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CROSSED PRODUCTS OF TOEPLITZ ALGEBRAS AND THE N-ADIC RATIONALS

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Abstract. Let G be a discrete group and G_+ be a generating subsemigroup with identity. If χ is a one-dimensional character of G, then it induces an automorphism on $C^*_r(G)$ and on the Toeplitz algebra T_{G_+} associated to (G, G_+). When (G, G_+) is the naturally ordered group of N-adic rationals, we find complete isomorphism invariants of $T_{G_+} \rtimes_{\alpha_\chi} \mathbb{Z}$.

1. Introduction. Let G be a discrete group and G_+ be a semigroup of G that contains the identity element. If $\{f_g \mid g \in G\}$ is the canonical orthonormal basis of the Hilbert space $l^2(G)$, then $l^2(G_+)$ can be viewed as the closed subspace of $l^2(G)$ with basis given by $\{f_g \mid g \in G_+\}$. One may construct the *left regular representation* U of G on $l^2(G)$ as follows: $U_g(f_h) = f_{g^{-1}h}$ for all g and h in G. Then the *Toeplitz operators* T_g are obtained by compressing U_g to the subspace $l^2(G_+)$, i.e., $T_g = PU_g$, where P is the orthogonal projection from $l^2(G)$ onto the subspace $l^2(G_+)$. The *Toeplitz C^*-algebra* associated to G_+ is the C^*-subalgebra of $B(l^2(G))$ generated by $\{T_g \mid g \in G_+\}$, and it is denoted by T_{G_+}.

When G is abelian and G_+ generates G, there is a short exact sequence (see [1])

$$0 \to [T_{G_+}, T_{G_+}] \to T_{G_+} \to C^*_r(G) \to 0 \tag{1.1}$$

where $[T_{G_+}, T_{G_+}]$ denotes the commutator ideal of T_{G_+}, and $C^*_r(G)$ the reduced group C^*-algebra, i.e., the C^*-subalgebra of $B(l^2(G))$ generated by the group $\{U_g \mid g \in G\}$.

Handelman and Yin [2] noticed that every character χ of G gives rise to a unitary operator W_χ on $l^2(G)$, namely $W_\chi(f_g) = \chi(g) f_g$ for all $g \in G$. Moreover, it was noted that W_χ restricts to a unitary on $l^2(G_+)$ (since the subspace $l^2(G_+)$ is invariant under W_χ and its adjoint) that will still be denoted by W_χ. The $*$-automorphism $\alpha_\chi := \operatorname{Ad} W_\chi$ of T_{G_+} induces the

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action $\mathbb{Z} \ni n \mapsto \alpha_\chi^n \in \mathrm{Aut}(T_{G_+})$, and thus one may form the crossed product $T_{G_+} \rtimes_{\alpha_\chi} \mathbb{Z}$. The main results in [2] assert that

$$0 \to [T_{G_+}, T_{G_+}] \rtimes_{\alpha_\chi} \mathbb{Z} \to T_{G_+} \rtimes_{\alpha_\chi} \mathbb{Z} \to C_r^*(G) \rtimes_{\alpha_\chi} \mathbb{Z} \to 0 \tag{1.2}$$

is an exact sequence, and that the following three objects are isomorphism invariants of $T_{G_+} \rtimes_{\alpha_\chi} \mathbb{Z}$:

- $C_r^*(G) \rtimes_{\alpha_\chi} \mathbb{Z}$,
- $\chi(G)$,
- $[T_{G_+}, T_{G_+}] \rtimes_{\alpha_\chi} \mathbb{Z}$.

The aim of this paper is to find complete isomorphism invariants when G is the group of N-adic rationals ($\mathbb{N} \ni N > 1$), i.e., $G = \{p/N^k \mid p \in \mathbb{Z}, k \in \mathbb{N}\}$, and $G_+ = \{x \in G \mid x \geq 0\}$. Complete isomorphism invariants for $(\mathbb{Z}, \mathbb{Z}_+)$ were found in [2]. In this case $G = \mathbb{Z}$ and $G_+ = \{x \in \mathbb{Z} \mid x \geq 0\}$, and $C_r^*(G) \rtimes_{\alpha_\chi} \mathbb{Z}$ is the rotation algebra A_θ, where $\chi(1) = e^{2\pi i\theta}$ for some $0 \leq \theta < 1$. Thus (see [2]) if χ, η are characters of $\mathbb{Z}$ (with $\eta(1) = e^{2\pi i\theta'}$), then the crossed products are isomorphic iff either $\theta = \theta'$ or $\theta = 1 - \theta'$.

If $g_n = 1/N^n$ ($n \in \mathbb{N}$) are the canonical generators of the group of N-adic rationals, it turns out that the crossed product $C_r^*(G) \rtimes_{\alpha_\chi} \mathbb{Z}$ is an inductive limit of the rotation algebras A_{θ_n}, where $e^{2\pi i\theta_n} = \chi(g_n)$, for some $0 \leq \theta_n < 1$. The K_0-group of each A_{θ_n} is $\mathbb{Z}^2$, and it is generated by the elements $[1]_0$ and $[p_n]_0$, where p_n is either a Rieffel projection ($0 < \theta_n < 1$) or the Bott projection ($\theta_n = 0$). Based on this description as an inductive limit we compute in Section 2 the group $K_0(C_r^*(G) \rtimes_{\alpha_\chi} \mathbb{Z})$.

If τ_n denotes the canonical trace on A_{θ_n}, the main result (Thm. 2.2) and its proof assert that the canonical sequence $\{K_0(\tau_n)([p_n]_0), n \in \mathbb{N}\}$ is an invariant for the crossed product $T_{G_+} \rtimes_{\alpha_\chi} \mathbb{Z}$, as well as is $C_r^*(G) \rtimes_{\alpha_\chi} \mathbb{Z}$. In fact, if η is another character of G, we have $T_{G_+} \rtimes_{\alpha_\chi} \mathbb{Z} \approx T_{G_+} \rtimes_{\alpha_\eta} \mathbb{Z}$ iff $C_r^*(G) \rtimes_{\alpha_\chi} \mathbb{Z} \approx C_r^*(G) \rtimes_{\alpha_\eta} \mathbb{Z}$ iff one canonical sequence is congruent modulo $\mathbb{Z}$ to a multiple of a subsequence of the other canonical sequence. The proof of Thm. 2.2 relies on the shape of the group automorphisms of G, on the description of the K_0-group of $C_r^*(G) \rtimes_{\alpha_\chi} \mathbb{Z}$ and on the description of the isomorphism (implemented from the C^*-algebra level) between $K_0(C_r^*(G) \rtimes_{\alpha_\chi} \mathbb{Z})$ and $K_0(C_r^*(G) \rtimes_{\alpha_\eta} \mathbb{Z})$. If N is a prime number, then the modulo $\mathbb{Z}$ congruence condition of Thm. 2.2 becomes an equality (see Cor. 2.3).

2. Crossed products. It is known that the dual group of G is given by $\{(z_n)_{n\in\mathbb{N}} \in \mathbb{T}^{\mathbb{N}} \mid z_{n+1}^N = z_n \text{ for all } n \in \mathbb{N}\}$, endowed with the induced topology from its injection into $\mathbb{T}^{\mathbb{N}}$. The pairing between G and its dual is given by $\langle p/N^k, (z_n)_{n\in\mathbb{N}}\rangle = z_k^p$.

Let $\chi = (z_n)_{n\in\mathbb{N}}$ be given. Define $g_n = 1/N^n$ for all $n \in \mathbb{N}$, and note that $U_{g_n} = U_{g_{n+1}}^N$, $\chi(g_n) = \chi(g_{n+1})^N$ for all $n \in \mathbb{N}$. Since G is generated by all g_n with $n \in \mathbb{N}$, the following lemma is trivial.

LEMMA 2.1. *The inductive limit of the sequence*

$$C^*(U_{g_0}) \rtimes_{\alpha_\chi} \mathbb{Z} \xrightarrow{i_0} C^*(U_{g_1}) \rtimes_{\alpha_\chi} \mathbb{Z} \xrightarrow{i_1} C^*(U_{g_2}) \rtimes_{\alpha_\chi} \mathbb{Z} \to \cdots$$

is $C_r^*(G) \rtimes_{\alpha_\chi} \mathbb{Z}$, where each i_n is the inclusion.

Since each g_n has infinite order in G, it follows that $C^*(U_{g_n}) \rtimes_{\alpha_\chi} \mathbb{Z} \approx A_{\theta_n}$, the rotation algebra, where $e^{2\pi i\theta_n} = \chi(g_n)$, $0 \le \theta_n < 1$, and $C^*(U_{g_n})$ is the C^*-sublagebra of $C_r^*(G)$ generated by U_{g_n}. The K_0-theory of A_{θ_n} is $\mathbb{Z}^2$, and it is generated by $[1]_0$ and $[p_n]_0$, where p_n is either a Rieffel projection $(0 < \theta_n < 1)$ or the Bott projection $(\theta_n = 0)$. The canonical unitaries in A_{θ_n} will be denoted by u_n and v_n, and the canonical trace will be denoted by τ_n.

If p_n is a Rieffel projection, then $p_n = f(u_n)v_n^* + g(u_n) + v_n f(u_n)$, where $f, g : \mathbb{T} \to \mathbb{R}$ are certain continuous functions (see [4, Exercise 5.8]), and $\tau(p_n) = \theta_n = \int_{\mathbb{T}} g(z)\,dz$. If p_n is the Bott projection, then $K_0(\tau_n)([p_n]_0) = 0$.

The K_0-theories of the inclusions in Lemma 2.1 are obtained from the commutative diagram

$$\begin{array}{ccc} K_0(A_{\theta_n}) & \xrightarrow{\delta_n} & K_1(C^*(U_{g_n})) \\ \downarrow & & \downarrow \\ K_0(A_{\theta_{n+1}}) & \xrightarrow{\delta_{n+1}} & K_1(C^*(U_{g_{n+1}})) \end{array}$$

where $\delta_n([1]_0) = 0$ and $\delta_n([p_n]_0) = [U_{g_n}]_1$. Assume that $K_0(i_n)([p_n]_0) = a_n[1]_0 + b_n[p_{n+1}]_0$ for some $a_n, b_n \in \mathbb{Z}$. From the definition of δ_n and from $U_{g_n} = U_{g_{n+1}}^N$ one sees that the right vertical map is multiplication by N. It follows that $b_n = N$, and thus $K_0(i_n)([p_n]_0) = a_n[1]_0 + N[p_{n+1}]_0$ for some $a_n \in \mathbb{Z}$. Taking into account the trace one obtains

$$a_n = K_0(\tau_n)([p_n]_0) - NK_0(\tau_{n+1})([p_{n+1}]_0).$$

Hence one has $K_0(C_r^*(G) \rtimes_{\alpha_\chi} \mathbb{Z}) = \varinjlim \mathbb{Z}^2 \xrightarrow{M_0} \mathbb{Z}^2 \xrightarrow{M_1} \mathbb{Z}^2 \to \cdots$ where $M_n = \left[\begin{smallmatrix} 1 & a_n \\ 0 & N \end{smallmatrix}\right]$ for all $n \ge 0$. Define

$$H_\chi = \left\{ \begin{bmatrix} a - (b/N^k)(a_0 + a_1 N + a_2 N^2 + \cdots + a_{k-1}N^{k-1}) \\ b/N^k \end{bmatrix} \;\middle|\; a, b \in \mathbb{Z},\ k \ge 1 \right\}$$

$$\cup \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \;\middle|\; a, b \in \mathbb{Z} \right\},$$

and note that it is a group with the usual addition. Next define $v_0 : \mathbb{Z}^2 \to H_\chi$ by $v_0\big(\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]\big) = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, and for $k \geq 1$ define $v_k : \mathbb{Z}^2 \to H_\chi$ by

$$v_k = \begin{bmatrix} 1 & a_0 \\ 0 & N \end{bmatrix}^{-1} \begin{bmatrix} 1 & a_1 \\ 0 & N \end{bmatrix}^{-1} \cdots \begin{bmatrix} 1 & a_{k-1} \\ 0 & N \end{bmatrix}^{-1}.$$

Since the diagram

$$\begin{array}{ccccccc} \mathbb{Z}^2 & \xrightarrow{M_0} & \mathbb{Z}^2 & \xrightarrow{M_1} & \mathbb{Z}^2 & \longrightarrow & \cdots \\ {\scriptstyle v_0}\big\downarrow & & {\scriptstyle v_1}\big\downarrow & & {\scriptstyle v_2}\big\downarrow & & \\ H_\chi & \xrightarrow{=} & H_\chi & \xrightarrow{=} & H_\chi & \longrightarrow & \cdots \end{array}$$

is commutative, it follows that $K_0(C_r^*(G) \rtimes_{\alpha_\chi} \mathbb{Z}) = H_\chi$.

Based on the fact that all tracial states of rotation algebras agree on K-theory, we determine the K-theory of any trace on the crossed product. Let τ be any trace on $C_r^*(G) \rtimes_{\alpha_\chi} \mathbb{Z}$, and note that all v_k take the first generator $(1, 0) \in \mathbb{Z}^2$ to $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \in H_\chi$, hence $K_0(\tau)\big(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\big) = 1$. Since v_0 takes the generator $(0, 1) \in \mathbb{Z}^2$ to $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \in H_\chi$ and since for $k \geq 1$ the map v_k takes the generator $(0, 1) \in \mathbb{Z}^2$ to

$$\begin{bmatrix} -(1/N^k)(a_0 + a_1 N + a_2 N^2 + \cdots + a_{k-1} N^{k-1}) \\ 1/N^k \end{bmatrix} \in H_\chi,$$

it follows that

$$K_0(\tau)\left(\begin{bmatrix} -(1/N^k)(a_0 + a_1 N + a_2 N^2 + \cdots + a_{k-1} N^{k-1}) \\ 1/N^k \end{bmatrix}\right) = \tau(p_k)$$

for $k \geq 1$, and $K_0(\tau)\big(\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\big) = 1$.

We shall denote by E_k the expression $a_0 + a_1 N + a_2 N^2 + \cdots + a_{k-1} N^{k-1}$ that defines H_χ, and let $\tilde{p}_k, \tilde{\tau}_k, \tilde{E}_k$ be the corresponding projections, traces and expressions arising from another character η of G. The proof of the following result is suggested by [3, Theorem 4.2].

THEOREM 2.2. *If* $\chi, \eta \in \hat{G}$, *then* $T_{G_+} \rtimes_{\alpha_\chi} \mathbb{Z} \approx T_{G_+} \rtimes_{\alpha_\eta} \mathbb{Z}$ *if and only if the following subsequence conditions hold:*

(i) *there exist* $k \in \mathbb{N}$ *and* $q \in \mathbb{Z}$, $q \,|\, N$, $q \notin \{N, -N\}$ *such that for all* $t \in \mathbb{N}$,
$$K_0(\tau_t)([p_t]_0) \equiv q K_0(\tilde{\tau}_{k+t})([\tilde{p}_{k+t}]_0) \bmod \mathbb{Z},$$

(ii) *there exist* $l \in \mathbb{N}$ *and* $r \in \mathbb{Z}$, $r \,|\, N$, $r \notin \{N, -N\}$ *such that for all* $t \in \mathbb{N}$,
$$K_0(\tilde{\tau}_t)([\tilde{p}_t]_0) \equiv r K_0(\tau_{l+t})([p_{l+t}]_0) \bmod \mathbb{Z}.$$

Proof. By the work of Handelman and Yin (see the introduction) one may assume that there is a group isomorphism $f : H_\chi \to H_\eta$. Since the $*$-isomorphism is unital, it follows that $f\big(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\big) = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$.

To a character η of G one may associate the surjective group morphism $\pi_\eta : H_\eta \to G$ given by

$$\pi_\eta\left(\begin{bmatrix} a - (b/N^k)E_k \\ b/N^k \end{bmatrix}\right) = \frac{b}{N^k}$$

when $k \geq 1$, and $\pi_\eta\big(\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]\big) = b$ when $a, b \in \mathbb{Z}$. Its kernel is $\mathrm{Ker}(\pi_\eta) = \big\{\left[\begin{smallmatrix}a\\0\end{smallmatrix}\right] \bigm| a \in \mathbb{Z}\big\}$, and if $\sigma : G \to G$ is the group morphism given by

$$\sigma\left(\frac{b}{N^k}\right) = \pi_\eta \circ f\left(\begin{bmatrix} -(b/N^k)E_k \\ b/N^k \end{bmatrix}\right),$$

then the following diagram is commutative:

$$\begin{array}{ccc} H_\chi & \xrightarrow{\ f\ } & H_\eta \\ {\scriptstyle \pi_\chi}\downarrow & & {\scriptstyle \pi_\eta}\downarrow \\ G & \xrightarrow{\ \sigma\ } & G \end{array}$$

Since f, π_η, π_χ are onto, so is σ. If $\sigma(b/N^k) = 0$, then

$$\pi_\eta \circ f\left(\begin{bmatrix} -(b/N^k)E_k \\ b/N^k \end{bmatrix}\right) = 0,$$

or in other words,

$$f\left(\begin{bmatrix} -(b/N^k)E_k \\ b/N^k \end{bmatrix}\right) \in \mathrm{Ker}(\pi_\eta) = \begin{bmatrix} a \\ 0 \end{bmatrix} = f\left(\begin{bmatrix} a \\ 0 \end{bmatrix}\right)$$

for some $a \in \mathbb{Z}$. Since f is 1-1, it follows that $b = 0$, thus σ is a group isomorphism.

By [3, Lemma 4.1] one has $\sigma(1) = q/N^k$ for some $k \in \mathbb{N}$, and some $q \in \mathbb{Z}$ with $q \mid N$, $q \notin \{-N, N\}$. In fact $\sigma(1/N^n) = q/N^{k+n}$ for all $n \in \mathbb{N}$.

For $t \geq 1$ suppose that

$$f\left(\begin{bmatrix} -(1/N^t)E_t \\ 1/N^t \end{bmatrix}\right) = \begin{bmatrix} x_t - (y_t/N^m)\tilde{E}_m \\ y_t/N^m \end{bmatrix}$$

for some $x_t, y_t \in \mathbb{Z}$ and some $m \geq 1$. Using the trace one deduces that

$K_0(\tau_t)([p_t]_0) = x_t + y_t K_0(\tilde{\tau}_t)([\tilde{p}_m]_0)$. Composing with π_η we get

$$\pi_\eta \circ f\left(\begin{bmatrix} -(1/N^t)E_t \\ 1/N^t \end{bmatrix}\right) = \pi_\eta\left(\begin{bmatrix} x_t - (y_t/N^m)\tilde{E}_m \\ y_t/N^m \end{bmatrix}\right),$$

$$\sigma \circ \pi_\chi\left(\begin{bmatrix} -(1/N^t)E_t \\ 1/N^t \end{bmatrix}\right) = \frac{y_t}{N^m}$$

and $\sigma(1/N^t) = y_t/N^m$.

We observe that $q/N^{k+t} = y_t/N^m$, so $q = y_t N^{k+t}/N^m$. If $k+t > m$, then $N \mid q$. From $q \mid N$ one gets $q \in \{-N, N\}$, a contradiction. Hence $k + t \le m$. The last inequality tells us that for all $t \ge 1$ one has

$$f\left(\begin{bmatrix} -(1/N^t)E_t \\ 1/N^t \end{bmatrix}\right) = \begin{bmatrix} x_t + x'_t - (q/N^{k+t})\tilde{E}_{k+t} \\ q/N^{k+t} \end{bmatrix}$$

for some $x'_t \in \mathbb{Z}$, hence finally $K_0(\tau_t)([p_t]_0) = x_t + x'_t + qK_0(\tilde{\tau}_{k+t})([\tilde{p}_{k+t}]_0) \equiv qK_0(\tilde{\tau}_{k+t})([\tilde{p}_{k+t}]_0) \bmod \mathbb{Z}$.

Considering f^{-1} instead of f one obtains the other subsequence condition.

To prove the "if" part it is enough to observe that

$$e^{2\pi i\theta_n} = e^{2\pi i(x_n + x'_n + p\tilde{\theta}_{k+n})} = e^{2\pi i p\tilde{\theta}_{k+n}}$$

and

$$\begin{aligned}\eta(1/N^n) &= e^{2\pi i\tilde{\theta}_n} = e^{2\pi i N^k \tilde{\theta}_{n+k}} \\ &= e^{(2\pi i p\tilde{\theta}_{n+k})N^k/p} = (e^{2\pi i\theta_n})^{N^k/p} = (\chi(1/N^n))^{N^k/p}.\end{aligned}$$

From this it follows that the crossed products are isomorphic: $T_{G_+} \rtimes_{\alpha_\chi} \mathbb{Z} \approx T_{G_+} \rtimes_{\alpha_\eta} \mathbb{Z}$ and $C^*_r(G) \rtimes_{\alpha_\chi} \mathbb{Z} \approx C^*_r(G) \rtimes_{\alpha_\eta} \mathbb{Z}$. ∎

If N is a prime number then the integers q, r in the above theorem are ± 1. Since for all n, s one has $0 \le \theta_n, \tilde{\theta}_s < 1$ and $x_t + x'_t \in \mathbb{Z}$, it follows that in the above theorem either $K_0(\tau_t)([p_t]_0) = K_0(\tilde{\tau}_{k+t})([\tilde{p}_{k+t}]_0)$ or $K_0(\tau_t)([p_t]_0) = 1 - K_0(\tilde{\tau}_{k+t})([\tilde{p}_{k+t}]_0)$. This has the following consequence.

Corollary 2.3. *Let N be a prime number and let $\chi, \eta \in \hat{G}$. Then $T_{G_+} \rtimes_{\alpha_\chi} \mathbb{Z} \approx T_{G_+} \rtimes_{\alpha_\eta} \mathbb{Z}$ if and only if:*

(i) *there exists $k \in \mathbb{N}$ such that for all $t \in \mathbb{N}$, either $K_0(\tau_t)([p_t]_0) = K_0(\tilde{\tau}_{k+t})([\tilde{p}_{k+t}]_0)$ or $K_0(\tau_t)([p_t]_0) = 1 - K_0(\tilde{\tau}_{k+t})([\tilde{p}_{k+t}]_0)$,*

(ii) *there exists $l \in \mathbb{N}$ such that for all $t \in \mathbb{N}$, either $K_0(\tilde{\tau}_t)([\tilde{p}_t]_0) = K_0(\tau_{l+t})([p_{l+t}]_0)$ or $K_0(\tilde{\tau}_t)([\tilde{p}_t]_0) = 1 - K_0(\tau_{l+t})([p_{l+t}]_0)$.*

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