer analysis of the available examples revealed their common properties, leading to the system of equations

$$\begin{aligned} x_1y_1z_1 &= x_2y_2z_2 = x_3y_3z_3,\\ x_1s_1t_1 &= x_2s_2t_2 = x_3s_3t_3,\\ x_1^2+y_1^2+z_1^2 &= x_2^2+y_2^2+z_2^2 = x_3^2+y_3^2+z_3^2\\ = x_1^2+s_1^2+t_1^2 &= x_2^2+s_2^2+t_2^2 = x_3^2+s_3^2+t_3^2, \end{aligned} \tag{3.3}$$

whose available numerical solutions satisfy the following auxiliary conditions:

$$\begin{aligned} s_1 &= y_1r_3r_4, \quad s_2 = y_2r_2r_4, \quad s_3 = y_3r_2r_3,\\ t_1 &= z_1r_1r_2, \quad t_2 = z_2r_1r_3, \quad t_3 = z_3r_1r_4, \end{aligned} \tag{3.4}$$

where r_1, r_2, r_3, r_4 are some rational numbers.

We further observed that on writing $q_i = -1/r_i$ for $i = 1, 2, 3, 4$, the numerical solutions of (3.3) also satisfy

$$\begin{aligned} x_1+y_1+z_1r_1 &= x_2+y_2+z_2r_1 = x_3+y_3+z_3r_1\\ = x_1+s_1+t_1q_1 &= x_2+s_2+t_2q_1 = x_3+s_3+t_3q_1,\\ x_1-y_1-z_1r_2 &= x_2-z_2-y_2r_2 = x_3+z_3+y_3r_2\\ = x_1-s_1-t_1q_2 &= x_2-t_2-s_2q_2 = x_3+t_3+s_3q_2,\\ x_1-z_1-y_1r_3 &= x_2+z_2+y_2r_3 = x_3-y_3-z_3r_3\\ = x_1-t_1-s_1q_3 &= x_2+t_2+s_2q_3 = x_3-s_3-t_3q_3,\\ x_1+z_1+y_1r_4 &= x_2-y_2-z_2r_4 = x_3-z_3-y_3r_4\\ = x_1+t_1+s_1q_4 &= x_2-s_2-t_2q_4 = x_3-t_3-s_3q_4. \end{aligned} \tag{3.5}$$

In view of the numerical evidence, it was expected that (3.3) should be solvable together with (3.4) and (3.5). By eliminating x_1, y_1, z_1 from (3.5), we get

$$r_1r_2r_3r_4 - r_1r_2 - r_1r_3 - r_1r_4 - r_2r_3 - r_2r_4 - r_3r_4 + 1 = 0, \tag{3.6}$$

which must necessarily be satisfied if there exists a nontrivial solution of (3.5). We obtain the same condition on eliminating x_2, y_2, z_2 from (3.5), and also on eliminating x_3, y_3, z_3 from (3.5).

In addition to (3.6), it was observed that r_1, r_2, r_3, r_4 also satisfy

$$r_1 + r_2 + r_3 + r_4 + r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4 = 0. \tag{3.7}$$

When the rational numbers r_1, r_2, r_3, r_4 satisfy (3.6) and (3.7), the simultaneous equations (3.3)–(3.5) can be solved. We omit the tedious details of solving these equations, and simply give the final solution below. We will show subsequently by direct verification that this is indeed a solution of (3.3)–(3.5).

We first define three functions:

$$
\begin{aligned}
&f_1(r_1,\, r_2,\, r_3) = r_1r_2 + r_1r_3 + r_2r_3 - 1,\\
&f_2(r_1,\, r_2,\, r_3) = r_1r_2r_3 - r_1 - r_2 - r_3,\\
&f_3(r_1,\, r_2,\, r_3) = r_1r_2^2 + 2r_1r_2r_3 + r_2^2r_3 - r_1 - 2r_2 - r_3.
\end{aligned}
\tag{3.8}
$$

We can now write the solution of (3.3)–(3.5) as follows:

$$
\begin{aligned}
x_1 &= (r_2+r_3)(r_2^2+1)f_1(-r_1,-r_2,1)f_1^2(-r_1,1,-r_3)f_3(r_1,r_2,r_3),\\
y_1 &= 2(r_2r_3+1)f_1(r_1,r_2,1)f_1(-r_1,-r_2,1)f_2(r_1,r_2,r_3)f_3(r_1,r_2,r_3),\\
z_1 &= 2(r_3^2+1)(r_2r_3+1)f_1(r_1,r_2,1)f_1(-r_1,-r_2,1)f_3(r_1,r_2,r_3),\\
x_2 &= (r_2^2+1)(r_3^2+1)(r_1r_3-1)(r_2r_3+1)f_1(r_1,r_2,1)f_1^2(-r_1,-r_2,1),\\
y_2 &= 2(r_1^2+1)(r_2r_3+1)f_1(r_1,r_2,1)f_1(1,r_2,r_3)f_3(r_1,r_2,r_3),\\
z_2 &= 2(r_2r_3+1)f_1(r_1,r_2,1)f_1(1,r_2,r_3)f_2(r_1,r_2,r_3)f_3(r_1,r_2,r_3),\\
x_3 &= (r_2+r_3)(r_3^2+1)f_1^2(r_1,r_2,1)f_1(-r_1,-r_2,1)f_3(r_1,r_2,r_3),\\
y_3 &= 2(r_2r_3+1)f_1(-r_1,-r_2,1)f_1(-r_1,1,-r_3)f_2(r_1,r_2,r_3)f_3(r_1,r_2,r_3),\\
z_3 &= 2(r_2^2+1)(r_2r_3+1)f_1(-r_1,-r_2,1)f_1(-r_1,1,-r_3)f_3(r_1,r_2,r_3),\\
s_1 &= 2r_3(r_2r_3+1)f_1(r_1,r_2,1)f_1(-r_1,-r_2,1)f_1(r_1,r_2,r_3)f_3(r_1,r_2,r_3),\\
t_1 &= 2r_1r_2(r_3^2+1)(r_2r_3+1)f_1(r_1,r_2,1)f_1(-r_1,-r_2,1)f_3(r_1,r_2,r_3),\\
s_2 &= 2r_2r_3(r_1^2+1)(r_2r_3+1)f_1(r_1,r_2,1)f_1(1,r_2,r_3)f_3(r_1,r_2,r_3),\\
t_2 &= 2r_1(r_2r_3+1)f_1(r_1,r_2,1)f_1(1,r_2,r_3)f_1(r_1,r_2,r_3)f_3(r_1,r_2,r_3),\\
s_3 &= 2r_2(r_2r_3+1)f_1(-r_1,-r_2,1)f_1(-r_1,1,-r_3)f_1(r_1,r_2,r_3)f_3(r_1,r_2,r_3),\\
t_3 &= 2r_1r_3(r_2^2+1)(r_2r_3+1)f_1(-r_1,-r_2,1)f_1(-r_1,1,-r_3)f_3(r_1,r_2,r_3).
\end{aligned}
\tag{3.9}
$$

When the rational numbers r_1, r_2, r_3, r_4 satisfy (3.6) and (3.7), we find, on eliminating r_4 from these conditions, that r_1, r_2, r_3 must satisfy

$$\phi(r_1,r_2,r_3) = 0 \tag{3.10}$$

where

$$
\begin{aligned}
\phi(r_1,r_2,r_3) = {}& r_1^2r_2^2r_3^2 + r_1^2r_2^2 + r_2^2r_3^2 + r_1^2r_3^2\\
&+ 2r_1r_2r_3(r_1+r_2+r_3) - (r_1+r_2+r_3)^2 - 1.
\end{aligned}
\tag{3.11}
$$

We will now verify that (3.9) is indeed a solution of (3.3)–(3.5). On substituting the values of x_i, y_i, z_i, s_i, t_i given by (3.9) into each of the equations of (3.3)–(3.5), we find that several of the equations are identically satisfied, and in the remaining cases, the difference of the two sides of the resulting equation is a multiple of $\phi(r_1,r_2,r_3)$, and is hence 0. This establishes that (3.9) is a solution of (3.3)–(3.5).

Since the solution (3.9) is expressed only in terms of r_1, r_2, r_3, it suffices to find three rational numbers r_1, r_2, r_3 satisfying (3.10). Moreover, in view of the symmetry of (3.10), each of the six permutations of r_1, r_2, r_3 will yield a solution, though these six solutions may not be distinct.

As an example, a solution of (3.10) is given by $r_1 = -1/p, r_2 = p, r_3 = (p+1)/(p-1)$, where p is an arbitrary parameter. This yields a semi-trivial solution of (3.3) in which the two chains

$$(x_1, y_1, z_1;\ x_2, y_2, z_2;\ x_3, y_3, z_3) \quad \text{and} \quad (x_1, s_1, t_1; x_2, s_2, t_2; x_3, s_3, t_3)$$

are in fact the same chain. This solution is as follows:

$$\begin{aligned}
&x_1 = 2(p^2+1)^3,\\
&y_1 = -s_1 = -(p-1)(p+1)(p^2-2p-1)(p^2+2p-1),\\
&z_1 = -t_1 = 2p(p^2-2p-1)(p^2+2p-1),\\
&x_2 = (p^2+1)(p^2+2p-1)^2,\\
&y_2 = -t_2 = -2(p-1)(p^2+1)(p^2-2p-1),\\
&z_2 = -s_2 = 2p(p+1)(p^2+1)(p^2-2p-1),\\
&x_3 = (p^2+1)(p^2-2p-1)^2,\\
&y_3 = t_3 = 2(p+1)(p^2+1)(p^2+2p-1),\\
&z_3 = s_3 = -2p(p-1)(p^2+1)(p^2+2p-1),
\end{aligned} \tag{3.12}$$

where p is an arbitrary parameter. The other permutations of $r_1 = -1/p$, $r_2 = p$, $r_3 = (p+1)/(p-1)$ yield only trivial solutions.

Another solution of (3.10), in terms of an arbitrary parameter p, is

$$r_1 = p, \qquad r_2 = \frac{3p^2+5}{(p-1)p(p+1)}, \qquad r_3 = \frac{(1-p)(p^3+3p^2+3p+5)}{p^4+6p^2+4p+5}.$$

In this case, all the six permutations of r_1, r_2, r_3 yield nontrivial solutions of (3.3) but not all are distinct. The simplest of the six solutions is

$$\begin{aligned}
&x_1 = -2(p^2+2p+5)(p^2-2p+5)(p^2+1)^3,\\
&y_1 = -(p+1)(p-1)(p^4+6p^2+4p+5)(p^4+6p^2-4p+5),\\
&z_1 = 2p(p-1)^2(p+1)^2(p^4+2p^2+5),\\
&x_2 = -(p-1)^2(p^2+1)(p^2-2p+5)(p^4+2p^2+5),\\
&y_2 = 2p(p+1)^2(p-1)(p^2+1)(p^4+6p^2+4p+5),\\
&z_2 = -2(p+1)(p^2+1)(p^2+2p+5)(p^4+6p^2-4p+5),\\
&x_3 = -(p+1)^2(p^2+1)(p^2+2p+5)(p^4+2p^2+5),\\
&y_3 = 2p(p-1)^2(p+1)(p^2+1)(p^4+6p^2-4p+5),\\
&z_3 = -2(p-1)(p^2+1)(p^2-2p+5)(p^4+6p^2+4p+5),\\
&s_1 = (p+1)^2(p-1)^2(p^3+3p^2+3p+5)(p^3-3p^2+3p-5),\\
&t_1 = 2p(p+1)(p-1)(3p^2+5)(p^4+2p^2+5),\\
&s_2 = -2(p+1)(p-1)(3p^2+5)(p^2+1)(p^3+3p^2+3p+5),\\
&t_2 = -2p(p+1)^2(p^2+1)(p^2+2p+5)(p^3-3p^2+3p-5),\\
&s_3 = 2(p+1)(p-1)(3p^2+5)(p^2+1)(p^3-3p^2+3p-5),\\
&t_3 = 2p(p^2+1)(p-1)^2(p^2-2p+5)(p^3+3p^2+3p+5),
\end{aligned} \tag{3.13}$$

where p is an arbitrary parameter.

We note that (3.10) may be considered as a quadratic equation in r_3 and its discriminant is a quartic function of r_1. Since we know a value of r_1 that makes the discriminant a perfect square, we can find infinitely many values of r_1 that make the discriminant a perfect square by repeatedly applying a method described by Fermat [8, p. 639]. These values yield infinitely many rational solutions of $\phi(r_1, r_2, r_3) = 0$, and we can thus obtain infinitely many parametric solutions of (3.3).

3.2. We will now obtain a parametric solution of the simultaneous diophantine chains (3.1) by solving the system of equations

$$x_1y_1z_1 = x_2y_2z_2 = x_3y_3z_3, \tag{3.14}$$

$$x_1^2 + y_1^2 + z_1^2 = x_2^2 + y_2^2 + z_2^2, \tag{3.15}$$

$$x_1^2 + y_1^2 + z_1^2 = x_3^2 + y_3^2 + z_3^2. \tag{3.16}$$

We write

$$\begin{aligned} x_1 &= p_1q_1m, & y_1 &= p_2q_2(m+1), & z_1 &= p_3q_3(m+n), \\ x_2 &= p_2q_3m, & y_2 &= p_3q_1(m+1), & z_2 &= p_1q_2(m+n), \\ x_3 &= p_3q_2m, & y_3 &= p_1q_3(m+1), & z_3 &= p_2q_1(m+n), \end{aligned} \tag{3.17}$$

where m, n, p_i, q_i, $i = 1, 2, 3$, are arbitrary parameters. With these values of x_i, y_i, z_i, it is easily seen that (3.14) is identically satisfied.

We now take

$$p_1 = p_3 + 1, \quad p_2 = p_3 + n, \quad q_1 = q_3 + 1, \quad q_2 = q_3 + n, \tag{3.18}$$

and on substituting the values of x_i, y_i, z_i, p_i, q_i given by (3.17) and (3.18) into (3.15), we get an equation which, on transposing all terms to the left-hand side and removing the factor $m - p_3$, reduces to a linear equation in m that is readily solved, and we thus get the following solution for m:

$$\begin{aligned} m = -\{&2(n^3 - n + 1)p_3q_3 + (n^4 - n^2 + 1)p_3 + 2n(n-1)q_3^2 \\ &+ 4n^2(n-1)q_3 + 2n^3(n-1)\}\{4(n^2 - n + 1)p_3q_3 \\ &+ 2(n^3 - n^2 + 1)p_3 + 2(n^3 - n + 1)q_3 + n^4 - n^2 + 1\}^{-1}. \end{aligned} \tag{3.19}$$

Similarly we substitute the values of x_i, y_i, z_i, p_i, q_i into (3.16) and get

$$\begin{aligned} m = -\{&2n(n-1)p_3^2 + 2(n^3 - n + 1)p_3q_3 + 4n^2(n-1)p_3 \\ &+ (n^4 - n^2 + 1)q_3 + 2n^3(n-1)\}\{4(n^2 - n + 1)p_3q_3 \\ &+ 2(n^3 - n + 1)p_3 + 2(n^3 - n^2 + 1)q_3 + n^4 - n^2 + 1\}^{-1}. \end{aligned} \tag{3.20}$$

On equating the two values of m given by (3.19) and (3.20), we get the condition

$$(3.21)\quad \begin{aligned}&\{8n(n-1)(n^2-n+1)q_3+4n(n-1)(n^3-n^2+1)\}p_3^2\\ &\quad+\{8n(n-1)(n^2-n+1)q_3^2-(4n^6-24n^5+40n^4-32n^3\\ &\qquad+12n^2+4)q_3-2(n^3-n^2+1)(n^4-4n^3+3n^2+1)\}p_3\\ &\quad+4n(n-1)(n^3-n^2+1)q_3^2-2(n^3-n^2+1)\\ &\qquad\times(n^4-4n^3+3n^2+1)q_3-(n^4-2n^3+n^2+1)^2=0.\end{aligned}$$

Now (3.21) is a quadratic equation in p_3, and it will have a rational solution for p_3 if its discriminant, which is a quartic function of q_3, becomes a perfect square. One value of q_3 which makes the discriminant a perfect square is given by

$$(3.22)\quad q_3=(n^4-2n^3+n^2+1)/\{2(n^3-2n^2+n-1)\},$$

and (3.21) can now be solved to get two rational solutions for p_3. One of them leads to a trivial solution of (3.14)–(3.16) but the second solution given by

$$(3.23)\quad p_3=(n^4-4n^3+3n^2+1)/\{2((n-1)n\}$$

leads to the following nontrivial solution:

$$(3.24)\quad \begin{aligned}x_1&=(n^4-4n^3+5n^2-2n+1)(n^4-3n^2+2n-1)(n^4-2n^3+n^2+1),\\ y_1&=(n^4-2n^3+n^2+1)(3n^4-6n^3+3n^2-2n+1)\\ &\quad\times(n^4-4n^3+5n^2-2n+3),\\ z_1&=(n^4-4n^3+3n^2+1)(n^4-2n^3+n^2+1)(n^4-2n^3+n^2-2n-1),\\ x_2&=(n^4-2n^3+n^2+1)^3,\\ y_2&=(n^4-4n^3+3n^2+1)(n^4-3n^2+2n-1)(n^4-4n^3+5n^2-2n+3),\\ z_2&=(n^4-4n^3+5n^2-2n+1)(3n^4-6n^3+3n^2-2n+1)\\ &\quad\times(n^4-2n^3+n^2-2n-1),\\ x_3&=(n^4-4n^3+3n^2+1)(3n^4-6n^3+3n^2-2n+1)(n^4-2n^3+n^2+1),\\ y_3&=(n^4-4n^3+5n^2-2n+1)(n^4-2n^3+n^2+1)\\ &\quad\times(n^4-4n^3+5n^2-2n+3),\\ z_3&=(n^4-2n^3+n^2+1)(n^4-3n^2+2n-1)(n^4-2n^3+n^2-2n-1),\end{aligned}$$

where n is an arbitrary parameter.

As a numerical example, taking $n=3$, we get the following three triads of integers with equal sums of squares and equal products:

$$(28379, 57165, 1073),\quad (50653, 885, 38831),\quad (3811, 7215, 63307).$$

As already observed, the discriminant of (3.21) is a quartic function of q_3, and since we know one value of q_3 that makes the discriminant a perfect square, we can, as in Section 3.1, find infinitely many such values by repeatedly applying Fermat's method. These values yield infinitely many rational solutions of (3.21), and we can thus obtain infinitely many parametric solutions of the system (3.14)–(3.16).

3.3. To obtain another parametric solution of (3.1), we first solve the simultaneous diophantine equations

$$x_1^2 + y_1^2 + z_1^2 = x_2^2 + y_2^2 + z_2^2, \tag{3.25}$$

$$x_1 y_1 z_1 = x_2 y_2 z_2, \tag{3.26}$$

together with an auxiliary equation

$$x_1 + y_1 + h z_1 = x_2 + y_2 + h z_2, \tag{3.27}$$

where h is an arbitrary parameter, and then we use this solution to obtain the simultaneous diophantine chains

$$\begin{aligned} x_1^2 + y_1^2 + z_1^2 &= x_2^2 + y_2^2 + z_2^2 = x_3^2 + y_3^2 + z_3^2, \\ x_1 y_1 z_1 &= x_2 y_2 z_2 = x_3 y_3 z_3, \\ x_1 + y_1 + h z_1 &= x_2 + y_2 + h z_2 = x_3 + y_3 + h z_3. \end{aligned} \tag{3.28}$$

The method is analogous to the one employed in [7] to obtain solutions of

$$\begin{aligned} x_1^2 + y_1^2 + z_1^2 &= x_2^2 + y_2^2 + z_2^2 = x_3^2 + y_3^2 + z_3^2, \\ x_1^3 + y_1^3 + z_1^3 &= x_2^3 + y_2^3 + z_2^3 = x_3^3 + y_3^3 + z_3^3, \\ x_1 + y_1 &= x_2 + y_2 = x_3 + y_3. \end{aligned} \tag{3.29}$$

To obtain a nontrivial solution of (3.25)–(3.27), we write

$$\begin{aligned} x_1 &= (at+1)p, \quad y_1 = (bt+1)q, \quad z_1 = r, \\ x_2 &= (bt+1)p, \quad y_2 = q, \quad z_2 = (at+1)r, \end{aligned} \tag{3.30}$$

where a, b, p, q, r and t are arbitrary nonzero parameters. With these values of $x_i, y_i, z_i, i = 1, 2$, it is seen that (3.26) is identically satisfied, while, on removing the factor t, (3.25) reduces to a linear equation in t, and (3.27) reduces to a linear equation in a, b. We can readily solve these equations, and thus obtain a parametric solution of (3.25)–(3.27) which may be written as $x_i = \alpha_i, y_i = \beta_i, z_i = \gamma_i, i = 1, 2$, where

$$\begin{aligned} \alpha_1 &= (2pq^2 - 2hpqr + h^2pr^2 - pr^2 - 2hq^2r + h^2qr^2 + qr^2)p, \\ \beta_1 &= (2hpq - h^2pr - pr - h^2qr - qr + 2hr^2)qr, \\ \gamma_1 &= (2p^2q - 2hp^2r - 2hpqr + h^2pr^2 + pr^2 + h^2qr^2 - qr^2)r, \\ \alpha_2 &= (2hpq - h^2pr - pr - h^2qr - qr + 2hr^2)pr, \\ \beta_2 &= (2p^2q - 2hp^2r - 2hpqr + h^2pr^2 + pr^2 + h^2qr^2 - qr^2)q, \\ \gamma_2 &= (2pq^2 - 2hpqr + h^2pr^2 - pr^2 - 2hq^2r + h^2qr^2 + qr^2)r. \end{aligned} \tag{3.31}$$

We will solve the simultaneous equations (3.28) by obtaining three distinct solutions of the simultaneous diophantine equations

$$\begin{aligned} X + Y + hZ &= s_1, \\ X^2 + Y^2 + Z^2 &= s_2, \\ XYZ &= s_3, \end{aligned} \tag{3.32}$$

where $s_1 = \alpha_1 + \beta_1 + h\gamma_1$, $s_2 = \alpha_1^2 + \beta_1^2 + \gamma_1^2$, $s_3 = \alpha_1\beta_1\gamma_1$, where $\alpha_1, \beta_1, \gamma_1$ are defined by (3.31), so that (3.32) already has two known solutions $(X, Y, Z) = (\alpha_1, \beta_1, \gamma_1)$ and $(X, Y, Z) = (\alpha_2, \beta_2, \gamma_2)$.

To obtain a third solution of (3.32), we eliminate X and Y from these three equations to get the following cubic equation in Z:

$$(Z - \gamma_1)(Z - \gamma_2)(Z - \gamma_3) = 0, \tag{3.33}$$

where

$$\gamma_3 = 2pq\{2hpq - (h^2 + 1)pr - (h^2 + 1)qr + 2hr^2\}/(h^2 + 1). \tag{3.34}$$

While the two roots $Z = \gamma_1$ and $Z = \gamma_2$ of (3.33) lead to the two known solutions of (3.28), the third root $Z = \gamma_3$ will yield a new solution. Substituting $Z = \gamma_3$ in the first two equations of (3.32) and eliminating Y from these equations, we get

$$\begin{aligned} &2(h^2 + 1)X^2 + \{4(h^2 - 1)p^2q^2 - 4h(h^2 + 1)p^2qr \\ &\quad + 2(h^2 + 1)^2p^2r^2 - 4h(h^2 + 1)pq^2r + 4h^2(h^2 + 3)pqr^2 \\ &\quad - 2h(h^2 + 1)^2pr^3 + 2(h^2 + 1)^2q^2r^2 - 2h(h^2 + 1)^2qr^3\}X \\ &+ (h^2 + 1)^2\{2pq^2 - 2hpqr + (h^2 - 1)pr^2 - 2hq^2r + (h^2 + 1)qr^2\} \\ &\quad \times \{2p^2q - 2hp^2r - 2hpqr + (h^2 + 1)pr^2 + (h^2 - 1)qr^2\}r^2 = 0. \end{aligned} \tag{3.35}$$

This is a quadratic equation in X and it will have two rational solutions if its discriminant is a perfect square. The discriminant is a quartic function of q which may be written as

$$\begin{aligned} &4(2h^2p^2 - 2p^2 - 2h^3pr - 2hpr + h^4r^2 + 2h^2r^2 + r^2)^2q^4 \\ &\quad - 8(4h^5p^4 - 4hp^4 - 4h^6p^3r - 4h^4p^3r + 20h^2p^3r + 4p^3r \\ &\quad - 8h^3p^2r^2 - 8hp^2r^2 + 2h^8pr^3 - 8h^4pr^3 - 8h^2pr^3 - 2pr^3 \\ &\quad - h^9r^4 + 6h^5r^4 + 8h^3r^4 + 3hr^4)q^3r + 4(8h^6p^4 + 12h^4p^4 \\ &\quad - 4p^4 + 16h^3p^3r + 16hp^3r - 10h^8p^2r^2 - 24h^6p^2r^2 \\ &\quad - 24h^4p^2r^2 - 32h^2p^2r^2 - 6p^2r^2 + 6h^9pr^3 + 12h^7pr^3 \\ &\quad + 8h^5pr^3 + 4h^3pr^3 + 2hpr^3 - h^{10}r^4 - 2h^8r^4 + 2h^6r^4 \\ &+ 8h^4r^4 + 7h^2r^4 + 2r^4)q^2r^2 - 8(h^2 + 1)^2(2h^3p^3 + 2hp^3 + 2h^4p^2r \\ &- 4h^2p^2r - 2p^2r - 3h^5pr^2 - hpr^2 + h^6r^3 + h^2r^3 + 2r^3)qpr^3 \\ &+ 4(h^2 + 1)^3(h^2p^2 + p^2 + 2h^3pr - 6hpr - h^4r^2 + h^2r^2 + 2r^2)p^2r^4. \end{aligned} \tag{3.36}$$

Since the coefficient of q^4 in (3.36) is a perfect square, a value of q that makes (3.36) a perfect square can be found by Fermat's method. One such value is

(3.37)
$$q = \{4(h^2+1)^2p^4r - 8h(h^4+2h^2+3)p^3r^2 + 8(h^6+2h^4+2h^2+1)p^2r^3 - 4h^3(h^2+1)^2pr^4 + (h^2-1)(h^2+1)^3r^5\} \times \{16hp^4 - 4(4h^2+4)p^3r + 8(h^2+1)^2pr^3 - 4h(h^2+1)^2r^4\}^{-1}.$$

We can now solve (3.35) and thus get a third solution of (3.32). We now have a solution of the simultaneous diophantine chains (3.28) in terms of three parameters h, p and r. As this solution is cumbersome to write down, we give a solution in terms of the single parameter p obtained by taking $h = 2$, $r = 1$:

(3.38)
$$\begin{aligned}
x_1 &= 2p(428p^5 - 2920p^4 + 8400p^3 - 13000p^2 + 10625p - 3750)\\
&\quad \times (12p^4 - 64p^3 + 168p^2 - 216p + 125),\\
y_1 &= 5(48p^5 - 340p^4 + 1040p^3 - 1680p^2 + 1460p - 535)\\
&\quad \times (100p^4 - 432p^3 + 840p^2 - 800p + 375),\\
z_1 &= 8(2p^2 - 8p + 5)(36p^4 - 248p^3 + 572p^2 - 620p + 225)\\
&\quad \times (2p^2 - 5)(2p^2 - 5p + 5),\\
x_2 &= 40p(48p^5 - 340p^4 + 1040p^3 - 1680p^2 + 1460p - 535)\\
&\quad \times (2p^2 - 5)(2p^2 - 5p + 5),\\
y_2 &= (100p^4 - 432p^3 + 840p^2 - 800p + 375)(2p^2 - 8p + 5)\\
&\quad \times (36p^4 - 248p^3 + 572p^2 - 620p + 225),\\
z_2 &= 2(428p^5 - 2920p^4 + 8400p^3 - 13000p^2 + 10625p - 3750)\\
&\quad \times (12p^4 - 64p^3 + 168p^2 - 216p + 125),\\
x_3 &= -8(428p^5 - 2920p^4 + 8400p^3 - 13000p^2 + 10625p - 3750)\\
&\quad \times (2p^2 - 5)(2p^2 - 5p + 5),\\
y_3 &= -5(12p^4 - 64p^3 + 168p^2 - 216p + 125)(2p^2 - 8p + 5)\\
&\quad \times (36p^4 - 248p^3 + 572p^2 - 620p + 225),\\
z_3 &= 2p(48p^5 - 340p^4 + 1040p^3 - 1680p^2 + 1460p - 535)\\
&\quad \times (100p^4 - 432p^3 + 840p^2 - 800p + 375).
\end{aligned}$$

The discriminant (3.36) is a quartic function of q, and as in the previous two subsections, we can find infinitely many values of q that make it a perfect square, thus obtaining infinitely many solutions of (3.28) in terms of h, p and r.

4. Multigrade chains. In this section we will obtain nontrivial solutions of the simultaneous multigrade chains (1.2).

If we impose the simplifying conditions $c_i = -a_i, d_i = -b_i, i = 1, \ldots, n$, equations (1.2) reduce to the single diophantine chain

$$a_1^2 + b_1^2 = a_2^2 + b_2^2 = \cdots = a_n^2 + b_n^2, \tag{4.1}$$

and it is well-known that solutions of (4.1) can be generated for any value of n [14, pp. 380–381]. Solutions of (1.2) obtained in this manner must be regarded as somewhat trivial.

To obtain nontrivial solutions of (1.2), we write

$$\begin{aligned} a_i &= x_i - y_i - z_i, \quad & b_i &= y_i - z_i - x_i, \\ c_i &= z_i - x_i - y_i, \quad & d_i &= x_i + y_i + z_i, \qquad i = 1, \ldots, n, \end{aligned} \tag{4.2}$$

and denote the sum $a_i^k + b_i^k + c_i^k + d_i^k$ by S_{ik}. It is readily verified that

$$\begin{aligned} S_{i1} &= 0, & S_{i2} &= 4(x_i^2 + y_i^2 + z_i^2), \\ S_{i3} &= 24x_iy_iz_i, \quad & S_{i5} &= 80x_iy_iz_i(x_i^2 + y_i^2 + z_i^2), \end{aligned} \tag{4.3}$$

and it follows from (4.3) that with the values of a_i, b_i, c_i, d_i defined by (4.2), the multigrade chains (1.2) reduce to the simultaneous diophantine chains (1.1).

As we have already obtained nontrivial parametric solutions of (1.1) with $n = 3$ and numerical solutions of (1.1) with $n = 3, 4, 5$, we immediately obtain nontrivial parametric multigrade chains (1.2) with $n = 3$ and numerical multigrade chains (1.2) with $n = 3, 4, 5$.

As a numerical example, the triads listed at No. 1 in Table 1 yield the following multigrade chain:

$$\begin{aligned} 69^r + (-137)^r + (-149)^r + 217^r &= 49^r + (-93)^r + (-179)^r + 223^r \\ &= (-11)^r + (-23)^r + (-197)^r + 231^r, \end{aligned}$$

where $r = 1, 2, 3, 5$.

5. Some open problems. We note that given any arbitrarily large positive integer n, we can find n triads of integers with equal sums and equal products ([6], [13]). Similarly there exist arbitrarily many triads of integers with equal sums of cubes and equal products [5]. It is also interesting to observe that if we write

$$x = 8r^2/(r^4-1), y = (r^2-1)^2/\{(r^2+1)r\}, z = (r^2+1)^2/\{(r^2-1)r\}, \tag{5.1}$$

then $x^2 + y^2 - z^2 = -12$ and $xyz = 8$, and so, by assigning distinct values to r, we immediately obtain infinitely many triads of rational numbers that satisfy the simultaneous diophantine chains

$$\begin{aligned} x_1y_1z_1 = {}& x_2y_2z_2 = \cdots = x_ny_nz_n = \cdots, \\ x_1^2 + y_1^2 - z_1^2 = {}& x_2^2 + y_2^2 - z_2^2 = \cdots = x_n^2 + y_n^2 - z_n^2 = \cdots. \end{aligned} \tag{5.2}$$

By appropriate scaling, we can obtain arbitrarily many triads of integers that satisfy (5.2).

The question naturally arises whether there exist arbitrarily many triads of integers with equal sums of squares and equal products. Our numerical data contains two examples of five triads with this property. We feel that there must exist an upper bound for the number of such triads, and in fact, our conjecture is that this upper bound is at most 6.

It would also be of interest to obtain parametric solutions of (1.1) when $n = 4$ or 5, and the complete solution of these diophantine chains when $n \geq 3$.

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