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DOI HOM-HOPF MODULES AND FROBENIUS TYPE PROPERTIES

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Abstract. We continue our study of the category of Doi Hom-Hopf modules introduced by Guo and Zhang [Colloq. Math. 143 (2016), 23–38]. Let (H, A, C) be a Doi Hom-Hopf datum. We find that the forgetful functor $F : \widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C \to \widetilde{\mathscr{H}}(\mathscr{M}_k)_A$ and its adjoint form a Frobenius pair if and only if (among other equivalent conditions) $A \otimes C$ and $C^* \otimes A$ are isomorphic as $(A; C^{*\mathrm{op}} \# A)$-bimodules.

1. Introduction. Hom-algebras and Hom-coalgebras were introduced by Makhlouf and Silvestrov [17] as generalizations of ordinary algebras and coalgebras in the following sense: the associativity of multiplication is replaced by Hom-associativity, and similarly for Hom-coassociativity. They also described the structures of Hom-bialgebras and Hom-Hopf algebras, and extended some important results from ordinary Hopf algebras to Hom-Hopf algebras in [18] and [19]. Recently, many properties and structure results for Hom-Hopf algebras have been developed (see [4], [7], [5], [8], [10], [11], [12], [13], [15] and references cited therein).

Caenepeel and Goyvaerts [1] studied Hom-bialgebras and Hom-Hopf algebras from a categorical view point, and called them monoidal Hom-bialgebras and monoidal Hom-Hopf algebras respectively, using slightly different definitions. Makhlouf and Panaite [16] defined Yetter–Drinfeld modules over Hom-bialgebras and showed that Yetter–Drinfeld modules over a Hom-bialgebra with bijective structure map provide solutions of the Hom-Yang–Baxter equation. Also Liu and Shen [14] studied Yetter–Drinfeld modules over monoidal Hom-bialgebras and called them Hom-Yetter–Drinfeld modules, and showed that the category of Hom-Yetter–Drinfeld modules is a braided monoidal category. Chen and Zhang [7] defined the category of Hom-Yetter–Drinfeld modules in a slightly different way to [14], and shown that it is a full monoidal subcategory of the left center of the left Hom-module category. Guo and Zhang [11] defined the category of Doi Hom-Hopf

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modules and gave a necessary and sufficient condition for the forgetful functor $F : \widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C \to \widetilde{\mathscr{H}}(\mathscr{M}_k)_A$ to be separable. At the same time, they proved that Doi Hom-Hopf modules unify many other types of Hom-module structures, for example Hom-Hopf modules, relative Hom-Hopf modules introduced by [10] or [6] and, last but not least, Hom-Yetter–Drinfeld modules.

In [12], induction functors between categories of Doi Hom-Hopf modules and their adjoints were discussed, and it turned out that many pairs of adjoint functors studied in the literature (the forgetful functor and its adjoint, extension and restriction of scalars, . . .) are special cases.

In this paper, we focus attention on the functor F from the category of entwined Hom-modules to the category of right (A, β)-modules forgetting the (C, γ)-coaction. This functor has a right adjoint $G = C \otimes \bullet$. A natural question is the following: when is G also a left adjoint of F? This is the motivation of this paper.

In this paper we study the generalization of the previous results of [2] or [3] to monoidal Hom-Hopf algebras. In Section 3, we obtain the main result of this paper: in Theorem 3.9 we prove that the forgetful functor $F : \widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C \to \widetilde{\mathscr{H}}(\mathscr{M}_k)_A$ and its adjoint form a Frobenius pair if and only if (among other equivalent conditions) $A\otimes C$ and $C^*\otimes A$ are isomorphic as $(A; C^{*\mathrm{op}} \# A)$-bimodules.

Throughout this paper we freely use the Hopf algebra and coalgebra terminology introduced in [9], [20], [21] and[22].

2. Preliminaries. Throughout this paper we work over a commutative ring k. We recall from [1] some information about Hom-structures which are needed in what follows.

Let $\mathcal{C}$ be a category. We introduce a new category $\widetilde{\mathscr{H}}(\mathcal{C})$ as follows: objects are couples (M, μ) with $M \in \mathcal{C}$ and $\mu \in \mathrm{Aut}_{\mathcal{C}}(M)$. A morphism $f : (M, \mu) \to (N, \nu)$ is a morphism $f : M \to N$ in $\mathcal{C}$ such that $\nu \circ f = f \circ \mu$.

Let $\mathscr{M}_k$ denote the category of k-modules and $\mathscr{H}(\mathscr{M}_k)$ the Hom-category associated to $\mathscr{M}_k$. If $(M, \mu) \in \mathscr{M}_k$, then $\mu : M \to M$ is obviously a morphism in $\mathscr{H}(\mathscr{M}_k)$. It is easy to show that $\widetilde{\mathscr{H}}(\mathscr{M}_k) = (\mathscr{H}(\mathscr{M}_k), \otimes, (I, I), \widetilde{a}, \widetilde{l}, \widetilde{r})$ is a monoidal category by [1, Proposition 1.1]: the tensor product of (M, μ) and (N, ν) in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ is given by the formula $(M, \mu) \otimes (N, \nu) = (M \otimes N, \mu \otimes \nu)$.

Assume that $(M, \mu), (N, \nu), (P, \pi) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)$. The associativity and unit constraints are given by the formulas

$$\widetilde{a}_{M,N,P}((m \otimes n) \otimes p) = \mu(m) \otimes (n \otimes \pi^{-1}(p)),$$
$$\widetilde{l}_M(x \otimes m) = \widetilde{r}_M(m \otimes x) = x\mu(m).$$

An algebra in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ will be called a monoidal Hom-algebra.

Definition 2.1. A *monoidal Hom-algebra* is an object $(A,\alpha)\in\widetilde{\mathscr{H}}(\mathscr{M}_k)$ together with a k-linear map $m_A : A\otimes A\to A$ and an element $1_A\in A$ such that

$$\alpha(ab)=\alpha(a)\alpha(b),\quad a(1_A)=1_A,$$
$$\alpha(a)(bc)=(ab)\alpha(c),\qquad a1_A=1_Aa=\alpha(a),$$

for all $a,b,c\in A$. Here we use the notation $m_A(a\otimes b)=ab$.

Definition 2.2. A *monoidal Hom-coalgebra* is an object $(C,\gamma)\in\widetilde{\mathscr{H}}(\mathscr{M}_k)$ together with k-linear maps $\Delta : C\to C\otimes C$, $\Delta(c)=c_{(1)}\otimes c_{(2)}$ (summation implicitly understood) and $\gamma : C\to C$ such that

$$\Delta(\gamma(c))=\gamma(c_{(1)})\otimes\gamma(c_{(2)}),\quad \varepsilon(\gamma(c))=\varepsilon(c),$$

and

$$\gamma^{-1}(c_{(1)})\otimes c_{(2)(1)}\otimes c_{(2)(2)}=c_{(1)(1)}\otimes c_{(1)(2)}\otimes\gamma^{-1}(c_{(2)}),$$
$$\varepsilon(c_{(1)})c_{(2)}=\varepsilon(c_{(2)})c_{(1)}=\gamma^{-1}(c),$$

for all $c\in C$.

Definition 2.3. A *monoidal Hom-bialgebra* $H=(H,\alpha,m,\eta,\Delta,\varepsilon)$ is a bialgebra in the symmetric monoidal category $\widetilde{\mathscr{H}}(\mathscr{M}_k)$. This means that (H,α,m,η) is a Hom-algebra, (H,Δ,α) is a Hom-coalgebra, and Δ and ε are morphisms of Hom-algebras, that is,

$$\Delta(ab)=a_{(1)}b_{(1)}\otimes a_{(2)}b_{(2)},\quad \Delta(1_H)=1_H\otimes 1_H,$$
$$\varepsilon(ab)=\varepsilon(a)\varepsilon(b),\qquad \varepsilon(1_H)=1_H.$$

Definition 2.4. A *monoidal Hom-Hopf algebra* is a monoidal Hom-bialgebra (H,α) together with a linear map $S : H\to H$ in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ such that

$$S*I=I*S=\eta\varepsilon,\quad S\alpha=\alpha S.$$

Definition 2.5. Let (A,α) be a monoidal Hom-algebra. A *right* (A,α)-*Hom-module* is an object $(M,\mu)\in\widetilde{\mathscr{H}}(\mathscr{M}_k)$ together with a morphism ψ : $M\otimes A\to M$, $\psi(m\cdot a)=m\cdot a$, in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ such that

$$(m\cdot a)\cdot\alpha(b)=\mu(m)\cdot(ab),\quad m\cdot 1_A=\mu(m),$$

for all $a\in A$ and $m\in M$. The fact that $\psi\in\widetilde{\mathscr{H}}(\mathscr{M}_k)$ means that

$$\mu(m\cdot a)=\mu(m)\cdot\alpha(a).$$

A morphism $f:(M,\mu)\to(N,\nu)$ in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ is called *right* A-*linear* if it preserves the A-action, that is, $f(m\cdot a)=f(m)\cdot a$. Let $\widetilde{\mathscr{H}}(\mathscr{M}_k)_A$ denote the category of right (A,α)-Hom-modules and A-linear morphisms.

Definition 2.6. Let (C,γ) be a monoidal Hom-coalgebra. A *right* (C,γ)-*Hom-comodule* is an object $(M,\mu)\in\widetilde{\mathscr{H}}(\mathscr{M}_k)$ together with a k-linear map $\rho_M : M \to M \otimes C$ (we write $\rho_M(m) = m_{[0]} \otimes m_{[1]}$) in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ such that

$$m_{[0][0]} \otimes (m_{[0][1]} \otimes \gamma^{-1}(m_{[1]})) = \mu^{-1}(m_{[0]}) \otimes \Delta_C(m_{[1]}); m_{[0]}\varepsilon(m_{[1]}) = \mu^{-1}(m)$$

for all $m \in M$. The fact that $\rho_M \in \widetilde{\mathscr{H}}(\mathscr{M}_k)$ means that

$$\rho_M(\mu(m)) = \mu(m_{[0]}) \otimes \gamma(m_{[1]}).$$

Morphisms of right (C, γ)-Hom-comodules are defined in the obvious way. The category of right (C, γ)-Hom-comodules will be denoted by $\widetilde{\mathscr{H}}(\mathscr{M}_k)^C$.

Definition 2.7. Let (H, α) be a monoidal Hom-Hopf algebra. A monoidal Hom-algebra (A, β) is called a *right* (H, α)*-Hom-comodule algebra* if (A, β) is a right (H, α) Hom-comodule with coaction $\rho_A : A \to A \otimes H$, $\rho^r_A(a) = a_{[0]} \otimes a_{[1]}$, such that

$$\rho^r_A(ab) = a_{[0]}b_{[0]} \otimes a_{[1]}b_{[1]}, \qquad \rho^r_A(1_A) = 1_A \otimes 1_H,$$

for all $a, b \in A$.

Definition 2.8. Let (H, α) be a monoidal Hom-Hopf algebra. A monoidal Hom-coalgebra (C, γ) is called a *right* (H, α)*-Hom-module coalgebra* if (C, γ) is a right (H, α)-Hom-module with action $\phi : C \otimes H \to C$, $\phi(c \otimes h) = c \cdot h$, such that

$$\Delta(c \cdot h) = c_{(1)} \cdot h_{(1)} \otimes c_{(2)} \cdot h_{(2)}, \qquad \varepsilon(c \cdot h) = \varepsilon(c)\varepsilon(h),$$

for all $c \in C$ and $g, h \in H$.

Definition 2.9. A pair of adjoint functors (F, G) is called a *Frobenius pair* if G is not only a right adjoint, but also a left adjoint of F. The following result can be found in any book on category theory: G is a left adjoint of F if and only if there exist natural transformations $v \in V = \mathbf{Nat}(GF, 1_{\mathcal{C}})$ and $\zeta \in W = \mathbf{Nat}(1_{\mathcal{D}}, FG)$ such that

$$F(v_M) \circ \zeta_{F(M)} = I_{F(M)}, \tag{2.1}$$

$$v_{G(N)} \circ G(\zeta_N) = I_{G(N)}. \tag{2.2}$$

3. Frobenius type properties. Recall from [11] that a *Doi Hom-Hopf module* (M, μ) is a right (A, β)-Hom-module which is also a right (C, γ)-Hom-comodule with the coaction structure $\rho_M : M \to M \otimes C$ defined by $\rho_M(m) = m_{[0]} \otimes m_{[1]}$ such that the following compatibility condition holds: for all $m \in M$ and $a \in A$,

$$\rho_M(m \cdot a) = m_{[0]} \cdot a_{[0]} \otimes m_{[1]}a_{[1]}.$$

A morphism between two right Doi Hom-Hopf modules is a k-linear map which is a morphism in the categories $\widetilde{\mathscr{H}}(\mathscr{M}_k)_A$ and $\widetilde{\mathscr{C}}(\mathscr{M}_k)^C$ at the same time. $\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C_A$ will denote the category of right Doi Hom-Hopf modules and morphisms between them.

We recall from [11] the following useful result.

PROPOSITION 3.1. *The forgetful functor* $F : \widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C \to \widetilde{\mathscr{H}}(\mathscr{M}_k)_A$ *has a right adjoint* $G : \widetilde{\mathscr{H}}(\mathscr{M}_k)_A \to \widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C$ *defined by*

$$G(N) = N \otimes C,$$

with structure maps

$$(n \otimes c) \cdot a = n \cdot a_{[0]} \otimes c \cdot a_{[1]},$$
$$\rho_{G(M)}(n \otimes c) = (\nu^{-1}(n) \otimes c_{(1)}) \otimes \gamma(c_{(2)}),$$

for all $a \in A$ *and* $n \in N$, $c \in C$.

Let V_1 be the k-module consisting of all k-linear maps $\theta : (C, \gamma) \otimes (C, \gamma) \to (A, \beta)$ such that

$$\beta^2(a_{[0][0]})\theta\big(\gamma^{-1}(d) \cdot a_{[0][1]} \otimes \gamma^{-1}(c) \cdot \alpha^{-1}(a_{[1]})\big) = \theta(d \otimes c)a, \tag{3.1}$$

$$\theta(\gamma^{-1}(d) \otimes c_{(1)}) \otimes \gamma(c_{(2)}) = \beta(\theta(d_{(2)} \otimes \gamma^{-1}(c))_{[0]}) \otimes d_{(1)} \cdot \theta(d_{(2)} \otimes \gamma^{-1}(c))_{[1]}. \tag{3.2}$$

PROPOSITION 3.2. *The map* $\Psi : V \to V_1$ *given by* $\Psi(v) = \theta$ *with*

$$\theta(c \otimes d) = \widetilde{r}_A(\mathrm{id}_A \otimes \varepsilon_C)\nu_{A\otimes C}((1_A \otimes c) \otimes \gamma(d)) \tag{3.3}$$

is an isomorphism of k*-modules. The inverse* $\Psi^{-1}(v) = \theta$ *is defined as follows:* $v_M : M \otimes C \to M$ *is given by*

$$v_M(m \otimes c) = \mu(m_{[0]})\theta(m_{[1]} \otimes \gamma^{-1}(c)). \tag{3.4}$$

Proof. In view of the naturality of v we get

$$\begin{aligned}\theta(c \otimes d) &= (\varepsilon_C \otimes \mathrm{id}_A)v_{C\otimes A}((c \otimes 1_A) \otimes \gamma(d)) \\ &= (\mathrm{id}_A \otimes \varepsilon_C)v_{A\otimes C}((1_A \otimes c) \otimes \gamma(d)).\end{aligned}$$

It is not hard to check that $GF(A \otimes C) = (A \otimes C) \otimes C \in {}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C$, the left (A, β)-action is induced by multiplication in (A, β) and $v_{A\otimes C}$ is a morphism in ${}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C$. Thus $v_{A\otimes C}$ and $(\mathrm{id}_A \otimes \varepsilon_C)v_{A\otimes C}$ are left and right (A, β)-linear, and

$$\begin{aligned}\theta(c \otimes d)a &= (\mathrm{id}_A \otimes \varepsilon_C)v_{A\otimes C}((1_A \otimes c) \otimes \gamma(d))a \\ &= (\mathrm{id}_A \otimes \varepsilon_C)v_{A\otimes C}((1_A \otimes c) \otimes \gamma(d))a \\ &= (\mathrm{id}_A \otimes \varepsilon_C)v_{A\otimes C}\big(\beta(a_{[0][0]})1_A \otimes [ca_{[0][1]} \otimes d\alpha^{-1}(a_{[1]})]\big) \\ &= \beta^2(a_{[0][0]})(\mathrm{id}_A \otimes \varepsilon_C)v_{A\otimes C}\big([1_A \otimes \gamma^{-1}(c)a_{[0][1]} \otimes d\alpha^{-1}(a_{[1]})]\big) \\ &= \beta^2(a_{[0][0]})(\mathrm{id}_A \otimes \varepsilon_C)v_{A\otimes C}\big([1_A \otimes \gamma^{-1}(c)a_{[0][1]}] \otimes d\alpha^{-1}(a_{[1]})\big) \\ &= \beta^2(a_{[0][0]})\theta\big(\gamma^{-1}(c)a_{[0][1]} \otimes \gamma^{-1}(d)a_{[0][1]}\big),\end{aligned}$$

proving (3.1).

The verification of (3.2) is more involved. It is not hard to check that $GF(C\otimes A)=C\otimes(C\otimes A)\in {}^C\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C_A$, and the left (C,γ)-coaction of $C\otimes A$ is given by

$$c\otimes a\mapsto \gamma(c_{(1)})\otimes(c_{(2)}\otimes\beta^{-1}(a)).$$

Moreover $\nu_{C\otimes A}:(C\otimes A)\otimes C\to C\otimes A$ is a morphism in ${}^C\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C_A$. Hence, $\nu_{C\otimes A}$ is left and right (C,γ)-colinear. Take $c,d\in C$, and set

$$\nu_{C\otimes A}((c\otimes 1_A)\otimes d)=\sum_i p_i\otimes b_i\in C\otimes A.$$

Writing down the condition that $\nu_{C\otimes A}$ is left (C,γ)-colinear, and then applying ε_C to the second factor, we find that

$$\gamma^2(c_{(1)})\otimes\theta(c_{(2)}\otimes\gamma^{-1}(d))=\sum_i c_i\otimes a_i.$$

Since $\nu_{C\otimes A}$ is also right (C,γ)-colinear,

$$v_{C\otimes A}\big(\gamma^{-1}(c)\otimes 1_A\otimes\gamma(d_{(1)})\big)\otimes\gamma^2(d_{(2)})=c_{i(1)}\otimes(a_{i[0])}\otimes c_{i(2)}a_{i[1]},$$

and, applying ε_C to the second factor, we obtain

$$\theta(\gamma^{-1}(c)\otimes d_{(1)})\otimes\gamma^2(d_{(2)})=\beta(a_{i[0]})\otimes\gamma^{-3}(c_i)\alpha^{-2}(a_{i[1]}).$$

Hence, (3.2) follows. This proves that there is a well-defined map $\Psi:V\to V_1$.

To show that Ψ^{-1} determined by (3.4) is well-defined, it is sufficient to show that $v_M\in\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C_A$, i.e., v_M is right (A,β)-linear and right (C,γ)-colinear, and that v is a natural transformation. The proof follows the arguments in [11, proof of Theorem 4.2].

Given any morphism $f:M\to N$ in $\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C_A$, one easily checks that for all $m\in M$ and $c\in C$, we have

$$\begin{aligned}v_N(f(m)\otimes c)&=f(\mu(m_{[0]}))\theta(m_{[1]}\otimes\gamma^{-1}(c))\\&=f\big(\mu(m_{[0]})\theta(m_{[1]}\otimes\gamma^{-1}(c))\big)=f(v_M(m\otimes c)),\end{aligned}$$

i.e., v is natural. The verification that Ψ and Ψ^{-1} are inverses of each other is left to the reader. ∎

Now we give a description of $W=\mathbf{Nat}(1_{\mathcal{M}_A},FG)$. Let

(3.5) $$W_1=\{z\in A\otimes C\mid az=za \text{ for all } a\in A,\ (\beta\otimes\gamma)(z)=z\},$$

i.e., $z=\sum_l a_l\otimes c_l\in W_1$ if and only if

(3.6) $$\sum_l\beta^{-1}(a)a_l\otimes\gamma(c_l)=\sum_l a_la_{[0]}\otimes c_l^{\psi}a_{[1]},\qquad(\beta\otimes\gamma)(z)=z.$$

Proposition 3.3. *Let (H,A,C) be a Doi Hom-Hopf datum. Then there is an isomorphism $\Phi:W\to W_1$ of k-modules given by*

(3.7) $$\Phi(\zeta)=\zeta(1_A).$$

The inverse of Φ *is* $\Phi^{-1}(\sum_l a_l \otimes c_l) = \zeta$ *with* $\zeta_N : N \to N \otimes C$ *given by*

$$\zeta_N(n) = \sum_l \nu^{-1}(n)a_l \otimes \gamma(c_l)$$

for any $(N,\nu) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C$ *and* $n \in N$.

Proof. Apply the fact that ζ_A is left and right (A,β)-linear. The details are left to the reader. ∎

THEOREM 3.4. *Let* $F : \widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C \to \widetilde{\mathscr{H}}(\mathscr{M}_k)_A$ *be the forgetful functor, and* $G : \widetilde{\mathscr{H}}(\mathscr{M}_k)_A \to \widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C$ *its adjoint. Then* F *is separable if and only if there exists* $\theta \in V_1$ *such that*

$$\theta \circ \Delta_C = \varepsilon_C,$$

and G *is separable if and only if there exists* $z = \sum_l a_l \otimes c_l \in W_1$ *such that*

$$\sum_l \varepsilon_C(c_l)a_l = 1_A.$$

Proof. This follows immediately from Propositions 3.2 and 3.3. ∎

Next we show that the fact that (F,G) is a Frobenius pair is also equivalent to the existence of $\theta \in V_1$ and $z \in W_1$, but now satisfying different normalizing conditions.

THEOREM 3.5. *Let* $F : \widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C \to \widetilde{\mathscr{H}}(\mathscr{M}_k)_A$ *be the forgetful functor, and* $G : \widetilde{\mathscr{H}}(\mathscr{M}_k)_A \to \widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C$ *its adjoint. Then* F *is separable if and only if there exist* $\theta \in V_1$ *and* $z = \sum_l a_l \otimes c_l \in W_1$ *such that the following normalizing condition holds for all* $d \in C$:

$$\varepsilon_C(d)1_A = \sum_l \beta(a_l)\theta(c_l \otimes \gamma^{-2}(d)) \tag{3.8}$$

$$= \beta(a_{l[0]})\theta(\gamma^{-2}(d)\alpha^{-1}(a_{l[1]}) \otimes \gamma^{-1}(c_l)). \tag{3.9}$$

Proof. Suppose that (F,G) is a Frobenius pair. Then there exist $v \in V$ and $\zeta \in W$ such that (2.1)–(2.2) hold. Let $\theta = \Psi(v) \in V_1$ and $z = \sum_l a_l \otimes c_l = \Phi(\zeta) \in W_1$. Then (2.1) can be rewritten as

$$v_M\Big(\sum_l \mu^{-1}(m)a_l \otimes \gamma(c_l)\Big) = (m_{[0]} \cdot \beta(a_{l[0]})) \cdot \theta(\gamma^{-1}(m_{[1]})a_{[1]} \otimes c_l) = m$$

for any $m \in M \in \widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C$. Taking $M = C \otimes A, m = d \otimes 1_A$, we have

$$\begin{aligned} d \otimes 1_A &= v_{C\otimes A}\Big(\sum_l \mu^{-1}(d \otimes 1_A)a_l \otimes \gamma(c_l)\Big) \\ &= \big((d \otimes 1_A)_{[0]} \cdot \beta(a_{l[0]})\big) \cdot \theta\big(\gamma^{-1}(d \otimes 1_A)_{[1]}a_{[1]} \otimes c_l\big) \\ &= \big((d_{(1)} \otimes 1_A) \cdot \beta(a_{l[0]})\big) \cdot \theta\big(\gamma^{-1}(d_{(2)})a_{l[1]} \otimes c_l\big) \\ &= \gamma^2(d_{(1)}) \otimes \big[\beta(a_{l[0]})\theta\big(\gamma^{-2}(d_{(2)})\alpha^{-1}(a_{l[1]}) \otimes \gamma^{-1}(c_l)\big)\big], \end{aligned}$$

thus

$$\begin{aligned}\varepsilon_C(d)1_A &= \widetilde{l}_A\big(\varepsilon_C(\gamma^2(d_{(1)}))\otimes\big[\beta(a_{l[0]})\theta\big(\gamma^{-2}(d_{(2)})\alpha^{-1}(a_{l[1]})\otimes\gamma^{-1}(c_l)\big)\big]\big)\\ &= \beta^2(a_{l[0]})\theta\big(\varepsilon_C(d_{(1)})\gamma^{-1}(d_{(2)})a_{l[1]}\otimes c_l\big)\\ &= \beta^2(a_{l[0]})\theta(\gamma^{-2}(d)a_{l[1]}\otimes c_l).\end{aligned}$$

Hence, (3.9) follows.

For all $n\in N\in\widetilde{\mathscr{H}}(\mathscr{M}_k)_A$ and $c\in C$, one has

$$\begin{aligned}&v_{G(N)}(G(\zeta_N)(n\otimes d)) = v_{G(N)}\big(\nu^{-1}(n)a_l\otimes\gamma(c_l)\otimes d\big)\\ &\quad= \nu^{-1}(n)a_l\otimes\gamma^2(c_{l(1)})\theta\big(\gamma^2(c_{l(2)})\otimes\gamma^{-1}(d)\big)\\ &\quad= n\big(a_l\beta\theta(c_{l(2)}\otimes\gamma^{-3}(d))_{[0]}\big)\otimes\gamma^3\big(\gamma^{-1}(c_{l(1)})\theta(c_{l(2)}\otimes\gamma^{-3}(d))_{[1]}\big)\\ &\quad= n(a_l\theta\big(\gamma^{-1}(c_{l(1)})\otimes\gamma^{-2}(d_{(1)})\big)\otimes\gamma(d_{(2)}) = n\otimes d,\end{aligned}$$

and (2.2) can be written as

$$n(a_l\theta\big(\gamma^{-1}(c_{l1})\otimes\gamma^{-2}(d_{(1)})\big)\otimes\gamma(d_{(2)}) = n\otimes d.$$

Taking $N=A$ and $n=1_A$, one obtains

$$\beta(a_l)\theta(c_{l1}\otimes\gamma^{-1}(d_{(1)}))\otimes\gamma(d_{(2)}) = 1_A\otimes d.$$

Applying ε_C to the second factor, we obtain (3.8). ■

In [11] it is shown that if (H,A,C) is a Doi Hom-Hopf datum, (A,β) is faithfully flat as a k-module, and (C,γ) is projective as a k-module, then (C,γ) is finitely generated. The next proposition shows that, in fact, one does not need the assumption that (C,γ) is projective.

PROPOSITION 3.6. *Let (H,A,C) be a Doi Hom-Hopf datum. If (F,G) is a Frobenius pair, then $A\otimes C$ is finitely generated and projective as a left (A,β)-Hom-module.*

Proof. Let θ and $z=\sum_l a_l\otimes c_l$ be as in Theorem 3.5. Then for $d\in C$,

$$\begin{aligned}1_A\otimes d &= \psi(d\otimes 1_A) = \psi(\gamma(d_1)\otimes\varepsilon(d_2)1_A)\\ &= \psi\Big(\gamma(d_1)\otimes\beta(a_{l[0]})\theta\big(\gamma^{-2}(d_2)\alpha^{-1}(a_{l[1]})\otimes\gamma^{-1}(c_l)\big)\Big)\\ &= \beta^2(a_{l[0][0]})\beta\theta\big(\gamma^{-2}(d_2)\alpha^{-1}(a_{l[1]})\otimes\gamma^{-1}(c_l)\big)_{[0]}\\ &\qquad\otimes d_1\beta(a_{l[0][1]})\theta(\gamma^{-2}(d_2)\alpha^{-1}(a_{l[1]}))\\ &= \beta(a_{l[0]})\theta\big(\gamma^{-3}(d)\gamma^{-2}(a_{l[1]})\otimes c_{l1}\big)\otimes\gamma(c_{l2}).\end{aligned}$$

Write $c_{ll}\otimes c_{l2}=\sum_{j=1}^{m_l}c_{lj}\otimes c'_{lj}$, and for all l,j consider the map

$$\sigma_{lj}:A\otimes C\to A,\qquad \sigma_{lj}(a\otimes d)=\beta^{-1}(a)\big[\beta(a_{l[0]})\theta\big(\gamma^{-3}(d)\alpha^{-2}(a_{l[1]})\otimes c_{lj}\big)\big].$$

Then for all $a\in A$ and $d\in C$,

$$a\otimes d=\sigma_{lj}(a\otimes d)(1\otimes c'_{lj}),$$

so $\{\sigma_{lj}, 1\otimes c'_{lj} \mid l=1,\ldots,n, j=1,\ldots,m_l\}$ is a finite dual basis for $A\otimes C$ as a left (A,β)-Hom-module. ∎

In some situations, one can conclude that (C,γ) is finitely generated and projective as a k-module.

Assume that (C,γ) is finitely generated and projective as a k-module, and let $\{d_i, d_i^* \mid l=1,\ldots,m\}$ be a finite dual basis for (C,γ). Then $C^*\otimes A$ can be made into an object of ${}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C$ as follows: for all $a,b,b'\in A$ and $c^*\in C^*$,

$$b(c^*\otimes a)=\sum_i\langle c^*,\gamma^{-1}(d_i)b_{[1]}\rangle\gamma^*(d_i^*)\otimes b_{[0]}a, \tag{3.10}$$

$$(c^*\otimes a)b'=\gamma^*(c^*)\otimes a\beta^{-1}(b'), \tag{3.11}$$

$$\rho^r(c^*\otimes a)=\sum_i\gamma^{*-1}(d_i^*)*\gamma^{*-2}(c^*)\otimes a_{[0]}\otimes\gamma^{-1}(d_i)a_{[1]}. \tag{3.12}$$

This can be checked directly. The map $\lambda: C\otimes A\otimes C\to A$ induces $\overline{\phi}: A\otimes C\to C^*\otimes A$. This is the map we need. At some place it is convenient to use $C^*\otimes A$ as the image space. Note that $\overline{\phi}$ is given by

$$\overline{\phi}(a\otimes c)=\gamma^*(d_i^*)\otimes a_{[0]}\theta\big(\gamma^{-2}(d_i)\alpha^{-1}(a_{[1]})\otimes\gamma^{-2}(c)\big). \tag{3.13}$$

It turns out that $\overline{\phi}$ is a morphism in ${}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C$. Let V_2 be the k-module consisting of all left (A,β)-linear, right (A,β)-linear, (C,γ)-colinear maps $\overline{\phi}: A\otimes C\to C^*\otimes A$. More specifically, one has

PROPOSITION 3.7. *Let (H,A,C) be a Doi Hom-Hopf datum, and assume that (C,γ) is finitely generated projective as a k-module. Then*

$$V\cong V_1\cong V_2.$$

The isomorphism is $\alpha_1: V_1\to V_2$ with $\alpha_1(\theta)=\overline{\phi}$ given by (4.13). The inverse of α_1 is

$$\alpha_1^{-1}(\overline{\phi})(d\otimes c)=\overline{\phi}(1_A\otimes c)d.$$

Proof. We first show that $\overline{\phi}\in V_2$. For all $a,b\in A$ and $c\in C$, we have

$$\begin{aligned}
b\overline{\phi}(a\otimes c)&=b\Big(\gamma^*(d_i{}^*)\otimes\big[a_{[0]}\theta\big(\gamma^{-2}(d_i)\alpha^{-1}(a_{[1]})\otimes\gamma^{-2}(c)\big)\big]\Big)\\
&=\sum\langle\gamma^*(d_i^*),\gamma^{-1}(d_j)b_{[1]}\rangle\gamma^*(d_j^*)\otimes b_{[0]}\Big(a_{[0]}\theta\big(\gamma^{-2}(d_i)\alpha^{-1}(a_{[1]})\otimes\gamma^{-2}(c)\big)\Big)\\
&=\sum\gamma^*(d_i^*)\otimes b_{[0]}\Big(a_{[0]}\theta\big(\gamma^{-3}(d_i)\alpha^{-2}((\beta^{-1}(b)a)_{[1]})\otimes\gamma^{-2}(c)\big)\Big)\\
&=\sum\gamma^*(d_i^*)\otimes(\beta^{-1}(b_{[0]})a_{[0]})\theta\big(\gamma^{-2}(d_i)\alpha^{-1}((\beta^{-1}(b)a)_{[1]})\otimes\gamma^{-1}(c)\big)\\
&=\sum\gamma^*(d_i^*)\otimes(\beta^{-1}(b)a_{[0]})_{[0]}\theta\big(\gamma^{-2}(d_i)\alpha^{-1}((\beta^{-1}(b)a)_{[1]})\otimes\gamma^{-1}(c)\big)\\
&=\overline{\phi}(\beta^{-1}(b)a\otimes\gamma(c))=\overline{\phi}(b\cdot(a\otimes c)),
\end{aligned}$$

proving that $\overline{\phi}$ is left (A,β)-linear. It is also right (A,β)-linear because

$$\begin{aligned}
\overline{\phi}(a\otimes c)b &= \sum \gamma^{*2}(d_i^*)\otimes \big[a_{[0]}\theta\big(\gamma^{-2}(d_i)\alpha^{-1}(a_{[1]})\otimes\gamma^{-2}(c)\big)\big]\beta^{-1}(b)\\
&= \sum \gamma^{*2}(d_i^*)\otimes \beta(a_{[0]})\big[\theta\big(\gamma^{-2}(d_i)\alpha^{-1}(a_{[1]})\otimes\gamma^{-2}(c)\big)\beta^{-2}(b)\big]\\
&= \sum \gamma^{*2}(d_i^*)\otimes [a_{[0]}\beta(\beta^{-1}(b_{[0][0]}))]\theta\big(\gamma([\gamma^{-4}(d_i)\alpha^{-2}(a_{[1]})]\alpha^{-1}(b_{[0][1]}))\\
&\qquad\qquad\qquad\qquad\qquad\qquad \otimes\ \gamma(\gamma^{-3}(c)\alpha^{-3}(b_{[1]}))\big)\\
&= \sum \gamma^{*2}(d_i^*)\otimes [a_{[0]}b_{[0][0]}]\theta\big(\gamma^{-2}(d_i)\alpha^{-1}(a_{[1]}b_{[0][1]})\otimes\gamma^{-2}(c)\alpha^{-2}(b_{[1]})\big)\\
&= \sum \gamma^{*2}(d_i^*)\otimes (ab_{[0]})_{[0]}\theta\big(\gamma^{-2}(d_i)\alpha^{-1}((ab_{[0]})_{[1]})\otimes\gamma^{-2}(c)\alpha^{-2}(b_{[1]})\big)\\
&= \overline{\phi}(ab_{[0]}\otimes cb_{[1]}) = \overline{\phi}((a\otimes c)b).
\end{aligned}$$

Notice that the dual basis for (C,γ) satisfies the equality

$$\sum \Delta(d_i)\otimes d_i^* = \sum d_i\otimes d_j\otimes d_i^* * d_j^*.$$

Using this equality one computes

$$\begin{aligned}
\rho^r(\overline{\phi}(a\otimes c)) &= \sum \rho^r\Big(\gamma^*(d_i^*)\otimes a_{[0]}\theta\big(\gamma^{-2}(d_i)\alpha^{-1}(a_{[1]})\otimes\gamma^{-2}(c)\big)\Big)\\
&= \sum \Big(\gamma^{*-1}(d_i^*)\otimes\beta^{-1}\big(a_{[0]}\theta(\gamma^{-1}(d_i)a_{[1]}\otimes\gamma^{-1}(c))_{[0]}\big)\\
&\qquad \otimes d_{i(1)}\theta(\gamma^{-1}(d_i)a_{[1]}\otimes\gamma^{-1}(c))_{[1]}\Big)\\
&= \sum \gamma^{*-1}(d_i^*)\otimes a_{[0][0]}\theta\big(\gamma^{-2}(d_{i(2)})\alpha^{-2}(a_{[1]})\otimes\gamma^{-2}(c)\big)_{[0]}\\
&\qquad \otimes\gamma^2\Big(\gamma^{-3}(d_{i(1)})\alpha^{-2}(a_{[0][1]})\theta\big(\gamma^{-1}(d_{i(2)})\alpha^{-1}(a_{[1]})\otimes\gamma^{-1}(c)\big)\Big)_{[1]}\\
&= \sum \Big(\gamma^{*-1}(d_i^*)\otimes\beta^{-1}(a_{[0]})\theta\big(\gamma^{-2}(d_i)\alpha^{-1}(a_{[1]})\otimes\gamma^{-2}(c)\big)_{[0]}\Big)\\
&\qquad \otimes\gamma\Big(\gamma\big(\gamma^{-3}(d_i)\alpha^{-2}(a_{[1]})\big)_{[1]}\theta\big(\gamma^{-1}(d_ia_{[1]})_{(2)}\otimes\gamma^{-1}(c)\big)\Big)_{[1]}\\
&= \sum \Big(\gamma^{*-1}(d_i^*)\otimes\beta^{-1}(a_{[0]})\theta\big(\gamma^{-3}(d_i)\alpha^{-2}(a_{[1]})\otimes\gamma^{-3}(c_{(1)})\big)\Big)\otimes\gamma(c_{(2)})\\
&= \sum \Big(\gamma^{*}(d_i^*)\otimes\beta^{-1}(a_{[0]})\theta\big(\gamma^{-2}(d_i)\alpha^{-2}(a_{[1]})\otimes\gamma^{-2}(c_{(1)})\big)\Big)\otimes\gamma(c_{(2)})\\
&= \sum \overline{\phi}(\beta^{-1}(a)\otimes c_{(1)})\otimes\gamma(c_{(2)}).
\end{aligned}$$

This proves that $\overline{\phi}$ is right C-colinear. Conversely, given $\phi\in V_2$, first one needs to show that $\theta=\alpha_1^{-1}(\overline{\phi})\in V_1$. It is now more convenient to work with $\mathrm{Hom}(C,A)$ rather than $C^*\otimes A$. For $f\in\mathrm{Hom}(C,A)$ and $b\in A$, we have

$$b_{[0]}f(\gamma^{-2}(c)\alpha^{-1}(b_{[1]})) = (b\cdot f)(c),\qquad f(\gamma^{-1}(c))b = (f\cdot b)(c).$$

Take any $c, d \in C$ and $a \in A$ and compute

$$\begin{aligned}\theta(c\otimes d)a &= (\overline{\phi}(1_A\otimes d)c)a = (\overline{\phi}(1_A\otimes d)a)\gamma(c) = \sum \overline{\phi}(\beta(a_{[0]})\otimes da_{[1]})\gamma(c)\\ &= \sum \big(\beta(a_{[0]})\overline{\phi}(1_A\otimes \gamma^{-1}(da_{[1]}))\gamma(c)\big)\\ &= \sum \beta^2(a_{[0][0]})\big(\overline{\phi}(1_A\otimes \gamma^{-1}(da_{[1]}))\gamma^{-1}(c\alpha(a_{[0][1]}))\big)\\ &= \sum \beta^2(a_{[0][0]})\Big(\overline{\phi}\big(\gamma^{-1}(c\alpha(a_{[0][1]}))\otimes \gamma^{-1}(da_{[1]})\big)\Big),\end{aligned}$$

thus (3.1) holds. Before proving (3.2), we write $\rho^r(f) = f_{[0]}\otimes f_{[1]}$ for $f = c^*\otimes a \in \mathrm{Hom}(C, A) \cong C^*\otimes A$, and we find that for all $c \in C$,

$$\begin{aligned}f_{[0]}(c)\otimes f_{[1]} &= \sum \big(\gamma^{*-1}(d_i^*) * \gamma^{*-2}(c^*)\otimes a_{[0]}\big)(c)\otimes d_i a_{[1]}\\ &= \sum \langle \gamma^{*-1}(d_i^*), c_1\rangle\langle \gamma^{*-2}(c^*), c_{(2)}\rangle \beta(a_{[0]})\otimes d_i a_{[1]}\\ &= \sum \langle c^*, \gamma^2(c_{(2)})\rangle \beta(a_{[0]})\otimes \gamma(c_{(1)})\alpha^2(a_{[1]})\\ &= \langle c^*, \gamma^2(c_{(2)})\rangle \psi(\gamma^2(c_{(1)})\otimes a) = \psi\big(\gamma^2(c_{(1)})\otimes f(\gamma(c_{(2)}))\big).\end{aligned}$$

Explicitly, we have

$$\begin{aligned}&\theta(d_{(2)}\otimes \gamma^{-1}(c))_{[0]}\otimes d_{(1)}\theta(\gamma(d_{(2)})\otimes c)_{[1]}\\ &\qquad = \psi\big(\gamma(d_{(1)})\otimes \overline{\phi}(1_A\otimes \gamma^{-1}(c))(d_{(2)})\big)\\ &\qquad = \psi(1_A\otimes \gamma^{-1}(c))_0(\gamma^{-1}(d))\otimes \psi(1_A\otimes \gamma^{-1}(c))_{(1)}\\ &\qquad = \psi(1_A\otimes \gamma^{-1}(c_{(1)}))(\gamma^{-1}(d))\otimes c_{(2)}\\ &\qquad = \theta(\gamma^{-1}(d)\otimes \gamma^{-1}(c_{(1)}))\otimes c_{(2)}.\end{aligned}$$

It remains to show that α_1 and α_1^{-1} are inverse to each other. For this purpose, take $\theta \in V_1$ and note that for all $c, d \in C$, we have

$$\begin{aligned}\alpha_1^{-1}(\alpha(\theta))(d\otimes c) &= \alpha_1(\theta)(1_A\otimes c)(d)\\ &= \big(\gamma^*(d_i^*)\otimes 1_A\theta(\gamma^{-1}(d_i)\otimes \gamma^{-2}(c))\big)(d)\\ &= \langle \gamma^*(d_i^*), d\rangle \theta(\gamma(d_i)\otimes c) = \theta(d\otimes c).\end{aligned}$$

Finally, for $\overline{\phi} \in V_2$, and $a \in A$, $c, d \in C$,

$$\begin{aligned}&\alpha_1(\alpha^{-1}(\overline{\phi}))(a\otimes c)(d)\\ &\quad = \sum \Big(\gamma^*(d_i^*)\otimes a_{[0]}\alpha_1^{-1}(\overline{\phi})(\gamma^{-2}(d_i)\alpha^{-1}(a_{[1]})\otimes \gamma^{-2}(c))\Big)(d)\\ &\quad = \sum \langle d_i^*, \gamma^{-1}(d)\rangle \beta(a_{[0]})\alpha_1^{-1}(\overline{\phi})(\gamma^{-2}(d_i)\alpha^{-1}(a_{[1]})\otimes \gamma^{-2}(c))\\ &\quad = \sum \beta(a_{[0]})\alpha_1^{-1}(\overline{\phi})(\gamma^{-1}(d)a_{[1]}\otimes \gamma^{-1}(c))\\ &\quad = \sum \beta(a_{[0]})\alpha_1^{-1}(\overline{\phi})(1_A\otimes \gamma^{-1}(c))\gamma^{-2}(d)\alpha^{-1}(a_{[1]})\\ &\quad = a\cdot \overline{\phi}(1_A\otimes \gamma^{-1}(c))d = \overline{\phi}(a\otimes c)d. \quad \blacksquare\end{aligned}$$

Now we give an alternative description for W_2.

PROPOSITION 3.8. *Let (C,γ) be finitely generated and projective as a k-module. Then*

$$W \cong W_1 \cong W_2 = \mathrm{Hom}^{kA}_{AA}(C^* \otimes A, A \otimes C).$$

The isomorphism $\beta_1 : W_1 \to W_2$ is given by $\beta_1(z) = \phi$ with

$$\phi(c^* \otimes a) = \sum_l a_l a_{[0]} \otimes c_{l(1)} a_{[1]} \langle \gamma^{*-2}(c^*), c_{l(2)}\rangle,$$

and the inverse of β_1 is given by

$$\beta_1^{-1}(\phi) = \phi(\varepsilon \otimes 1).$$

Proof. We have to show that $\beta_1(z) = \phi$ is left and right (A,β)-linear and right (C,γ)-colinear. For all $c^* \in C^*$ and $a, b \in A$,

$$\begin{aligned}
\phi((c^* \otimes a)b) &= \phi(\gamma^*(c^*) \otimes a\beta^{-1}(b)) \\
&= \sum_l a_l (a\beta^{-1}(b))_{[0]} \otimes c_{l(1)} (a\beta^{-1}(b))_{[1]} \langle \gamma^{*-1}(c^*), c_{l(2)}\rangle \\
&= \sum_l a_l [a_{[0]}\beta^{-1}(b)_{[0]}] \otimes c_{l(1)} [a_{[1]}\beta^{-1}(b)_{[1]}] \langle \gamma^{*-1}(c^*), c_{l(2)}\rangle \\
&= \sum_l [\beta^{-1}(a_l) a_{[0]}] b_{[0]} \otimes [\gamma^{-1}(c_{l(1)}) a_{[1]}] b_{[1]} \langle \gamma^{*-1}(c^*), c_{l(2)}\rangle \\
&= \sum_l [\beta^{-1}(a_l) a_{[0]} \otimes \gamma^{-1}(c_{l(1)}) a_{[1]} \langle \gamma^{*-1}(c^*), c_{l(2)}\rangle] \cdot b \\
&= \sum_l [a_l a_{[0]} \otimes c_{l(1)} a_{[1]} \langle \gamma^{*-1}(c^*), \gamma(c_{l(2)})\rangle] \cdot b \\
&= \sum_l [a_l a_{[0]} \otimes c_{l(1)} a_{[1]} \langle \gamma^{*-2}(c^*), c_{l(2)}\rangle] \cdot b = \phi(c^* \otimes a)b,
\end{aligned}$$

proving that ϕ is right (A,β)-linear. The proof of left (A,β)-linearity goes as follows:

$$\begin{aligned}
\phi(b(c^* \otimes a)) &= \phi\big(\langle c^*, \gamma^{-1}(d_i) b_{[1]}\rangle \gamma^*(d_i^*) \otimes b_{[0]} a\big) \\
&= \sum_l \langle c^*, \gamma^{-1}(d_i) b_{[1]}\rangle a_l (b_{[0]} a)_{[0]} \otimes c_{l(1)} (b_{[0]} a)_{[1]} \langle \gamma^{*-1}(d_i^*), c_{l(2)}\rangle \\
&= \sum_l \langle c^*, \gamma^{-1}(d_i) b_{[1]}\rangle a_l [b_{[0][0]} a_{[0]}] \otimes c_{l(1)} [b_{[0][1]} a_{[1]}] \langle \gamma^{*-1}(d_i^*), c_{l(2)}\rangle \\
&= \sum_l \langle c^*, [c_l \alpha(b_{[1]})]_{(2)}\rangle a_l [\beta^{-1}(b_{[0]}) a_{[0]}] \otimes [\gamma^{-1}(c_l) b_{[1]}]_{(1)} \alpha(a_{[1]}) \\
&= \sum_l \langle c^*, [c_l \alpha(b_{[1]})]_{(2)}\rangle [\beta^{-1}(a_l) \beta^{-1}(b_{[0]})] \beta(a_{[0]}) \otimes [\gamma^{-1}(c_l) b_{[1]}]_{(1)} \alpha(a_{[1]})
\end{aligned}$$

$$= \sum_l \langle c^*, \gamma(c_{l(2)})\rangle [\beta^{-1}(a_l)\beta^{-2}(b)]\beta(a_{[0]}) \otimes c_{l(1)}\alpha(a_{[1]})$$
$$= \sum_l \langle \gamma^{*-1}(c^*), c_{l(2)}\rangle \beta^{-1}(b)[\beta^{-1}(a_l)a_{[0]}] \otimes c_{l(1)}\alpha(a_{[1]})$$
$$= b \cdot \Big[\sum_l \langle \gamma^{*-1}(c^*), c_{l(2)}\rangle \beta^{-1}(a_l)a_{[0]} \otimes \gamma^{-1}(c_{l(1)})a_{[1]}\Big]$$
$$= b \cdot \Big[\sum_l a_l a_{[0]} \otimes c_{l(1)}a_{[1]}\langle \gamma^{*-2}(c^*), c_{l(2)}\rangle\Big] = b \cdot \phi(c^* \otimes a).$$

Next one needs to show that ϕ is right (C,γ)-colinear:

$$\phi((c^*\otimes a)_{[0]}) \otimes (c^*\otimes a)_{[1]} = \phi(\gamma^{*-1}(d_i^*) * \gamma^{*-2}(c^*) \otimes a_{[0]}) \otimes \gamma^{-1}(d_i)a_{[1]})$$
$$= \sum_{i,l} a_l a_{[0][0]} \otimes c_{l(1)}a_{[0][1]}\langle \gamma^{*-2}(\gamma^{*-1}(d_i^*) * \gamma^{*-2}(c^*)), c_{l(2)}\rangle \otimes \gamma^{-1}(d_i)a_{[1]}$$
$$= \sum_{i,l} a_l a_{[0][0]} \otimes c_{l(1)}a_{[0][1]}\langle \gamma^{*-3}(d_i^*), c_{l(2)(1)}\rangle\langle \gamma^{*-4}(c^*), c_{l(2)(2)}\rangle \otimes \gamma^{-1}(d_i)a_{[1]})$$
$$= \sum_{i,l} a_l\beta^{-1}(a_{[0]}) \otimes c_{l(1)}a_{[1](1)}\langle \gamma^{*-4}(c^*), c_{l(2)(2)}\rangle \otimes \gamma^2(c_{l(2)(1)})\alpha(a_{[1](2)})$$
$$= \sum_{i,l} a_l\beta^{-1}(a_{[0]}) \otimes \gamma(c_{l(1)(1)})a_{[1](1)}\langle \gamma^{*-3}(c^*), c_{l(2)}\rangle \otimes \gamma^2(c_{l(2)(1)})\alpha(a_{[1](2)})$$
$$= \sum_{i,l} \beta^{-1}(a_l)\beta^{-1}(a_{[0]}) \otimes c_{l(1)(1)}a_{[1](1)}\langle \gamma^{*-2}(c^*), c_{l(2)}\rangle \otimes \gamma(c_{l(2)(1)})\alpha(a_{[1](2)})$$
$$= \rho^r(\phi((c^*\otimes a))).$$

Conversely, let $\phi \in W_2$ and set $z = \phi(\varepsilon \otimes 1_A) = \sum_l a_l \otimes c_l$. We see that $a(\varepsilon\otimes 1_A) = (\varepsilon \otimes 1_A)a$ for all $a \in A$, hence

$$az = a\phi(\varepsilon\otimes 1_A) = \phi(a(\varepsilon\otimes 1_A)) = \phi((\varepsilon\otimes 1_A)a) = \phi(\varepsilon\otimes 1_A)a = za,$$

and $z \in W_1$. Take $z = \sum_l a_l \otimes c_l \in W_1$. Then

$$\beta_1^{-1}(\beta_1(z)) = \beta_1^{-1}(\phi) = \phi(\varepsilon\otimes 1_A)$$
$$= \sum_l a_l\beta^{-1}(1_A) \otimes \gamma^2(c_{l(1)})\langle \varepsilon, c_{l(2)}\rangle$$
$$= \sum_l \beta(a_l) \otimes \gamma^2(c_{l(1)})\langle \varepsilon, c_{l(2)}\rangle = \sum_l \beta(a_l)\otimes\gamma(c_l) = z.$$

Finally, take $\phi \in W_2$, and write $\beta_1^{-1}(\phi) = \phi(\varepsilon\otimes 1_A) = \sum_l a_l\otimes c_l$. Now, $C^*\otimes A$ and $A\otimes C$ are right (C,γ)-Hom-comodules and left (C^*,γ^*)-Hom-modules. Since ϕ is right (A,β)-linear, right (C,γ)-colinear and left (C^*,γ^*)-linear,

$$
\begin{aligned}
\phi(c^* \otimes a) &= \phi(\gamma^{*-1}(c^*) \otimes 1_A)a = [\gamma^{*-1}(c^*)\phi(\varepsilon \otimes 1_A)]a \\
&= \Big[\gamma^{*-1}(c^*) \cdot \sum_l a_l \otimes c_l\Big]a \\
&= \sum_l a_l a_{[0]} \otimes \gamma(c_{l(1)})\alpha(a_{[1]})\langle \gamma^{*-3}(c^*), c_{l(2)}\rangle = \beta_1(z)(c^* \otimes a),
\end{aligned}
$$

and it follows that $\phi = \beta_1(z) = \beta_1(\beta_1^{-1}(\phi))$, as required. ■

THEOREM 3.9. *Let (H, A, C) be a Doi Hom-Hopf datum, and assume that (C, γ) is finitely generated projective as a k-module. Let $F : \widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C \to \widetilde{\mathscr{H}}(\mathscr{M}_k)_A$ be the forgetful functor, and $G : \widetilde{\mathscr{H}}(\mathscr{M}_k)_A \to \widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C$ its adjoint. Then the following statements are equivalent:*

(1) *(F, G) is a Frobenius pair.*
(2) *There exist $z = \sum a_l \otimes c_l \in W_1$ and $\theta \in V_1$ such that the maps*

$$\phi : C^* \otimes A \to A \otimes C \quad \textit{and} \quad \overline{\phi} : A \otimes C \to C^* \otimes A,$$

given by

$$\phi(c^* \otimes a) = \sum_l a_l a_{[0]} \otimes c_{l(1)} a_{[1]} \langle \gamma^{*-2}(c^*), c_{l(2)}\rangle, \tag{3.14}$$

$$\overline{\phi}(a \otimes c) = \gamma^*(d_i^*) \otimes a_{[0]}\theta(\gamma^{-2}(d_i)\alpha^{-1}(a_{[1]}) \otimes \gamma^{-2}(c)), \tag{3.15}$$

are inverses of each other.
(3) *$C^* \otimes A$ and $A \otimes C^*$ are isomorphic as objects in ${}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C$.*

Proof. (1)⇒(2). Let $z \in W_1$ and $\theta \in V_1$ be as in Theorem 3.5. Then $\phi = \beta_1(z)$ and $\overline{\phi} = \alpha_1(\theta)$ are morphisms in ${}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C$, and

$$
\begin{aligned}
\overline{\phi}(\phi(\varepsilon \otimes 1_A)) = \overline{\phi}(z) &= \sum_l \gamma^*(d_i^*) \otimes a_{l[0]}\theta\big(\gamma^{-2}(d_i)\alpha^{-1}(a_{l[1]}) \otimes \gamma^{-2}(c_l)\big) \\
&= \sum_l \gamma^*(d_i^*) \otimes \beta(a_{l[0]})\theta\big(\gamma^{-2}(d_i)a_{l[1]} \otimes \gamma^{-1}(c_l)\big) \\
&= \sum_l \gamma^*(d_i^*) \otimes \varepsilon(d_i)1_A = \varepsilon \otimes 1_A.
\end{aligned}
$$

The fact that ϕ and $\overline{\phi}$ are right (A, β)-linear and left (C^*, γ^*)-linear implies that $\overline{\phi} \circ \phi = I_{C^* \otimes A}$. Similarly, for all $c \in C$,

$$
\begin{aligned}
\phi(\overline{\phi}(1_A \otimes c)) &= \phi(\gamma^*(d_i^*) \otimes \theta(d_i \otimes c)) \\
&= \sum_l a_l\theta(d_i \otimes c)_{[0]} \otimes c_{l(1)}\theta(d_i \otimes c)_{[1]}\langle d_i^*, c_{l(2)}\rangle \\
&= \sum_l a_l\theta(c_{l(2)} \otimes c)_{[0]} \otimes c_{l(1)}\theta(d_i \otimes c)_{[1]}
\end{aligned}
$$

$$= \sum_l a_l\theta(\gamma^{-1}(c_l) \otimes \gamma^{-2}(d)) \otimes \gamma(c_{(2)})$$
$$= \varepsilon(c_{(1)})1_A \otimes \gamma(c_{(2)}) = 1_A \otimes c.$$

(2)⇒(3). Obvious, since ϕ and $\overline{\phi}$ are in ${}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C_A$.

(3)⇒(1). Let $\phi : C^* \otimes A \to A \otimes C$ be an isomorphism, and write $z = \phi(\varepsilon \otimes 1_A) = \sum_l a_l \otimes c_l \in W_1$ and $\theta = \alpha_1^{-1}(\phi^{-1}) \in V_1$. One finds

$$\varepsilon \otimes 1_A = \phi^{-1}(\phi(\varepsilon \otimes 1_A)) = \gamma^*(d_i^*) \otimes a_{l[0]}\theta\big(\gamma^{-2}(d_i)\alpha^{-1}(a_{l[1]}) \otimes \gamma^{-2}(c_l)\big).$$

Evaluating this equality at $c \in C$, one obtains (3.15). For all $c \in C$,

$$1_A \otimes C = \phi(\phi^{-1}(1_A \otimes c)) = \sum_l a_l\theta(\gamma^{-1}(c_l) \otimes \gamma^{-2}(d)) \otimes \gamma(c_{(2)}).$$

Applying ε to the second factor, one finds (3.14). Thus (F, G) is a Frobenius pair. ■

If we take $A = H = k$ in Theorem 3.9, we obtain the following result.

COROLLARY 3.10. *Let (C, γ) be a k-projective monoidal Hom-coalgebra. The following statements are equivalent:*

(1) *The functor $G = \bullet \otimes C : \widetilde{\mathscr{H}}(\mathscr{M}_k)_k \to \widetilde{\mathscr{H}}(\mathscr{M}_k)^C$ is a left adjoint of the forgetful functor $F : \widetilde{\mathscr{H}}(\mathscr{M}_k)^C \to \widetilde{\mathscr{H}}(\mathscr{M}_k)_k$.*
(2) *(C, γ) is finitely generated as a k-module and there exists an isomorphism $\phi : C^* \to C$ of right (C^*, γ^*)-modules.*

If we apply Theorem 3.9 in the case when $C = H$, we obtain the following result.

COROLLARY 3.11. *Let (H, α) be a monoidal Hom-Hopf algebra and (A, β) a left (H, α)-Hom-comodule algebra. The following statements are equivalent:*

(1) *The functor $G = \bullet \otimes C : \widetilde{\mathscr{H}}(\mathscr{M}_k)_A \to \widetilde{\mathscr{H}}(\mathscr{M}_k)^H_A$ is a left adjoint of the forgetful functor $F : \widetilde{\mathscr{H}}(\mathscr{M}_k)^H_A \to \widetilde{\mathscr{H}}(\mathscr{M}_k)_A$.*
(2) *(H, α) is finite-dimensional.*

Proof. Apply the arguments used in [2] or [3]. The details are left to the reader. ■

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REFERENCES

[1] S. Caenepeel and I. Goyvaerts, *Monoidal Hom-Hopf algebras*, Comm. Algebra 39 (2011), 2216–2240.

[2] S. Caenepeel, G. Militaru and S. Zhu, *Doi-Hopf modules, Yetter–Drinfel'd modules and Frobenius type properties*, Trans. Amer. Math. Soc. 349 (1997), 4311–4342.

[3] S. Caenepeel, G. Militaru and S. Zhu, *Frobenius and Separable Functors for Generalized Module Categories and Nonlinear Equations*. Lecture Notes in Math. 1787, Springer, Berlin, 2002.

[4] Y. Y. Chen, Z. W. Wang and L. Y. Zhang, *Integrals for monoidal Hom-Hopf algebras and their applications*, J. Math. Phys. 54 (2013), 073515, 22 pp.

[5] Y. Y. Chen, Z. W. Wang and L. Y. Zhang, *Quasitriangular Hom-Hopf algebras*, Colloq. Math. 137 (2014), 67–88.

[6] Y. Y. Chen, Z. W. Wang and L. Y. Zhang, *The fundamental theorem and Maschke theorem in the category of relative Hom-Hopf modules*, Colloq. Math. 144 (2016), 55–71.

[7] Y. Y. Chen and L. Y. Zhang, *The category of Yetter–Drinfel'd Hom-modules and the quantum Hom-Yang–Baxter equation*, J. Math. Phys. 55 (2014), 031702, 18 pp.

[8] Y. Y. Chen and X. Zhou, *Separable and Frobenius monoidal Hom-algebras*, Colloq. Math. 137 (2014), 229–251.

[9] S. Dăscălescu, C. Năstăsescu and S. Raianu, *Hopf Algebras. An Introduction*, Lecture Notes in Pure Appl. Math. 235, Dekker, New York, 2001.

[10] S. J. Guo and X. L. Chen, *A Maschke type theorem for relative Hom-Hopf modules*, Czechoslovak Math. J. 64 (2014), 783–799.

[11] S. J. Guo and X. H. Zhang, *Separable functors for the category of Doi Hom-Hopf modules*, Colloq. Math. 143 (2016), 23–37.

[12] S. J. Guo, X. H. Zhang and S. X. Wang, *Braided monoidal categories and Doi Hopf modules for monoidal Hom-Hopf algebras*, Colloq. Math. 143 (2016), 79–104.

[13] H. Li and T. Ma, *A construction of the Hom-Yetter–Drinfeld category*, Colloq. Math. 137 (2014), 43–65.

[14] L. Liu and B. L. Shen, *Radford's biproducts and Yetter–Drinfeld modules for monoidal Hom-Hopf algebras*, J. Math. Phys. 55 (2014), 031701,

[15] T. Ma, H. Li and T. Yang, *Cobraided Hom-smash product Hopf algebras*, Colloq. Math. 134 (2014), 75–92.

[16] A. Makhlouf and F. Panaite, *Yetter–Drinfeld modules for Hom-bialgebras*, J. Math. Phys. 55 (2014), 013501,

[17] A. Makhlouf and S. D. Silvestrov, *Hom-algebra structures*, J. Gen. Lie Theory Appl. 2 (2008), 51–64.

[18] A. Makhlouf and S. Silvestrov, *Hom-Lie admissible Hom-coalgebras and Hom-Hopf algebras*, in: Generalized Lie Theory in Mathematics, Physics and Beyond, Springer, Berlin, 2009, 189–206.

[19] A. Makhlouf and S. D. Silvestrov, *Hom-algebras and Hom-coalgebras*, J. Algebra Appl. 9 (2010), 553–589.

[20] D. Radford, *Hopf Algebras*, Ser. Knots and Everything 49, World Sci., 2012.

[21] D. Simson, *Coalgebras of tame comodule type, comodule categories, and a tame-wild dichotomy problem*, in: Representation of Algebras and Related Topics (Tokyo, 2010), A. Skowroński and K. Yamagata (eds.), EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2011, 561–660.
[22] M. E. Sweedler, *Hopf Algebras*, Benjamin, New York, 1969.

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