STUDIA MATHEMATICA 237 (3) (2017)

On the set of limit points of conditionally convergent series

by

SZYMON GŁĄB and JACEK MARCHWICKI (Łódź)

Abstract. Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series in a Banach space and let τ be a permutation of the natural numbers. We study the set $\mathrm{LIM}(\sum_{n=1}^{\infty} x_{\tau(n)})$ of all limit points of the sequence $(\sum_{n=1}^{p} x_{\tau(n)})_{p=1}^{\infty}$ of partial sums of the rearranged series $\sum_{n=1}^{\infty} x_{\tau(n)}$. We give a full characterization of such limit sets in finite-dimensional spaces. Namely, every such limit set in $\mathbb{R}^m$ is either compact and connected, or closed with all connected components unbounded. On the other hand, each set of one of these types is the limit set of some rearranged conditionally convergent series. Moreover, this characterization does not hold in infinite-dimensional spaces.

We show that if $\sum_{n=1}^{\infty} x_n$ has the Rearrangement Property and A is a closed subset of the closure of the sum range of $\sum_{n=1}^{\infty} x_n$ and it is ε-chainable for every $\varepsilon > 0$, then there is a permutation τ such that $A = \mathrm{LIM}(\sum_{n=1}^{\infty} x_{\tau(n)})$.

1. Introduction. Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series of real numbers. For any $a < b$ one can find a permutation $\sigma \in S_\infty$ of the natural numbers such that the sequence $(\sum_{n=1}^{k} x_{\sigma(n)})_{k=1}^{\infty}$ of partial sums of the rearrangement $\sum_{n=1}^{\infty} x_{\sigma(n)}$ has lower limit a and upper limit b. Consequently, a and b are limit points of the sequence $(\sum_{n=1}^{k} x_{\sigma(n)})_{k=1}^{\infty}$. Since $|x_{\sigma(n)}|$ tends to zero, the whole interval $[a, b]$ consists of such limit points. This simple observation shows that the set of all limit points of $(\sum_{n=1}^{k} x_{\sigma(n)})_{k=1}^{\infty}$ is closed and connected, and for any closed connected subset I of $\mathbb{R}$ and any conditionally convergent series $\sum_{n=1}^{\infty} x_n$ one can find a rearrangement $\sum_{n=1}^{\infty} x_{\sigma(n)}$ such that the set of all limit points of its partial sums equals I. If $\sum_{n=1}^{\infty} x_{\sigma(n)}$ diverges to ∞ or to $-\infty$, then the set of its limit points is empty.

The situation becomes more complicated for conditionally convergent series in multidimensional Euclidean spaces. One could expect that the limit sets of all rearrangements are connected or even arcwise connected. It turns

2010 *Mathematics Subject Classification*: Primary 40A05; Secondary 46B15, 46B20.
Key words and phrases: sum range, Steinitz Theorem, set of limit points, conditionally convergent series, series in Banach spaces.
Received 15 December 2015; revised 19 July 2016.
Published online 17 February 2017.

DOI: 10.4064/sm8480-10-2016

out that this is not the case. However, some result concerning connectedness can be proved for multidimensional spaces (see Theorem 3.5).

Let $\sum_{n=1}^\infty x_n$ be a series in a Banach space X. Denote by $\mathrm{LIM}(\sum_{n=1}^\infty x_n)$ the set of all limit points of the sequence of partial sums of the series, let

$$\mathrm{LPS}\Big(\sum_{n=1}^\infty x_n\Big) = \bigcup_{\sigma\in S_\infty} \mathrm{LIM}\Big(\sum_{n=1}^\infty x_{\sigma(n)}\Big)$$

where S_∞ stands for the collection of all bijective maps $\sigma : \mathbb{N} \to \mathbb{N}$, and denote by $\mathrm{SR}(\sum_{n=1}^\infty x_n)$ the *sum range* of the series, that is, the set of sums of all convergent rearrangements of the series. Let us also introduce the following notation. Denote by $\mathbf{\Sigma}$ the collection of all series in X, by $\mathbf{\Sigma}_0$ the collection of those $\sum_{n=1}^\infty x_n \in \mathbf{\Sigma}$ with $\lim_{n\to\infty} x_n = 0$, and by $\mathbf{\Sigma}_c$ the collection of those $\sum_{n=1}^\infty x_n \in \mathbf{\Sigma}$ that have a convergent rearrangement, i.e. with $\mathrm{SR}(\sum_{n=1}^\infty x_n) \neq \emptyset$.

It is easy to see that $\mathrm{LIM}(\sum_{n=1}^\infty x_n)$ is a closed separable set, and that for every closed separable set $A \subseteq X$ there is a series $\sum_{n=1}^\infty x_n \in \mathbf{\Sigma}$ with $\mathrm{LIM}(\sum_{n=1}^\infty x_n) = A$.

Now, let $\sum_{n=1}^\infty x_n \in \mathbf{\Sigma}_c(\mathbb{R}^m)$. By the Steinitz Theorem the sum range $\mathrm{SR}(\sum_{n=1}^\infty x_n)$ is an affine subspace of $\mathbb{R}^n$. The limit sets of series were studied by Victor Klee [3], who claimed that if $A = \mathrm{LIM}(\sum_{n=1}^\infty x_n)$, then for every $\varepsilon > 0$ the ε-shell $A(\varepsilon) = \{x : \|x - y\| < \varepsilon \text{ for some } y \in A\}$ of A is connected. Our Example 2.2 shows that this is not true.

A metric space (Y, ρ) is *ε-chainable* if any points $a, b \in Y$ can be joined by a path $x_0, x_1, \dots, x_k \in Y$ such that $x_0 = a$, $x_k = b$ and $\rho(x_i, x_{i-1}) < \varepsilon$. Each connected metric space is ε-chainable for every $\varepsilon > 0$ [1, 6.1.D(a)]; moreover, if (Y, ρ) is compact and ε-chainable for every $\varepsilon > 0$, then Y is connected, the compactness assumption being essential [1, 6.1.D(b)]. Klee also proved that if $\sum_{n=1}^\infty x_n \in \mathbf{\Sigma}_c(\mathbb{R}^m)$ and $A \subseteq \mathrm{SR}(\sum_{n=1}^\infty x_n)$ is closed and ε-chainable for every $\varepsilon > 0$, then there is $\sigma \in S_\infty$ such that $A = \mathrm{LIM}(\sum_{n=1}^\infty x_{\sigma(n)})$.

In this article we complete Klee's result by giving a full characterization of limit sets $\mathrm{LIM}(\sum_{n=1}^\infty x_n)$ for $\sum_{n=1}^\infty x_n \in \mathbf{\Sigma}_c(\mathbb{R}^m)$. Namely we prove the following dichotomy (Theorem 3.5): the limit set is either compact and connected, or any of its connected components is unbounded; moreover, the closure of the limit set in the one-point compactification of $\mathbb{R}^m$ is connected. The proof uses the fact that the underlying space has finite dimension. Moreover, this dichotomy does not hold for all Banach spaces. More precisely, in every infinite-dimensional Banach space we construct an example of a conditionally convergent series such that the limit set of some of its rearrangements consists of two points.

Theorem 3.5 cannot be reversed in the sense that there is an unbounded, closed set in the one-dimensional Euclidean space $\mathbb{R}$ each of whose connected components is unbounded but it cannot be a limit set. Namely, set $X := (-\infty,-1] \cup [1,\infty)$. As mentioned at the beginning, any limit set on the real line must be connected, and therefore X is not a limit set. However, Theorem 3.5 can be reversed in higher dimensions. This means that any compact connected set (or even any closed ε-chainable set for every $\varepsilon > 0$) in $\mathbb{R}^m$, $m \geq 1$, and any closed set in $\mathbb{R}^m$, $m \geq 2$, all of whose components are unbounded, is the limit set of some rearrangement of a conditionally convergent series.

In the last section we show that if $\sum_{n=1}^{\infty} x_n$ is a series in an arbitrary Banach space such that $\sum_{n=1}^{\infty} x_n$ has the Rearrangement Property, and $A \subseteq \overline{\mathrm{SR}(\sum_{n=1}^{\infty} x_n)}$ is closed and ε-chainable for every $\varepsilon > 0$, then there is $\tau \in S_\infty$ such that $A = \mathrm{LIM}(\sum_{n=1}^{\infty} x_{\tau(n)})$.

2. Counterexample to Klee's claim. As mentioned in the Introduction, Victor Klee [3] claimed that if $A = \mathrm{LIM}(\sum_{n=1}^{\infty} x_n)$ for some $\sum_{n=1}^{\infty} x_n \in \mathbf{\Sigma}_c$, then its ε-shell $A(\varepsilon)$ is connected for every $\varepsilon > 0$. This is equivalent to saying that A is ε-chainable for every $\varepsilon > 0$. Klee used different notation, but the gap in his argument can be translated into our language as follows. He argued that it was "evident" that $\mathrm{LIM}(\sum_{n=1}^{\infty} x_n)$ cannot consist of two sets X and Y having disjoint ε-shells $X(\varepsilon)$ and $Y(\varepsilon)$. However, the following example shows that this is not true.

According to [4], in finite-dimensional spaces every series $\sum_{n=1}^{\infty} x_n \in \mathbf{\Sigma}_0$ with $\mathrm{LIM}(\sum_{n=1}^{\infty} x_n) \neq \emptyset$ has a convergent rearrangement. Therefore the problem of characterizing those $A \subseteq \mathbb{R}^m$ with $\mathrm{LIM}(\sum_{n=1}^{\infty} x_n) = A$ for some $\sum_{n=1}^{\infty} x_n \in \mathbf{\Sigma}_c$ is equivalent to characterizing those $A \subseteq \mathbb{R}^m$ with $\mathrm{LIM}(\sum_{n=1}^{\infty} x_n) = A$ for some $\sum_{n=1}^{\infty} x_n \in \mathbf{\Sigma}_0$. This permits us not to care about the existence of a convergent rearrangement, but only about the condition $\lim_{n\to\infty} x_n = 0$.

For natural numbers $n < m$ we denote by $[n,m]$ the discrete interval $\{n, n+1, n+2, \ldots, m\}$, and by $[n,\infty)$ the set $\{n, n+1, \ldots\}$. Let $\sum_{n=1}^{\infty} x_n \in \mathbf{\Sigma}_0$. The partial sums sequence (s_n), $s_n = \sum_{k=1}^{n} x_k$, will be called a *walk*. Note that $a \in \mathrm{LIM}(\sum_{n=1}^{\infty} x_n)$ if for every $\varepsilon > 0$ the walk (s_n) hits the ball $B(a,\varepsilon)$. If $(s_n)_{n=1}^{\infty}$ is a sequence in $\mathbb{R}^m$, then we call it a walk if some rearrangement of the series $\sum_{n=1}^{\infty}(s_{n+1} - s_n)$ is convergent. A sequence (s_n) of elements of a set X is called an X-*walk* if

(i) the set $\{s_n : n \in \mathbb{N}\}$ is dense in X;
(ii) $\|s_{n+1} - s_n\| \to 0$.

Observe that conditions (i) and (ii) imply that X is ε-chainable for every $\varepsilon > 0$.

PROPOSITION 2.1. *Let $X \subseteq \mathbb{R}^m$ be dense-in-itself and let (s_n) be an X-walk. Then $\sum_{n=1}^{\infty} x_n := \sum_{n=1}^{\infty}(s_n - s_{n-1}) \in \mathbf{\Sigma}_c$, where $s_0 = 0$, and $\mathrm{LIM}(\sum_{k=1}^{\infty} x_k) = \overline{X}$.*

Proof. From (ii) we get $\sum_{n=1}^{\infty} x_n \in \mathbf{\Sigma}_0$. Fix $x \in X$. From (i) we know that $x = \lim_{n\to\infty} s_{m_n}$ for some increasing sequence $\{m_n\}_{n=1}^{\infty}$ of natural numbers. Hence $x \in \mathrm{LIM}(\sum_{n=1}^{\infty} x_n)$, so $X \subseteq \mathrm{LIM}(\sum_{n=1}^{\infty} x_n)$. Since $\mathrm{LIM}(\sum_{n=1}^{\infty} x_n)$ is closed, we have $\overline{X} \subseteq \mathrm{LIM}(\sum_{n=1}^{\infty} x_n)$. We also know that each element of (s_n) is in X, so $\mathrm{LIM}(\sum_{k=1}^{\infty} x_k) \subseteq \overline{X}$. Finally, by the above mentioned paper [4] of Rosenthal we have $\sum_{n=1}^{\infty} x_n \in \mathbf{\Sigma}_c$. ∎

EXAMPLE 2.2. We define a series $\sum_{k=1}^{\infty} y_k \in \mathbf{\Sigma}(\mathbb{R}^2)$.

STEP 1. The first two y_n's are $(1/2,0),(1/2,0)$.

STEP 2. We define the next $1\cdot 4 + 4 + 1\cdot 4$ elements:

$$(0,1/4),\ldots,(0,1/4),(-1/4,0),\ldots,(-1/4,0),(0,-1/4),\ldots,(0,-1/4).$$

STEP $2k+1$. In this step we define $2k\cdot 2^{2k+1} + 2^{2k+1} + 2k\cdot 2^{2k+1}$ elements:

$$\underbrace{(0,2^{-2k-1}),\ldots,(0,2^{-2k-1})}_{2k\cdot 2^{2k+1}},\underbrace{(2^{-2k-1},0),\ldots,(2^{-2k-1},0)}_{2^{2k+1}},$$
$$\underbrace{(0,-2^{-2k-1}),\ldots,(0,-2^{-2k-1})}_{2k\cdot 2^{2k+1}}.$$

STEP $2k+2$. In this step we define $(2k+1)\cdot 2^{2k+2} + 2^{2k+2} + (2k+1)\cdot 2^{2k+2}$ elements:

$$\underbrace{(0,2^{-2k-2}),\ldots,(0,2^{-2k-2})}_{(2k+1)\cdot 2^{2k+2}},\underbrace{(-2^{-2k-2},0),\ldots,(-2^{-2k-2},0)}_{2^{2k+2}},$$
$$\underbrace{(0,-2^{-2k-2}),\ldots,(0,-2^{-2k-2})}_{(2k+1)\cdot 2^{2k+2}}.$$

Since $\sum_{k=1}^{\infty} y_k \in \mathbf{\Sigma}_0(\mathbb{R}^2)$ and $(0,0) \in \mathrm{LIM}(\sum_{k=1}^{\infty} y_k)$, we have $\sum_{k=1}^{\infty} y_k \in \mathbf{\Sigma}_c(\mathbb{R}^2)$. Note that $\mathrm{LIM}(\sum_{n=1}^{\infty} y_n)$ equals $\{0,1\} \times [0,\infty)$, so its ε-shell is disconnected for all $\varepsilon < 1/2$.

EXAMPLE 2.3. Now we describe a construction in which the limit set of the series is the closure of the union of infinitely many pairwise disjoint half-lines $\{a_n : n \in \mathbb{N}\} \times [0,\infty)$ where (a_n) is a sequence of distinct real numbers. This example is similar to Example 2.2, so we only specify the walk (s_n). Since at each step of the construction the walk goes from one point to another and then back along the same path, the steps of the walk can be rearranged into an alternating series. Since the lengths of the walk's steps tend to zero, the resulting series is convergent. We describe the first three steps of the construction:

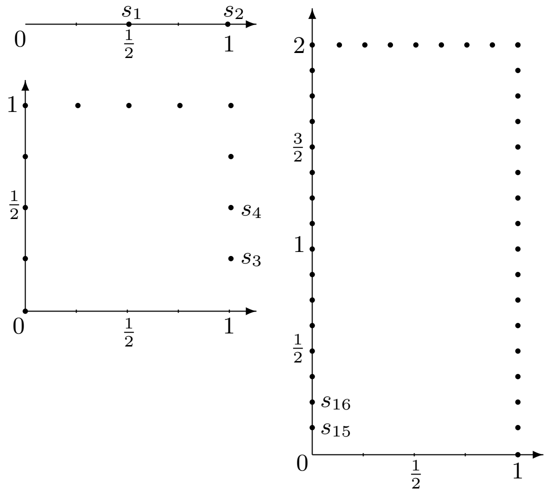

Fig. 1. The first three steps of the construction of the walk $(\sum_{n=1}^{m} x_n)_{m=1}^{\infty}$ from Example 2.2.

STEP 1. We start the walk at $(a_1, 0)$. Then we move to $(a_2, 0)$ along the line $y = 0$ using steps of length not greater than 1. Then we go back to $(a_1, 0)$ along the same path.

STEP 2. We go upwards to $(a_1, 1)$, then along the line $y = 1$ to $(a_2, 1)$, next downwards to $(a_2, 0)$ and back upwards to $(a_2, 1)$, then again along $y = 1$ to $(a_3, 1)$ and downwards to $(a_3, 0)$, always using steps no greater than $1/2$. Finally, we go back to $(a_1, 0)$ using the same path.

STEP 3. In this step first four points $(a_1, 0), \ldots, (a_4, 0)$ are involved, steps are no greater than $1/4$, and to move between the vertical lines $x = a_i$ we use the horizontal line $y = 2$, etc.

Clearly $A := \mathrm{LIM}(\sum_{n=1}^{\infty} x_n) \supseteq \{a_n : n \in \mathbb{N}\} \times [0, \infty)$. Since A is closed, we obtain $A \supseteq \overline{\{a_n : n \in \mathbb{N}\} \times [0, \infty)} = \overline{\{a_n : n \in \mathbb{N}\}} \times [0, \infty)$. To show the opposite inclusion, let $(u, v) \notin \overline{\{a_n : n \in \mathbb{N}\}} \times [0, \infty)$. If $v < 0$ then $(u, v) \notin A$, because our walk is in $\mathbb{R}^2$ and has a nonnegative second coordinate. If $v \geq 0$ and $u \notin \overline{\{a_n : n \in \mathbb{N}\}}$ then $\inf_{n\in\mathbb{N}} |u - a_n| = \delta > 0$. Fix a natural number $m > v+\delta$. Then the ball $B((u, v), \delta)$ contains no elements of our walk defined at the kth step of the construction for any $k \geq m$. Hence $(u, v) \notin A$. Finally, $A = \overline{\{a_n : n \in \mathbb{N}\}} \times [0, \infty)$.

Using Example 2.3 we can show that the limit set of a rearrangement of a conditionally convergent series can have uncountably many unbounded components. Let $E = \{a_1, a_2, \ldots\}$ be a countable dense subset of the ternary Cantor set C. By Example 2.3 one can find a series $\sum_{n=1}^{\infty} x_n \in \mathbf{\Sigma}_c$ such that $\mathrm{LIM}(\sum_{n=1}^{\infty} x_n) = \overline{\{a_n\}_{n=1}^{\infty} \times [0, \infty)} = C \times [0, \infty)$. Since C is totally disconnected, i.e. each of its components is a singleton, the half-lines $\{x\} \times [0,\infty)$, $x \in C$, are the components of $C \times [0,\infty)$.

3. Characterization of limit sets in Euclidean spaces. Let $B(0,R) = \{v \in \mathbb{R}^m : \|v\| \le R\}$ and $S(0,R) = \{v \in \mathbb{R}^m : \|v\| = R\}$. For a topological space X, we denote by $\mathcal{K}(X)$ the set of all nonempty compact subsets of X equipped with the Vietoris topology (for details see for example [6, p. 66]). It is well known that the compactness (metrizability, separability) of X implies the compactness (metrizability, separability) of the hyperspace $\mathcal{K}(X)$ and that the family of all nonempty compact connected subsets of X forms a closed subset of $\mathcal{K}(X)$.

LEMMA 3.1. *Let $X \subseteq \mathbb{R}^m$ be a closed set and let $R > 0$. Then*

$$Z := \bigcup\{C : C \text{ is a component of } X \cap B(0,R) \text{ such that } C \cap S(0,R) \ne \emptyset\}$$

is a compact subset of $\mathbb{R}^m$.

Proof. Let $(v_n) \subseteq Z$. Find components C_n of $X \cap B(0,R)$ such that $C_n \cap S(0,R) \ne \emptyset$ and $v_n \in C_n$. Pick $x_n \in C_n \cap S(0,R)$. Since $\mathcal{K}(X \cap B(0,R))$ is compact, we may assume that C_n tends to some C, $v_n \to v$ and $x_n \to x$. Then $v, x \in C$ and C is connected. Therefore v and x are in the same component of $X \cap B(0,R)$ which intersects the sphere $S(0,R)$. Thus $v \in Z$, and consequently Z is compact. ∎

Let $X \subseteq \mathbb{R}^m$ be a closed set. We define an equivalence relation E on X as follows:

$$xEy \;\Leftrightarrow\; x \text{ and } y \text{ belong to the same component of } X.$$

We denote by X/E the set of all equivalence classes of E, and by q the mapping from X to X/E assigning to a point $x \in X$ the equivalence class $[x]_E \in X/E$. On X/E we consider the so-called *quotient topology* consisting of those $U \subseteq X/E$ such that $q^{-1}(U)$ is open in X. The set X/E equipped with this topology is called the *quotient space*, and $q : X \to X/E$ is the *natural quotient mapping*. The following result important for us can be found in [1].

THEOREM 3.2. *For every compact space X, the quotient space X/E is compact and zero-dimensional.*

LEMMA 3.3. *Let $\sum_{n=1}^\infty x_n \in \mathbf{\Sigma}_c(\mathbb{R}^m)$. Assume that Y is a nonempty bounded subset of $X := \mathrm{LIM}(\sum_{n=1}^\infty x_n)$. If $Y(\varepsilon)$ is disjoint from $X \setminus Y$ for some $\varepsilon > 0$, then $X = Y$.*

Proof. Note that the closure Z of $Y(\varepsilon) \setminus Y(\varepsilon/2)$ is a compact set disjoint from X. Suppose that $X \setminus Y \ne \emptyset$. Consider the set $A := \{\sum_{n=1}^k x_n : k \in \mathbb{N}\} \cap Z$ of those partial sums of $\sum_{n=1}^\infty x_n$ which are in Z. Since all elements of the nonempty sets Y and $X \setminus Y$ are limit points of the partial sums sequence $\{\sum_{n=1}^{k} x_n\}_{k=1}^{\infty}$, the elements of that sequence walk from Y to $X \setminus Y$ and back infinitely many times. Since the lengths of steps $\|x_n\|$ taken during this walk tend to zero, the set A is infinite. By compactness of Z we know that A has a limit point, which in turn is in Z, but this contradicts the fact that $Z \cap X = \emptyset$. Thus $X \setminus Y = \emptyset$ and consequently $X = Y$. ∎

Denote by $a(\mathbb{R}^m)$ the one-point compactification of $\mathbb{R}^m$, that is, to $\mathbb{R}^m$ we add a point ∞. A neighborhood base at each $x \in \mathbb{R}^m$ consists of open balls centered at x, and a neighborhood base at ∞ consists of all sets of the form $(\mathbb{R}^m \setminus C) \cup \{\infty\}$ where C is compact in $\mathbb{R}^m$. For $A \subset a(\mathbb{R}^m)$ we denote by $\overline{A}^{\infty}$ the closure of A in $a(\mathbb{R}^m)$.

LEMMA 3.4. *Let $\{C_i : i \in I\}$ be a family of connected and unbounded subsets of $\mathbb{R}^m$ and let $C := \bigcup_{i\in I} C_i$. Then*

(1) $\overline{C}^{\infty} = \overline{C} \cup \{\infty\}$;
(2) $\overline{C}^{\infty}$ *is connected.*

Proof. (1) The set $\overline{C} \cup \{\infty\}$ is closed in $a(\mathbb{R}^m)$, since $(\mathbb{R}^m \cup \{\infty\}) \setminus (\overline{C} \cup \{\infty\}) = \mathbb{R}^m \setminus \overline{C}$ is open in $\mathbb{R}^m$. Thus $\overline{C}^{\infty} \subseteq \overline{C} \cup \{\infty\}$. Since C is unbounded, we have $\infty \in \overline{C}^{\infty}$, and consequently $\overline{C} \cup \{\infty\} \subseteq \overline{C}^{\infty}$.

(2) Note that $\overline{C \cup \{\infty\}}^{\infty} = \overline{C} \cup \{\infty\}$—this follows from (1) and the inclusions $C \subseteq C \cup \{\infty\} \subseteq \overline{C} \cup \{\infty\}$. It is enough to show that $A := C \cup \{\infty\}$ is connected. Suppose to the contrary that there are two nonempty disjoint open sets U and V with $A = (A \cap U) \cup (A \cap V)$ and $\infty \in U$. Set $U' := U \setminus \{\infty\}$. Then U' is open in $\mathbb{R}^m$. There is a compact set $D \subseteq \mathbb{R}^m$ such that $(X \setminus D) \cup \{\infty\} = U$. Then $X \setminus D = U'$ and $V \subseteq D$. Since V is nonempty, there is $i \in I$ with $V \cap C_i \neq \emptyset$. But then $C_i = (V \cap C_i) \cup (U \cap C_i)$ and by the connectedness of C_i we obtain $C_i \subseteq V \subseteq D$, which contradicts the unboundedness of C_i. ∎

THEOREM 3.5. *Let $\sum_{n=1}^{\infty} x_n \in \mathbf{\Sigma}_c(\mathbb{R}^m)$. Then the set $X = \mathrm{LIM}(\sum_{n=1}^{\infty} x_n)$ is either compact connected, or a union (finite, countably infinite or uncountable) of unbounded closed connected sets; in particular, $\overline{X}^{\infty}$ is compact and connected.*

Proof. Since X is closed in $\mathbb{R}^m$, we have $\overline{X}^{\infty} = X$ if X is bounded and $\overline{X}^{\infty} = X \cup \{\infty\}$ if X is unbounded. On $\overline{X}^{\infty}$ define an equivalence relation E given by the decomposition of $\overline{X}^{\infty}$ into components.

Assume that C is a bounded component of X. There is $R > 0$ such that $C \subseteq B(0, R)$ and $C \cap S(0, R) = \emptyset$. Set $Z := \bigcup\{C' : C'$ is a component of $X \cap B(0, R)$ such that $C' \cap S(0, R) \neq \emptyset\}$. By Lemma 3.1, Z is compact in $\mathbb{R}^m$. Set $U := (\overline{X}^{\infty} \cap B(0, R)) \setminus Z$. Then U is open in $\overline{X}^{\infty}$ and $U = q^{-1}(q(U))$; therefore $q(U)$ is open in $\overline{X}^{\infty}/E$. Since $C \in q(U)$ and $\overline{X}^{\infty}/E$

is zero-dimensional, there is a clopen set $V \subseteq \overline{X}^\infty/E$ with $C \in V \subseteq q(U)$. Since Z and $Y := q^{-1}(V)$ are compact, there is $\varepsilon > 0$ with $Y(\varepsilon) \cap Z = \emptyset$ and $(Y(\varepsilon) \setminus Y) \cap \overline{X}^\infty = \emptyset$; consequently, $Y(\varepsilon) \cap X \setminus Y = \emptyset$. By Lemma 3.3 we find that $Z \subseteq X \setminus Y = \emptyset$. Therefore no component of $X \cap B(0,R)$ intersects $S(0,R)$. Thus $X \cap B(0,R) = q^{-1}(q(X \cap B(0,R)))$, and consequently $q(X \cap B(0,R))$ is open in $\overline{X}^\infty/E$. Since $\overline{X}^\infty/E$ is zero-dimensional, there is a clopen V with $C \in V \subseteq q(X \cap B(0,R))$. Thus $Y := q^{-1}(V)$ is clopen and it contains C. There is $\varepsilon > 0$ such that $Y(\varepsilon) \subseteq B(0,R)$, which means that $Y(\varepsilon)$ is disjoint from $X \setminus Y$. By Lemma 3.3 we deduce that X is bounded.

We have thus proved that if X has a bounded component, then X is bounded itself. That means that if X has an unbounded component, then each of its components is unbounded and, by Lemma 3.4, $\overline{X}^\infty$ is connected, or equivalently $q(\overline{X}^\infty) = [\infty]_E$. Thus $\overline{X}^\infty$ is connected in $a(\mathbb{R}^m)$ if X is unbounded.

To finish the proof we need to show that if X is bounded, then it is connected. If not, there would be two disjoint nonempty clopen subsets Y and $X \setminus Y$ of X. But then there would be $\varepsilon > 0$ with $Y(\varepsilon) \cap (X \setminus Y) = \emptyset$, contrary to Lemma 3.3. ∎

4. Theorem 3.5 does not hold in infinite-dimensional spaces. Now we will define a series $\sum_{n=1}^\infty x_n \in \mathbf{\Sigma}_c(X)$ in an infinite-dimensional Banach space X such that $\mathrm{LIM}(\sum_{n=1}^\infty x_n)$ consists of two points.

Example 4.1. Let X be an infinite-dimensional Banach space and $Y \subset X$ be a linear subspace with a normalized Schauder basis $\{e_i : i \in \mathbb{N}\}$. Such a subspace Y exists by Mazur's Theorem (see for example [2, Theorem 6.3.3]). For $x \in Y$ we write $x = (x(1), x(2), \dots)$ instead of $x = \sum_{i=1}^\infty x(i)e_i$.

Step 1. We define the first six elements $x_1, \dots, x_6$ to be e_2, e_1, $-e_2$, e_2, $-e_1$, $-e_2$.

Step $k+1$. In this step we define six consecutive groups of elements of the series, each consisting of 2^k elements: the elements of the first group are all equal to $2^{-k}e_{k+2}$, of the second are all $2^{-k}e_1$, of the third 2^k are $-2^{-k}e_{k+2}$, of the fourth are $2^{-k}e_{k+2}$, of the fifth are $-2^{-k}e_1$, and of the last one are $-2^{-k}e_{k+2}$. Observe that the series can be rearranged to become an alternating series, and since its term tends to zero, the rearranged series is convergent.

The sequence of partial sums $s_m = \sum_{n=1}^m x_n$ is the following:

$$e_2, e_2+e_1, e_1, e_2+e_1, e_2, \theta,$$

$$\tfrac{1}{2}e_3, e_3, e_3+\tfrac{1}{2}e_1, e_3+e_1, \tfrac{1}{2}e_3+e_1, e_1,$$

$$\tfrac{1}{2}e_3+e_1, e_3+e_1, e_3+1/2e_1, e_3, \tfrac{1}{2}e_3, \theta, \dots,$$

$$\frac{1}{2^k}e_{k+2}, \frac{2}{2^k}e_{k+2}, \dots, e_{k+2}, e_{k+2}+\frac{1}{2^k}e_1, \dots, e_{k+2}+e_1, \frac{2^k-1}{2^k}e_{k+2}+e_1, \dots, e_1,$$
$$e_1+\frac{1}{2^k}e_{k+2}, \dots, e_1+e_{k+2}, e_{k+2}+\frac{2^k-1}{2^k}e_1, \dots, e_{k+2}, \frac{2^k-1}{2^k}e_{k+2}, \dots, \theta, \dots,$$

where $\theta = (0, 0, \dots)$.

The walk (s_m) has the following properties:

(i) θ and e_1 appear in the sequence (s_m) infinitely many times;
(ii) for every natural $j \geq 2$ there exists $p \in \mathbb{N}$ such that $s_m(j) = 0$ for all natural $m \geq p$;
(iii) the distance between $z = (z(1), z(2), \dots)$ with $z(1) \notin [0,1]$ and the set $\{s_m : m \in \mathbb{N}\}$ is positive;
(iv) if $s_m(1) \notin \{0,1\}$ then there exists a natural $k \geq 2$ such that $s_m(k) = 1$.

We claim that $A := \mathrm{LIM}(\sum_{n=1}^{\infty} x_n) = \{\theta, e_1\}$. By (i) we get $\theta, e_1 \in A$. Conditions (ii) and (iii) imply $A \subseteq \{(a, 0, 0, \dots) : a \in [0,1]\}$. Indeed, since $z = (z(1), z(2), \dots) \in A$, by (ii) we get $z(i) = 0$ for every $i \geq 2$. Moreover, if $z(1) > 1$ or $z(1) < 0$ then by (iii) we have $z \notin A$.

Now, let $a \in (0,1)$. We will show that $\underline{a} := (a, 0, 0, \dots) \notin A$. One can find $\varepsilon > 0$ such that $(a-\varepsilon, a+\varepsilon) \cap \{0,1\} = \emptyset$. We consider the ball $B(\underline{a}, \varepsilon)$ in X. If $z \in B(\underline{a}, \varepsilon) \cap \{s_m : m \in \mathbb{N}\}$ then $z(1) \in (a-\varepsilon, a+\varepsilon)$, hence the first coordinate of z is neither 0 nor 1. Then by (iv) there exists $k \geq 2$ such that $z(k) = 1$, which contradicts the fact that $z \in B(\underline{a}, \varepsilon)$. Hence $B(\underline{a}, \varepsilon) \cap \{s_m : m \in \mathbb{N}\} = \emptyset$, so $\underline{a} \notin \overline{\{s_m : m \in \mathbb{N}\}}$. Since $A \subseteq \overline{\{s_m : m \in \mathbb{N}\}}$, we have $\underline{a} \notin A$. Finally, $A = \{\theta, e_1\}$.

Remark. Roman Wituła called our attention to the fact that he and his co-authors had found a very similar example of a series with two-point limit set (see [7]).

5. On the converse of Theorem 3.5. In this section we will prove that Theorem 3.5 can be reversed: for any compact and connected subset X of $\mathbb{R}^m$ there is a series $\sum_{n=1}^{\infty} x_n \in \mathbf{\Sigma}_c$ with $X = \mathrm{LIM}(\sum_{n=1}^{\infty} x_n)$, and for any closed subset Y of $\mathbb{R}^m$ with each component unbounded there is a series $\sum_{n=1}^{\infty} y_n \in \mathbf{\Sigma}_c$ with $Y = \mathrm{LIM}(\sum_{n=1}^{\infty} y_n)$. This shows that Theorem 3.5 gives a full characterization of limit sets in finite-dimensional Banach spaces.

Theorem 5.1. *Let $m \in \mathbb{N}$. Assume that $X \subseteq \mathbb{R}^m$ is closed and ε-chainable for every $\varepsilon > 0$. Then there is a series $\sum_{n=1}^{\infty} x_n \in \mathbf{\Sigma}_c(\mathbb{R}^m)$ such that $X = \mathrm{LIM}(\sum_{n=1}^{\infty} x_n)$. In particular, the assertion holds if X is compact and connected.*

Proof. Let (d_n) be dense in X. We will construct an X-walk. In the first step we find a 1-chain inside X between d_1 and d_2 and denote it $a_1 = d_1, a_2, \ldots, a_p = d_2$. We define $s_i = a_i$ for every $i \in \{1, \ldots, p\}$. In the second step let $a_p = d_2, a_{p+1}, \ldots, a_{p+r} = d_3$ be a 2^{-1}-chain between d_2 and d_3. We define $s_i = a_i$ for $i \in \{p+1, \ldots, p+r\}$. In the third step we consider a 2^{-2}-chain between d_3 and d_4 and define the next s_n's as before, and so on. By Proposition 2.1, we obtain the assertion. Finally, note that connected sets are ε-chainable for every $\varepsilon > 0$. ■

THEOREM 5.2. *Let $m \geq 2$. Assume that $X \subseteq \mathbb{R}^m$ is closed and any component of X is unbounded. Then there is a series $\sum_{n=1}^{\infty} x_n \in \mathbf{\Sigma}_c(\mathbb{R}^m)$ such that $X = \mathrm{LIM}(\sum_{n=1}^{\infty} x_n)$.*

Proof. Let $X = \bigcup_{t \in T} A_t$, where for every $t \in T$ the set A_t is an unbounded component of X. Clearly each A_t is closed and ε-chainable for every $\varepsilon > 0$. Let (d_n) be dense in X. If $d_i \in A_s$, $d_j \in A_t$, $A_t \cap A_s = \emptyset$, then, by the connectedness and unboundedness of A_s and A_t, there is a sphere $S(0, R)$ intersecting A_s and A_t. Let $a_s \in A_s \cap S(0, R)$ and $a_t \in A_t \cap S(0, R)$. By an *ε-chain via* $S(0, R)$ from d_i to d_j we mean a concatenation of three ε-chains: from d_i to a_s using elements of A_s, from a_s to a_t using elements of S_R and from d_j to a_t using elements of A_t. If $A_t = A_s$, then by an ε-chain via $S(0, R)$ from d_i to d_j we mean just an ε-chain from d_i to d_j using elements of A_s.

Let R_n be a sequence of radii tending to ∞ such that $S(0, R_n)$ intersects each component containing $d_1, \ldots, d_{n+1}$.

Now let us describe a walk (s_n), which, in general, need not be an X-walk: The first elements of (s_n) form a 1/2-chain via $S(0, R_1)$ from d_1 to d_2. In the kth step of the construction the subsequent elements of (s_n) are elements of a 2^{-k}-chain via $S(0, R_k)$ from d_k to d_{k+1}.

We have defined the sequence of partial sums $s_n = \sum_{i=1}^{n} x_i$ of a series $\sum_{n=1}^{\infty} x_n \in \mathbf{\Sigma}_0(\mathbb{R}^m)$. Clearly $X \subseteq \mathrm{LIM}(\sum_{n=1}^{\infty} x_n) \subseteq X \cup \bigcup_{k=1}^{\infty} S(0, R_k)$. Since $R_k \to \infty$ and the sequence (s_n) contains at most finitely many elements of $S(0, R_k) \setminus X$, we obtain the reverse inclusion $X \supseteq \mathrm{LIM}(\sum_{n=1}^{\infty} x_n)$. In particular $\mathrm{LIM}(\sum_{n=1}^{\infty} x_n) \neq \emptyset$. Thus $\sum_{n=1}^{\infty} x_n \in \mathbf{\Sigma}_c(\mathbb{R}^m)$. ■

As mentioned in the Introduction, the assertion of Theorem 5.2 is not true if $m = 1$.

6. When the limit set is a singleton. By definition, if $\sum_{n=1}^{\infty} x_n = x_0$, then $\mathrm{LIM}(\sum_{n=1}^{\infty} x_n) = \{x_0\}$, since every subsequence of the sequence of partial sums is convergent to x_0. In general the converse need not be true, which is illustrated by Proposition 6.2 below. However, in finite-dimensional spaces the above implication can be reversed.

THEOREM 6.1. *Let $\sum_{n=1}^{\infty} x_n \in \mathbf{\Sigma}_0(\mathbb{R}^m)$. If $\mathrm{LIM}(\sum_{n=1}^{\infty} x_n)$ is a singleton $\{x_0\}$, then $\sum_{n=1}^{\infty} x_n$ converges to x_0.*

Proof. Suppose that $\sum_{n=1}^{\infty} x_n$ does not converge to x_0, so there exists $\varepsilon > 0$ such that for every $k_0 \in \mathbb{N}$ one can find $l \geq k_0$ such that $\|\sum_{n=1}^{l} x_n - x_0\| > \varepsilon$. This means that there are infinitely many indices p such that $\sum_{n=1}^{p} x_n \notin B(x_0, \varepsilon)$. On the other hand, since x_0 is a limit point of the series $\sum_{n=1}^{\infty} x_n$, there exist infinitely many $r \in \mathbb{N}$ such that $\sum_{n=1}^{r} x_n \in \operatorname{int} B(x_0, \varepsilon/2)$. Hence there are infinitely many elements of a walk (s_n) of partial sums in the interior of $B(x_0, \varepsilon/2)$ and infinitely many outside the ball $B(x_0, \varepsilon)$. Since $x_n \to 0$, there are infinitely many s_n's in $B = B(x_0, \varepsilon) \setminus \operatorname{int} B(x_0, \varepsilon/2)$. By the compactness of B, it contains a limit point of (s_n), contrary to $\mathrm{LIM}(\sum_{n=1}^{\infty} x_n) = \{x_0\}$. ∎

Note that the assumption $x_n \to 0$ cannot be omitted. To see this consider the series $2^{-1} + 2^1 - 2^1 + 2^{-2} + 2^2 - 2^2 + 2^{-3} + 2^3 - 2^3 + \cdots$. Clearly 1 is its only limit point, but the series is not convergent.

PROPOSITION 6.2. *Let X be an infinite-dimensional Banach space. There is a series $\sum_{n=1}^{\infty} x_n \in \mathbf{\Sigma}_c(X)$ such that $\mathrm{LIM}(\sum_{n=1}^{\infty} x_n) = \{\theta\}$, where $\theta = (0, 0, \dots)$, but $\sum_{n=1}^{\infty} x_n$ diverges.*

Proof. The construction we present is very similar to that in the proof of [2, Theorem 6.4.1]. Let $Y \subset X$ be a closed linear subspace of X with normalized Schauder basis $\{e_n\}_{n \in \mathbb{N}}$. As in Example 4.1, for $x \in Y$ we write $x = (x(1), x(2), \dots)$ instead of $x = \sum_{i=1}^{\infty} x(i) e_i$.

STEP 1. Firstly we define $x_1 = e_1$, $x_2 = -e_1$.

STEP k. In this step we define 2^k elements; the first 2^{k-1} of them are all equal to $2^{-n+1} e_k$ and the others are equal to $-2^{-n+1} e_k$.

The series $\sum_{n=1}^{\infty} x_n$ is

$$e_1 - e_1 + \tfrac{1}{2} e_2 + \tfrac{1}{2} e_2 - \tfrac{1}{2} e_2 - \tfrac{1}{2} e_2 + \tfrac{1}{4} e_3 + \tfrac{1}{4} e_3 + \tfrac{1}{4} e_3 + \tfrac{1}{4} e_3 - \tfrac{1}{4} e_3 - \tfrac{1}{4} e_3 - \tfrac{1}{4} e_3 - \tfrac{1}{4} e_3 + \cdots.$$

It can be rearranged to become the alternating, convergent series

$$e_1 - e_1 + \tfrac{1}{2} e_2 - \tfrac{1}{2} e_2 + \tfrac{1}{2} e_2 - \tfrac{1}{2} e_2 + \tfrac{1}{4} e_3 - \tfrac{1}{4} e_3 + \tfrac{1}{4} e_3 - \tfrac{1}{4} e_3 + \tfrac{1}{4} e_3 - \tfrac{1}{4} e_3 + \tfrac{1}{4} e_3 - \tfrac{1}{4} e_3 + \cdots,$$

so $\sum_{n=1}^{\infty} x_n \in \mathbf{\Sigma}_c(X)$.

Thus the walk $s_n = \sum_{k=1}^{n} x_k$ has the following properties:

(1) $s_{2^{k+1}-2} = \theta$ for every $k \in \mathbb{N}$;
(2) for every $j \in \mathbb{N}$ there exists $p \in \mathbb{N}$ such that $s_m(j) = 0$ for all $m \geq p$;
(3) $s_{2^k + 2^{k-1} - 2} = e_k$ for every $k \in \mathbb{N}$.

From (1) we have $\{\theta\} \subseteq \mathrm{LIM}(\sum_{n=1}^{\infty} x_n)$ and from (2) we get the reverse inclusion. Hence $\mathrm{LIM}(\sum_{n=1}^{\infty} x_n) = \{\theta\}$. From (1) and (3) we obtain $\|s_{2^k+2^{k-1}-2}(k) - s_{2^{k+1}-2}(k)\| = \|e_k\| = 1$ for every $k \in \mathbb{N}$. This means that the sequence of partial sums of the series is not a Cauchy sequence, and consequently it diverges. ∎

7. Rearrangement property. For a series $\sum_{n=1}^{\infty} x_n$ in $\mathbb{R}^k$ Klee proved the following fact: if $A \subseteq \mathrm{SR}(\sum_{n=1}^{\infty} x_n)$ is closed and ε-chainable for every $\varepsilon > 0$, then there is $\tau \in S_\infty$ such that $A = \mathrm{LIM}(\sum_{n=1}^{\infty} x_{\tau(n)})$. This is a strengthening of Theorem 5.1—to see this, take any conditionally convergent series $\sum_{n=1}^{\infty} x_n$ with $\mathrm{SR}(\sum_{n=1}^{\infty} x_n) = \mathbb{R}^m$.

We will show that the above fact holds true in every Banach space provided $\sum_{n=1}^{\infty} x_n$ has the so-called Rearrangement Property; in fact, we then prove that if $A \subseteq \overline{\mathrm{SR}(\sum_{n=1}^{\infty} x_n)}$ is closed and ε-chainable for every $\varepsilon > 0$, then there is $\tau \in S_\infty$ such that $A = \mathrm{LIM}(\sum_{n=1}^{\infty} x_{\tau(n)})$.

LEMMA 7.1. *Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series in a Banach space X. Then*

$$\mathrm{SR}\Big(\sum_{n=1}^{\infty} x_n\Big) = \mathrm{SR}\Big(\sum_{n=k+1}^{\infty} x_n\Big) + \sum_{n=1}^{k} x_n$$

for every $k \in \mathbb{N}$.

Proof. "$\supseteq$" Let $k \in \mathbb{N}$ and $x \in \mathrm{SR}(\sum_{n=k+1}^{\infty} x_n) + \sum_{n=1}^{k} x_n$. Then there exists a permutation $\sigma : [k+1, \infty) \to [k+1, \infty)$ such that $x = \sum_{n=k+1}^{\infty} x_{\sigma(n)} + \sum_{n=1}^{k} x_n$. Define $\pi(n) = n$ for $n \le k$ and $\pi(n) = \sigma(n)$ for $n \ge k+1$. Hence $x = \sum_{n=1}^{\infty} x_{\pi(n)}$, so $x \in \mathrm{SR}(\sum_{n=1}^{\infty} x_n)$.

"$\subseteq$" Let $x \in \mathrm{SR}(\sum_{n=1}^{\infty} x_n)$ and $k \in \mathbb{N}$. Then there exists a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that $x = \sum_{n=1}^{\infty} x_{\pi(n)}$. Let $M = \pi^{-1}(\{1, \ldots, k\})$. Then for every $\varepsilon > 0$ there exists $m_0 \ge \max M$ such that $\|x - \sum_{n=1}^{m} x_{\pi(n)}\| < \varepsilon$ for every $m > m_0$. This means that $\|x - \sum_{n=1}^{k} x_n - \sum_{n \in \{1,\ldots,m\} \setminus M} x_{\pi(n)}\| < \varepsilon$ for every $m > m_0$. Define a permutation $\sigma : [k+1, \infty) \to [k+1, \infty)$ as follows: $\sigma(k+l) = \pi(n)$ where n is the lth number in the set $\mathbb{N} \setminus M$. Then $x = \sum_{n=k+1}^{\infty} x_{\sigma(n)} + \sum_{n=1}^{k} x_n$. Hence $x \in \mathrm{SR}(\sum_{n=k+1}^{\infty} x_n) + \sum_{n=1}^{k} x_n$. ∎

We say that a conditionally convergent series $\sum_{k=1}^{\infty} x_k$ has the *Rearrangement Property*, or (RP), if for every $\varepsilon > 0$ there are a natural number $N(\varepsilon)$ and a positive real number $\delta(\varepsilon)$ such that the implication

$$\Big\|\sum_{i=1}^{n} y_i\Big\| < \delta(\varepsilon) \Rightarrow \Big(\max_{j \le n} \|\sum_{i=1}^{j} y_{\sigma(i)}\| < \varepsilon \text{ for some permutation } \sigma \in S_n\Big)$$

holds for every finite sequence $(y_i)_{i=1}^{n} \subseteq (x_i)_{i=N(\varepsilon)}^{\infty}$. The Rearrangement

Property is widely known and used implicitly by many authors. It appears explicitly in [2, Lemma 2.3.1] as Property (A), or in [5, p. 65]. Note that if $\varepsilon > \varepsilon' > 0$, then we can find numbers $\delta(\varepsilon), N(\varepsilon)$ and $\delta(\varepsilon'), N(\varepsilon')$ as in the definition of (RP) such that $\delta(\varepsilon) \geq \delta(\varepsilon')$ and $N(\varepsilon) \leq N(\varepsilon')$. Similarly, having a decreasing sequence (ε_n) of positive real numbers, we can find $\delta(\varepsilon_n), N(\varepsilon_n)$ as in the definition of (RP) such that $\delta(\varepsilon_n) \geq \delta(\varepsilon_{n+1})$ and $N(\varepsilon_n) \leq N(\varepsilon_{n+1})$ for every $n \in \mathbb{N}$.

Lemma 7.2. *Assume that $\sum_{n=1}^{\infty} x_n$ is a conditionally convergent series with* (RP) *in a Banach space X. Let $\varepsilon \geq \varepsilon' > 0$ and let $\delta(\varepsilon/2), N(\varepsilon/2)$ and $\delta(\varepsilon'/2), N(\varepsilon'/2)$ be as in the definition of* (RP). *Let $k \in \mathbb{N}$, $a, b \in \mathrm{SR}(\sum_{n=1}^{\infty} x_n)$ with*

$$\|a-b\| < \min\left\{\frac{\varepsilon}{12}, \frac{1}{3}\delta\left(\frac{\varepsilon}{2}\right)\right\},$$

and $\tau : [1,k] \to \mathbb{N}$ be a partial permutation such that

$$\Big\|\sum_{n=1}^{k} x_{\tau(n)} - a\Big\| \leq \min\left\{\frac{\varepsilon}{12}, \frac{1}{3}\delta\left(\frac{\varepsilon}{2}\right)\right\} \quad \text{and} \quad \operatorname{rng}\tau \supseteq [1, N(\varepsilon/2)].$$

Then there exist $k' > k$ and a partial permutation $\tau' : [1,k'] \to \mathbb{N}$ such that:

1. $\tau'|_{[1,k]} = \tau$ *and* $[1, \max\operatorname{rng}\tau] \subseteq \operatorname{rng}\tau'$;
2. $\|\sum_{n=1}^{p} x_{\tau'(n)} - a\| \leq \varepsilon$ *for* $p \in [k+1, k']$;
3. $\|\sum_{n=1}^{k'} x_{\tau'(n)} - b\| \leq \min\{\frac{\varepsilon'}{12}, \frac{1}{3}\delta(\frac{\varepsilon'}{2})\}$;
4. $\operatorname{rng}\tau' \supseteq [1, N(\varepsilon'/2)]$.

Proof. Let $k_0 = \max\{N(\varepsilon'/2), N(\varepsilon/2), \max\operatorname{rng}\tau\}$. Define $y = \sum_{n=1}^{k} x_{\tau(n)}$ and $z = \sum_{n\in\{1,\dots,k_0\}\setminus\{\tau(1),\dots,\tau(k)\}} x_n$. Hence $y + z = \sum_{n=1}^{k_0} x_n$. From the assumption that $b \in \mathrm{SR}(\sum_{n=1}^{\infty} x_n)$ by Lemma 7.1 we obtain $b - (y+z) \in \mathrm{SR}(\sum_{n=k_0+1}^{\infty} x_n)$. Thus we can find $k_0 < n_1 < \dots < n_l$ such that

$$y + z + w \in B\left(b, \min\left\{\frac{\varepsilon'}{12}, \frac{1}{3}\delta\left(\frac{\varepsilon'}{2}\right), \frac{1}{3}\delta\left(\frac{\varepsilon}{2}\right)\right\}\right),$$

where $w = x_{n_1} + \dots + x_{n_l}$.

Enumerate the set $([1,k_0] \setminus \{\tau(1),\dots,\tau(k)\}) \cup \{n_1,\dots,n_l\}$ as $\{m_1 < \dots < m_{k'-k}\}$, where $k' = k_0 + l$. Hence,

$$\Big\|\sum_{i=1}^{k'-k} x_{m_i}\Big\| = \|z+w\| \leq \|y-a\| + \|a-b\| + \|b-(y+z+w)\|.$$

Consequently,

$$\Big\|\sum_{i=1}^{k'-k} x_{m_i}\Big\| \le \min\Big\{\frac{\varepsilon}{12}, \frac{1}{3}\delta\Big(\frac{\varepsilon}{2}\Big)\Big\} + \min\Big\{\frac{\varepsilon}{12}, \frac{1}{3}\delta\Big(\frac{\varepsilon}{2}\Big)\Big\}$$
$$+ \min\Big\{\frac{\varepsilon'}{12}, \frac{1}{3}\delta\Big(\frac{\varepsilon'}{2}\Big), \frac{1}{3}\delta\Big(\frac{\varepsilon}{2}\Big)\Big\} \le \delta\Big(\frac{\varepsilon}{2}\Big).$$

Since $m_i \ge N(\varepsilon/2)$ for $i \in [1, k'-k]$ and $\|\sum_{i=1}^{k'-k} x_{m_i}\| \le \delta(\varepsilon/2)$, by (RP) there is a permutation $\sigma \in S_{k'-k}$ such that $\|\sum_{i=1}^{j} x_{m_{\sigma(i)}}\| \le \varepsilon/2$ for every $j \in [1, k'-k]$. Define $\tau'(n) = \tau(n)$ for $n \le k$ and $\tau'(n) = m_{\sigma(n-k)}$ for $n \in [k+1, k']$. Then for every $p \in [k+1, k']$ we have

$$\Big\|\sum_{n=1}^{p} x_{\tau'(n)} - a\Big\| = \Big\|\sum_{n=1}^{k} x_{\tau(n)} + \sum_{n=k+1}^{p} x_{\tau'(n)} - a\Big\|$$
$$\le \|y - a\| + \Big\|\sum_{n=k+1}^{p} x_{\tau'(n)}\Big\|$$
$$\le \min\Big\{\frac{\varepsilon}{12}, \frac{1}{3}\cdot\delta\Big(\frac{\varepsilon}{2}\Big)\Big\} + \frac{\varepsilon}{2} < \varepsilon,$$

which gives (2).

Now we check (1), (3) and (4). Note that the numbers $1, \dots, k_0$ are among $\tau'(1), \dots, \tau'(k')$ and $k_0 \ge \max \operatorname{rng} \tau$. Therefore we have (1). Since $\sum_{n=1}^{k'} x_{\tau'(n)} = y + z + w$ and $\|y + z + w - b\| \le \min\{\frac{\varepsilon'}{12}, \frac{1}{3}\delta(\frac{\varepsilon'}{2})\}$, we obtain (3). Condition (4) follows from the fact that if $n \notin \operatorname{rng} \tau'$, then $n > k_0 \ge N(\varepsilon'/2)$. ∎

Lemma 7.3. *Let A be a separable subset of a Banach space such that A is ε-chainable for every $\varepsilon > 0$. Let (η_i) be a sequence of positive numbers. Then there is a sequence (d_n) dense in A with the property that there is an increasing sequence (l_i) of natural numbers such that $\{d_{l_i}, d_{l_i+1}, \dots, d_{l_{i+1}}\}$ is an η_i-chain for every i.*

Proof. Since A is separable, there are $v_1, v_2, \dots$ such that $A = \overline{\{v_n : n \in \mathbb{N}\}}$. Then one can find an η_i-chain $d_{l_i}, d_{l_i+1}, \dots, d_{l_{i+1}}$ of elements of A with $d_{l_i} = v_i$ and $d_{l_{i+1}} = v_{i+1}$ for any $i \in \mathbb{N}$. Clearly the sequence $(d_n)_{n=1}^{\infty}$ is as desired. ∎

Lemma 7.4. *Let A be a separable and ε-chainable (for every $\varepsilon > 0$) subset of a Banach space. Assume that $\{d_i : i \in \mathbb{N}\}$ is a dense subset of A and (ε_i) is a sequence of positive numbers tending to zero. If (x_i) is such that $\|x_i - d_i\| < \varepsilon_i$ for every $i \in \mathbb{N}$, then $\mathrm{LIM}(x_i) = \overline{A}$ where $\mathrm{LIM}(x_i)$ denotes the set of all limit points of the sequence (x_i).*

Proof. If A is a singleton, then $d_i = a$, $A = \{a\}$ and $x_i \to a$. Then $\mathrm{LIM}(x_i) = \{a\} = \overline{A}$. Assume that A has at least two elements. Clearly A is dense-in-itself. Fix $i \in \mathbb{N}$. There is a sequence $(d_{j_k})_{k=1}^{\infty}$ such that $j_1 < j_2 < \cdots$ and $\|d_{j_k} - d_i\| < \varepsilon_{j_k}$. Then $x_{j_k} \to d_i$, and consequently $d_i \in \mathrm{LIM}(x_i)$. Since the set $\mathrm{LIM}(x_i)$ is closed, we have $\overline{A} \subseteq \mathrm{LIM}(x_i)$.

Note that for every k almost every element of (x_i) is in the ε_k-shell of A. Thus $\overline{A} \supseteq \mathrm{LIM}(x_i)$. ∎

THEOREM 7.5. *Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series with* (RP) *in a Banach space X. Then for every $A \subseteq \overline{\mathrm{SR}(\sum_{n=1}^{\infty} x_n)}$ which is closed and ε-chainable for every $\varepsilon > 0$, there exists a permutation $\tau \in S_\infty$ such that $A = \mathrm{LIM}(\sum_{n=1}^{\infty} x_{\tau(n)})$.*

Proof. Let $\varepsilon_i = 2^{-i}$. We fix numbers $\delta(\varepsilon_i/2), N(\varepsilon_i/2)$ as in the definition of (RP) such that $\delta(\varepsilon_i/2) \geq \delta(\varepsilon_{i+1}/2)$ and $N(\varepsilon_i/2) \leq N(\varepsilon_{i+1}/2)$ for every $i \in \mathbb{N}$. Since A is separable and ε-chainable for every $\varepsilon > 0$, using Lemma 7.3, let $A = \overline{\{d_n : n \in \mathbb{N}\}}$, where for every $i \in \mathbb{N}$ the elements $d_{l_i}, d_{l_i+1}, \ldots, d_{l_{i+1}}$ form an η_i-chain for some $1 = l_1 < l_2 < \cdots$, where $\eta_i = \min\{\frac{\varepsilon_i}{48}, \frac{1}{12} \cdot \delta(\frac{\varepsilon_i}{2})\}$. Note that (η_i) is a nonincreasing sequence of positive real numbers.

Inductively we define natural numbers $1 = k_1 < k_2 < \cdots$, one-to-one functions $\tau_i : [1, k_{i+1}] \to \mathbb{N}$ and $d_1', d_2', \ldots$ fulfilling the following conditions:

(i) $\tau_i \subseteq \tau_{i+1}$;
(ii) $[1, \max \operatorname{rng} \tau_i] \subseteq \operatorname{rng} \tau_{i+1}$;
(iii) $\|\sum_{n=1}^{p} x_{\tau_i(n)} - d_{i-1}'\| < \varepsilon_j$ for $p \in [k_i + 1, k_{i+1}]$ and $i \in [l_j + 1, l_{j+1}]$;
(iv) $d_i' \in \mathrm{SR}(\sum_{n=1}^{\infty} x_n)$, $\|d_i' - d_i\| < \eta_j$ for $i \in [l_j, l_{j+1} - 1]$;
(v) $\|\sum_{n=1}^{k_{i+1}} x_{\tau_i(n)} - d_i'\| < 4\eta_j$ for $i \in [l_j, l_{j+1} - 1]$;
(vi) $\operatorname{rng} \tau_i \supseteq [1, N(\varepsilon_j/2)]$ for $i \in [l_j, l_{j+1} - 1]$.

Define $x = \sum_{n=1}^{N(\varepsilon_1/2)} x_n$. Let $d_1' \in \mathrm{SR}(\sum_{n=1}^{\infty} x_n)$ be such that $\|d_1 - d_1'\| < \eta_1$. Hence from Lemma 7.1 we get $d_1' - x \in \mathrm{SR}(\sum_{N(\varepsilon_1/2)+1}^{\infty} x_n)$. Let $\pi : [N(\varepsilon_1/2) + 1, \infty) \to [N(\varepsilon_1/2) + 1, \infty)$ be a bijection such that $d_1' - x = \sum_{n=N(\varepsilon_1/2)+1}^{\infty} x_{\pi(n)}$. One can find a natural number $k_2 > N(\varepsilon_1/2)$ which satisfies $\|d_1' - x - \sum_{n=N(\varepsilon_1/2)+1}^{k_2} x_{\pi(n)}\| \leq \eta_1$. Define $\tau_1(k) = k$ for $k \leq N(\varepsilon_1/2)$ and $\tau_1(k) = \pi(k)$ for $k \in [N(\varepsilon_1/2) + 1, k_2]$. Conditions (i)–(vi) are fulfilled for τ_1, d_1', k_1, k_2: we need not check (i) and (ii), condition (iii) has to be checked for $i \geq l_1 + 1 = 2$, and (iv)–(vi) are fulfilled since $l_1 = 1$.

Assume now that we have already defined $\tau_1, \ldots, \tau_i$, $k_1 < \cdots < k_{i+1}$ and $d_1', \ldots, d_i'$ fulfilling (i)–(vi). Find d_{i+1}' such that (iv) holds. We use Lemma 7.2 for $a = d_i'$, $b = d_{i+1}'$, $\tau = \tau_i$, $\varepsilon = \varepsilon_j$ where $l_j \leq i \leq l_{j+1} - 1$, $\varepsilon' = \varepsilon_q$ where $l_q - 1 \leq i \leq l_{q+1} - 2$, and $k = k_{i+1}$; note that $j = q$ if $l_j \leq i < l_{j+1} - 1$, that is, if $i \notin \{l_s - 1 : s \geq 1\}$, otherwise $i = l_{j+1} - 1$ implies that $q = j + 1$.

Let us check the assumptions of Lemma 7.2. By (iv) and (vi) we obtain $a, b \in \mathrm{SR}(\sum_{n=1}^{\infty} x_n)$ and $\operatorname{rng}\tau \supseteq [1, N(\varepsilon/2)]$. Since $d_{l_j}, d_{l_j+1}, \dots, d_{l_{j+1}}$ form an η_j-chain, by (iv) we obtain

$$\begin{aligned}\|a-b\| &\le \|d_i - d_i'\| + \|d_i - d_{i+1}\| + \|d_{i+1} - d_{i+1}'\| \le \eta_j + \eta_j + \eta_j < 4\eta_j \\ &= \min\left\{\frac{\varepsilon}{12}, \frac{1}{3}\delta\left(\frac{\varepsilon}{2}\right)\right\}.\end{aligned}$$

By (v) we obtain $\|\sum_{n=1}^{k} x_{\tau(n)} - a\| \le 4\eta_j = \min\{\frac{\varepsilon}{12}, \frac{1}{3}\delta(\frac{\varepsilon}{2})\}$. Now, using Lemma 7.2 we find $k_{i+2} > k_{i+1}$ and a function $\tau_{i+1} : [1, k_{i+2}] \to \mathbb{N}$ such that

(1) $\tau_{i+1}|_{[1,k_{i+1}]} = \tau_i$ and $[1, \max\operatorname{rng}\tau_i] \subseteq \operatorname{rng}\tau_{i+1}$;
(2) $\|\sum_{n=1}^{p} x_{\tau_{i+1}(n)} - d_i'\| \le \varepsilon_j$ for $p \in [k_{i+1}+1, k_{i+2}]$ and $i \in [l_j, l_{j+1}-1]$;
(3) $\|\sum_{n=1}^{k_{i+2}} x_{\tau_{i+1}(n)} - d_{i+1}'\| \le 4\eta_q$ for $i \in [l_q - 1, l_{q+1} - 2]$;
(4) $\operatorname{rng}\tau_{i+1} \supseteq [1, N(\varepsilon_q/2)]$ for $i \in [l_q - 1, l_{q+1} - 2]$.

Note that $\tau_1, \dots, \tau_{i+1}$, $k_1 < \dots < k_{i+2}$ and $d_1', \dots, d_{i+1}'$ fulfill (i)–(vi): By (1) we obtain (i) and (ii). Since the condition $i+1 \in [l_j+1, l_{j+1}]$ is equivalent to $i \in [l_j, l_{j+1}-1]$, we obtain (iii). The element d_{i+1}' has already been chosen to fulfill (iv). Conditions (3) and (4) are exactly (v) and (vi) for $i+1$.

Let $\tau = \bigcup_{i\ge 1} \tau_i : \mathbb{N} \to \mathbb{N}$. Then (i) implies that τ is one-to-one. Condition (ii) implies that τ is onto $\mathbb{N}$, and consequently $\tau \in S_\infty$. By (iii) and (iv) the distance between A and $\sum_{n=1}^{p} x_{\tau(n)}$ is less than $1/2^j$ for almost every $p \in \mathbb{N}$. Thus $\mathrm{LIM}(\sum_{n=1}^{\infty} x_{\tau(n)}) \subseteq \overline{A}$. By (iv) and (v) we obtain $\|\sum_{n=1}^{k_{i+1}} x_{\tau(n)} - d_i\| < 5\eta_j < \varepsilon_j$ for $i \in [l_j, l_{j+1}-1]$. Thus by Lemma 7.4 we get $\overline{A} = \mathrm{LIM}((\sum_{n=1}^{k_{i+1}} x_{\tau(n)})_{i=1}^{\infty}) \subseteq \mathrm{LIM}(\sum_{n=1}^{\infty} x_{\tau(n)})$. ∎

It is well-known that every conditionally convergent series of elements in a finite-dimensional Banach space has (RP) (for details see [2, Chapter 2]). Thus, Klee's result mentioned at the beginning of this section is a particular case of Theorem 7.5. Combining the methods used in the proofs of Theorems 7.5 and 5.2 one can prove the following strengthening of Theorem 5.2.

Corollary 7.6. *Let $m \ge 2$. Assume that $\mathrm{SR}(\sum_{n=1}^{\infty} x_n) = \mathbb{R}^m$, $X \subseteq \mathbb{R}^m$ is closed and any component of X is unbounded. Then $X = \mathrm{LIM}(\sum_{n=1}^{\infty} x_{\sigma(n)})$ for some $\sigma \in S_\infty$.*

Note that singletons are trivially ε-chainable for every $\varepsilon > 0$. Fix $a \in \overline{\mathrm{SR}(\sum_{n=1}^{\infty} x_n)}$. Using Theorem 7.5 for $A \subseteq \overline{\mathrm{SR}(\sum_{n=1}^{\infty} x_n)}$ such that $A = \{a\}$, we deduce that there is $\tau \in S_\infty$ such that $\mathrm{LIM}(\sum_{n=1}^{\infty} x_{\tau(n)}) = \{a\}$. As we have seen in Proposition 6.2, this does not necessarily mean that $\sum_{n=1}^{\infty} x_{\tau(n)} = a$. However, if we set $d_i = a$, then $d_i' \to a$, and by (iii) almost all elements of the sequence $(\sum_{n=1}^{p} x_{\tau(n)})_{p=1}^{\infty}$ are in every neighborhood of a. Therefore

$\sum_{n=1}^{\infty} x_{\tau(n)}$ converges to a. Thus $a \in \mathrm{SR}(\sum_{n=1}^{\infty} x_n)$. Hence as a byproduct of the proof of Theorem 7.5 we obtain the following.

COROLLARY 7.7. *Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series in a Banach space X, which has* (RP). *Then* $\mathrm{SR}(\sum_{n=1}^{\infty} x_n)$ *is a closed set.*

The referee pointed out that Corollary 7.7 is widely known: see for example [5, Theorem 3.3]. It follows from the fact that (RP) implies that $\mathrm{SR}(\sum_{n=1}^{\infty} x_n) = \mathrm{LPS}(\sum_{n=1}^{\infty} x_n)$, and the latter set is always closed.

Now, we will discuss the problem of whether Corollary 7.7 can be reversed, namely whether or not the closedness of $\mathrm{SR}(\sum_{n=1}^{\infty} x_n)$ implies (RP) for the series $\sum_{n=1}^{\infty} x_n$. We denote by S_n the set of all permutations of the set $[1,n]$.

LEMMA 7.8. *Let $k \in \mathbb{N}$ and $n = \binom{2k}{k}$. Then there exists a finite sequence $x_1, \ldots, x_{2k} \in \mathbb{R}^n$ such that:*

(1) $\|x_i\|_{\sup} = 1$ *for every* $i \le 2k$.
(2) $\|\sum_{i=1}^{k} x_{\sigma(i)}\|_{\sup} \ge k$ *for every* $\sigma \in S_{2k}$.
(3) $\sum_{i=1}^{2k} x_i = 0$.

Proof. Let $k \in \mathbb{N}$. There are $n = \binom{2k}{k}$ sequences of length $2k$ consisting of k many 1's and k many -1's. Enumerate all such sequences as $t_1, \ldots, t_n$. Define $x_i(j) = t_j(i)$ for $j = 1, \ldots, n$ and $i = 1, \ldots, 2k$. Now, if $\sigma \in S_{2k}$, then there is a sequence t_{j_σ} such that $t_{j_\sigma}(\sigma(i)) = 1$ for $i = 1, \ldots, k$ and $t_{j_\sigma}(\sigma(i)) = -1$ for $i = k+1, \ldots, 2k$. Thus

$$\sum_{i=1}^{k} x_{\sigma(i)}(j_\sigma) = k,$$

and consequently

$$\Big\| \sum_{i=1}^{k} x_{\sigma(i)} \Big\|_{\sup} \ge k. \ \blacksquare$$

Before we state the last result, first note that if a series $\sum_{i=1}^{\infty} x_i$ in $\mathbb{R}^m$ does not have (RP), then one can find $\varepsilon > 0$ such that for every $\delta > 0$ and $N \in \mathbb{N}$ there exists a finite subsequence $(y_i)_{i=1}^{n} \subseteq (x_i)_{i=N}^{\infty}$ for which two conditions hold:

- $\|\sum_{i=1}^{n} y_i\| < \delta$;
- for every $\sigma \in S_n$ there is $j \le n$ such that $\|\sum_{i=1}^{j} y_{\sigma(i)}\|_{\sup} \ge \varepsilon$.

The following theorem shows that Corollary 7.7 cannot be reversed.

THEOREM 7.9. *There is a conditionally convergent series $\sum_{n=1}^{\infty} z_n$ in c_0 that does not have* (RP) *and for which* $\mathrm{SR}(\sum_{n=1}^{\infty} z_n)$ *is a singleton, in particular it is a closed set.*

Proof. Define $e_n = (\delta_{in})_{i=1}^{\infty}$ for every $n \in \mathbb{N}$, where $\delta_{in} = 1$ if $i = n$, and $\delta_{in} = 0$ otherwise. Let $n_0 = 0$ and $n_k = \binom{2^{k+1}}{2^k} + n_{k-1}$. For every $k \in \mathbb{N}$ let $x_1^{(k)}, \dots, x_{2^{k+1}}^{(k)} \in \mathbb{R}^{n_k - n_{k-1}}$ be the sequence constructed in Lemma 7.8. Define

$$y_i^{(k)} = \frac{1}{2^k} \cdot \sum_{j=1}^{n_k - n_{k-1}} x_i^{(k)}(j) \cdot e_{n_{k-1}+j} \quad \text{for } i, k \in \mathbb{N}.$$

It is easy to see that $y_i^{(k)} \in c_0$ for all $k, i \in \mathbb{N}$. Define the series $\sum_{n=1}^{\infty} z_n$ as follows:

$$z_1 = y_1^{(1)}, z_2 = -y_1^{(1)}, z_3 = y_2^{(1)}, z_4 = -y_2^{(1)}, z_5 = y_3^{(1)}, z_6 = -y_3^{(1)},$$
$$z_7 = y_4^{(1)}, z_8 = -y_4^{(1)}, z_9 = y_1^{(2)}, z_{10} = -y_1^{(2)}, \dots.$$

It is easy to see that $\sum_{n=1}^{\infty} z_n$ converges to $\theta = (0, 0, \dots)$.

Let $\varepsilon = 1$, $N \in \mathbb{N}$, and $\delta > 0$. One can find $k \in \mathbb{N}$ such that $n_{k-1} > N$. Then by Lemma 7.8 for $(y_1^{(k)}, \dots, y_{2^{k+1}}^{(k)}) \subseteq (z_i)_{i \geq N}$ and every permutation $\sigma \in S_{2^{k+1}}$ we have

$$\left\| \frac{1}{2^k} \cdot \sum_{i=1}^{2^k} y_{\sigma(i)}^{(k)} \right\|_{\sup} = \left\| \frac{1}{2^k} \cdot \sum_{i=1}^{2^k} x_{\sigma(i)}^{(k)} \right\|_{\sup} \geq \frac{1}{2^k} \cdot 2^k = 1 = \varepsilon.$$

Moreover $\| \sum_{i=1}^{2^{k+1}} y_i^{(k)} \| = 0 < \delta$. This proves that the series $\sum_{n=1}^{\infty} z_n$ does not have (RP).

Since the projection of the series on each coordinate contains only finitely many nonzero terms and a finite sum does not change under rearrangements, we have $\mathrm{SR}(\sum_{n=1}^{\infty} z_n) = \{\theta\}$. ∎

Let us finish the paper with a list of open questions.

1. Let $\sum_{n=1}^{\infty} x_n \in \mathbf{\Sigma}_0$ be a series in an infinite-dimensional Banach space. Is there $\sum_{n=1}^{\infty} y_n \in \mathbf{\Sigma}_c$ with $\mathrm{LIM}(\sum_{n=1}^{\infty} y_n) = \mathrm{LIM}(\sum_{n=1}^{\infty} x_n)$?
2. Assume that A is a closed and separable subset of an infinite-dimensional Banach space. Is there $\sum_{n=1}^{\infty} x_n \in \mathbf{\Sigma}_c$ with $A = \mathrm{LIM}(\sum_{n=1}^{\infty} x_n)$? (Note that a positive answer to this question answers affirmatively the first question as well.)
3. Is the assertion of Theorem 3.5 true in some infinite-dimensional topological vector spaces?

Acknowledgements. We would like to thank Filip Strobin for a very careful analysis of this article, shortening the proof of Lemma 3.1 and providing Lemma 3.4. We would also like to thank Robert Stegliński for fruitful discussions on our paper. Finally, we would like to thank the referee for bringing papers [4] and [5] to our attention, noting that we can simplify the proofs in the finite-dimensional case by taking care only of $\lim_{n\to\infty} x_n = 0$

and not of the existence of a convergent rearrangement, and other remarks on the manuscript.

The first author has been supported by the National Science Centre Poland Grant no. DEC-2012/07/D/ST1/02087.

References

[1] R. Engelking, *General Topology*, 2nd ed., Sigma Ser. Pure Math. 6, Heldermann, Berlin, 1989.
[2] M. I. Kadets and V. M. Kadets, *Series in Banach Spaces. Conditional and Unconditional Convergence*, Oper. Theory Adv. Appl. 94, Birkhäuser, Basel, 1997.
[3] V. Klee, *Rearrangements of series of vectors*, Math. Z. 81 (1963), 46–51.
[4] P. Rosenthal, *The remarkable theorem of Lévy and Steinitz*, Amer. Math. Monthly 94 (1987), 342–351.
[5] M. A. Sofi, *Lévy–Steinitz theorem in infinite dimension*, New Zealand J. Math. 38 (2008), 63–73.
[6] S. M. Srivastava, *A Course on Borel Sets*, Grad. Texts in Math. 180, Springer, New York, 1998.
[7] R. Wituła, E. Hetmaniok and K. Kaczmarek, *On series whose rearrangements possess discrete sets of limit points*, J. Appl. Anal. 20 (2014), 93–96.

Szymon Głąb, Jacek Marchwicki
Institute of Mathematics
Łódź University of Technology
Wólczańska 215
93-005 Łódź, Poland
E-mail: szymon.glab@p.lodz.pl
marchewajaclaw@gmail.com