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A PROBLEM OF G. Q. WANG ON THE DAVENPORT CONSTANT OF THE MULTIPLICATIVE SEMIGROUP OF QUOTIENT RINGS OF $\mathbb{F}_2[x]$

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Abstract. Given a finite commutative semigroup $\mathcal{S}$ (written multiplicatively), denote by $\mathrm{D}(\mathcal{S})$ the Davenport constant of $\mathcal{S}$, the least positive integer ℓ such that for any $x_1, \ldots, x_\ell \in \mathcal{S}$ there exists a set $I \subsetneq [1, \ell]$ for which $\prod_{i \in I} x_i = \prod_{i=1}^{\ell} x_i$, the equality being interpreted in the conditional unitization of $\mathcal{S}$ to make sense of the left-hand side also in the case when $I = \emptyset$ and $\mathcal{S}$ is not unitary.

Then, let R be the quotient ring of $\mathbb{F}_2[x]$ by the principal ideal generated by a non-constant polynomial $f \in \mathbb{F}_2[x]$. Moreover, let $\mathcal{S}_R$ be the multiplicative semigroup of the cosets in R, and $\mathrm{U}(\mathcal{S}_R)$ the group of units of $\mathcal{S}_R$.

We prove that

$$\mathrm{D}(\mathrm{U}(\mathcal{S}_R)) \le \mathrm{D}(\mathcal{S}_R) \le \mathrm{D}(\mathrm{U}(\mathcal{S}_R)) + \delta_f,$$

where

$$\delta_f = \begin{cases} 0 & \text{if } \gcd(x * (x + 1_{\mathbb{F}_2}), f) = 1_{\mathbb{F}_2}, \\ 1 & \text{if } \gcd(x * (x + 1_{\mathbb{F}_2}), f) \in \{x, x + 1_{\mathbb{F}_2}\}, \\ 2 & \text{if } \gcd(x * (x + 1_{\mathbb{F}_2}), f) = x * (x + 1_{\mathbb{F}_2}). \end{cases}$$

This gives a partial answer to an open problem of G. Q. Wang.

1. Introduction. The additive properties of sequences in abelian groups have been widely studied within zero-sum theory (see [2] for a survey), since K. Rogers [4] in 1963 pioneered the investigation of a combinatorial invariant associated with an arbitrary finite abelian group G, here denoted by $\mathrm{D}(G)$ and called the Davenport constant of G, which can be defined as the smallest $\ell \in \mathbb{N}$ such that every sequence T of elements of G of length at least ℓ contains a nonempty subsequence T' the sum of whose terms is equal to the identity element of the group G. (Note that neither the term 'Davenport constant' nor the notation $\mathrm{D}(G)$ appear in Rogers' paper.) The Davenport

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constant is a central concept of zero-sum theory and has been investigated by many researchers in the scope of finite abelian groups.

In 2008, Gao and Wang [8] extended the definition of the Davenport constant to commutative semigroups, and subsequently obtained several related additive results (see [1, 5–7]). As for the case of *finite* commutative semigroups, in which we are interested in the present work, their definition can be phrased as follows:

Definition A ([6, 8]). Let $\mathcal{S}$ be a finite commutative semigroup. Let T be a sequence of elements of $\mathcal{S}$. We call T *reducible* if T contains a proper subsequence T' ($T' \neq T$) such that the sum of all terms of T' equals the sum of all terms of T. Define the *Davenport constant* of the semigroup $\mathcal{S}$, denoted $\mathrm{D}(\mathcal{S})$, to be the smallest $\ell \in \mathbb{N}$ such that every sequence T in $\mathcal{S}$ of length at least ℓ is reducible.

In 2006, A. Geroldinger and F. Halter-Koch introduced another combinatorial invariant, which they denoted by $\mathrm{d}(\mathcal{S})$ (see [3, Definition 2.8.12]) and which is now called the *small Davenport constant* of $\mathcal{S}$ after [1]; this is closely related to the Davenport constant of $\mathcal{S}$, as it is known from [1, Proposition 1.2] that $\mathrm{D}(\mathcal{S}) = \mathrm{d}(\mathcal{S}) + 1$ for any finite commutative semigroup $\mathcal{S}$.

For any unitary finite ring R, we shall always denote by $\mathcal{S}_R$ the multiplicative semigroup of the ring R (that is, the ground set of R equipped with the operation of multiplication of the ring), and by $\mathrm{U}(\mathcal{S}_R)$ the group of units of the semigroup $\mathcal{S}_R$.

Very recently, Wang obtained the following result on the Davenport constant of $\mathcal{S}_R$, where R is a quotient ring of the polynomial ring $\mathbb{F}_q[x]$.

Proposition B ([5]). *Let $q > 2$ be a prime power, and let R be a quotient ring of $\mathbb{F}_q[x]$ with $0 \neq R \neq \mathbb{F}_q[x]$. Then $\mathrm{D}(\mathcal{S}_R) = \mathrm{D}(\mathrm{U}(\mathcal{S}_R))$.*

As for the case $q = 2$, Wang proposed the following:

Problem C (see [5, concluding remarks]). *Let R be a quotient ring of $\mathbb{F}_2[x]$ with $0 \neq R \neq \mathbb{F}_2[x]$. Determine $\mathrm{D}(\mathcal{S}_R) - \mathrm{D}(\mathrm{U}(\mathcal{S}_R))$.*

In this paper, we considered this open problem. By refining Wang's methods, we obtain the following result, which is a partial solution of Problem C.

Theorem 1.1. *Let $R = \mathbb{F}_2[x]/(f)$, where $f \in \mathbb{F}_2[x]$ and $0 \neq R \neq \mathbb{F}_2[x]$. Then*

$$\mathrm{D}(\mathrm{U}(\mathcal{S}_R)) \leq \mathrm{D}(\mathcal{S}_R) \leq \mathrm{D}(\mathrm{U}(\mathcal{S}_R)) + \delta_f,$$

where

$$\delta_f = \begin{cases} 0 & \text{if } \gcd(x * (x + 1_{\mathbb{F}_2}), f) = 1_{\mathbb{F}_2}, \\ 1 & \text{if } \gcd(x * (x + 1_{\mathbb{F}_2}), f) \in \{x, x + 1_{\mathbb{F}_2}\}, \\ 2 & \text{if } \gcd(x * (x + 1_{\mathbb{F}_2}), f) = x * (x + 1_{\mathbb{F}_2}). \end{cases}$$

2. Preliminaries. Notation and terminology used in this section are consistent with the ones used in [1, 5–7], but for the reader's convenience we recall some necessary definitions.

- Throughout, $\mathcal{S}$ is a *unitary* finite commutative semigroup.

The operation on $\mathcal{S}$ is denoted by $+$. The identity element of $\mathcal{S}$, denoted $0_{\mathcal{S}}$, is the unique element e of $\mathcal{S}$ such that $e + a = a$ for every $a \in \mathcal{S}$. Let $\mathrm{U}(\mathcal{S}) = \{a \in \mathcal{S} : a + a' = 0_{\mathcal{S}} \text{ for some } a' \in \mathcal{S}\}$ be the group of units of $\mathcal{S}$. For any $c \in \mathcal{S}$, let $\mathrm{St}(c) = \{a \in \mathrm{U}(\mathcal{S}) : a + c = c\}$ denote the stabilizer of c in the group $\mathrm{U}(\mathcal{S})$. Green's preorder on the semigroup $\mathcal{S}$, denoted $\leqq_{\mathcal{H}}$, is defined by

$$a \leqq_{\mathcal{H}} b \;\Leftrightarrow\; a = b + c \text{ for some } c \in \mathcal{S}.$$

Green's congruence on $\mathcal{S}$, denoted $\mathcal{H}$, is defined by:

$$a \;\mathcal{H}\; b \;\Leftrightarrow\; a \leqq_{\mathcal{H}} b \text{ and } b \leqq_{\mathcal{H}} a.$$

We write $a <_{\mathcal{H}} b$ to mean that $a \leqq_{\mathcal{H}} b$ but $a \;\mathcal{H}\; b$ does not hold.

A sequence T in $\mathcal{S}$ is denoted by

$$T = a_1 \cdots a_\ell = \coprod_{a \in \mathcal{S}} a^{[\mathrm{v}_a(T)]},$$

where $[\mathrm{v}_a(T)]$ means that the element a occurs $\mathrm{v}_a(T)$ times in the sequence T. We denote by $\cdot$ the operation of concatenation in the free abelian monoid over $\mathcal{S}$. We let $|T|$ denote the length of the sequence, i.e., $|T| = \sum_{a \in \mathcal{S}} \mathrm{v}_a(T) = \ell$. Let T_1, T_2 be sequences in $\mathcal{S}$. We call T_2 a *subsequence* of T_1, written $T_2 \,|\, T_1$, if $\mathrm{v}_a(T_2) \le \mathrm{v}_a(T_1)$ for every $a \in \mathcal{S}$. In particular, if $T_2 \ne T_1$, we call T_2 a *proper* subsequence of T_1, and write $T_3 = T_1 T_2^{[-1]}$ for the unique subsequence of T_1 with $T_2 \cdot T_3 = T_1$. If T is a nonempty sequence of $\mathcal{S}$, then we let $\sigma(T) = \sum_{a \in \mathcal{S}} [v_a(T)]a$. We also define $\sigma(\varepsilon) = 0_{\mathcal{S}}$, where ε is the empty word. We say that T is *reducible* if $\sigma(T') = \sigma(T)$ for some proper subsequence T' of T.

In what follows, we shall always assume that $R = \mathbb{F}_2[x]/(f)$ is the quotient of $\mathbb{F}_2[x]$ by some *fixed* nonconstant polynomial $f \in \mathbb{F}_2[x]$. For notational convenience, we write

$$f = f_1^{n_1} * f_2^{n_2} * \cdots * f_r^{n_r}, \tag{2.1}$$

where $f_1, \ldots, f_r$ are pairwise nonassociate irreducible polynomials in $\mathbb{F}_2[x]$ with $r \ge 2$, $f_1 = x$, $f_2 = x + 1_{\mathbb{F}_2}$, $n_1 \ge 0$ and $n_2 \ge 0$, and $n_i \ge 1$ for $i > 2$.

For any $a \in \mathcal{S}_R$, we let θ_a be the unique polynomial in $\mathbb{F}_2[x]$ such that $a = \theta_a + (f)$ with $\deg(\theta_a) < \deg(f)$. We denote by $\gcd(\theta_a, f)$ the greatest common divisor of θ_a and f in $\mathbb{F}_2[x]$. For any polynomial g and any irreducible polynomial h of $\mathbb{F}_2[x]$, let $\mathrm{pot}_h(g)$ be the largest integer k such that $h^k \,|\, g$.

By the previous definitions, it is immediate that for any $a, b \in \mathcal{S}_R$, $a \leqq_{\mathcal{H}} b$ if and only if there exists $h \in \mathbb{F}_2[x]$ such that $\theta_a + (f) = (\theta_b + (f)) * (h + (f))$ in R, or equivalently $\theta_a \equiv \theta_b * h \pmod{f}$.

To prove Theorem 1.1, we need some lemmas.

LEMMA 2.1 ([3, Lemma 6.1.3]). *Let G be a finite abelian group, and let H be a subgroup of G. Then* $\mathrm{D}(G) \geq \mathrm{D}(G/H) + \mathrm{D}(H) - 1$.

LEMMA 2.2 (see [8, Proposition 1.2]). *Let $\mathcal{S}$ be a finite unitary commutative semigroup. Then* $\mathrm{D}(\mathrm{U}(\mathcal{S})) \leq \mathrm{D}(\mathcal{S})$.

LEMMA 2.3. *Let $a, b \in \mathcal{S}_R$ with $a \leqq_{\mathcal{H}} b$. Let $\alpha_i = \mathrm{pot}_{f_i}(\gcd(\theta_a, f))$ and $\beta_i = \mathrm{pot}_{f_i}(\gcd(\theta_b, f))$ for each $i \in [1, r]$. Then:*

(i) $\mathrm{St}(b) \subseteq \mathrm{St}(a)$ *and $\beta_i \leq \alpha_i$ for each $i \in [1, r]$; in particular, if $a \; \mathcal{H} \; b$ then* $\mathrm{St}(b) = \mathrm{St}(a)$ *and $\beta_i = \alpha_i$ for each $i \in [1, r]$;*

(ii) *if $\beta_i = \alpha_i$ for each $i \in [1, r]$, then $a \; \mathcal{H} \; b$;*

(iii) *if $a <_{\mathcal{H}} b$ and*

$$(\alpha_1 - \beta_1)(2n_1 - 1 - \alpha_1 - \beta_1) + (\alpha_2 - \beta_2)(2n_2 - 1 - \alpha_2 - \beta_2) + \sum_{i=3}^{r} (\alpha_i - \beta_i) > 0, \tag{2.2}$$

then $\mathrm{St}(b) \subsetneq \mathrm{St}(a)$.

Proof of Lemma 2.3. (i) Note that $a \leqq_{\mathcal{H}} b$ implies $a = b + c$ for some $c \in \mathcal{S}_R$, or equivalently $\theta_a \equiv \theta_b * \theta_c \pmod{f}$. Then (i) follows from a routine verification.

(ii) Assume $\beta_i = \alpha_i$ for each $i \in [1, r]$, that is, $\gcd(\theta_b, f) = \gcd(\theta_a, f)$. Then there exist $h, h' \in \mathbb{F}_2[x]$ such that $\theta_a * h \equiv \theta_b \pmod{f}$ and $\theta_b * h' \equiv \theta_a \pmod{f}$, and (ii) follows readily.

(iii) Now assume $a <_{\mathcal{H}} b$ and (2.2) holds. It is sufficient to find $d \in \mathrm{U}(\mathcal{S}_R)$ such that $d \in \mathrm{St}(a) \setminus \mathrm{St}(b)$. Let

$$X_i = \begin{cases} (\alpha_i - \beta_i)(2n_i - 1 - \alpha_i - \beta_i) & \text{if } i \in \{1, 2\}, \\ \alpha_i - \beta_i & \text{if } i \in [3, r]. \end{cases}$$

By (2.2), we have $X_t > 0$ for some $t \in [1, r]$. Trivially,

$$\alpha_t - \beta_t > 0. \tag{2.3}$$

Let

$$h = f / f_t^{\beta_t + 1}. \tag{2.4}$$

We show that

$$\gcd(h + 1_{\mathbb{F}_2}, f) = 1_{\mathbb{F}_2} \tag{2.5}$$

or

$$\gcd(x * h + 1_{\mathbb{F}_2}, f) = 1_{\mathbb{F}_2}. \tag{2.6}$$

If $t \in \{1,2\}$, we have $\alpha_t > \beta_t$ and $n_t > \beta_t+1$, and (2.5) holds immediately. Hence, we suppose $t \in [3,r]$ and to the contrary that $\gcd(h+1_{\mathbb{F}_2}, f) \neq 1_{\mathbb{F}_2}$ and $\gcd(x*h+1_{\mathbb{F}_2}, f) \neq 1_{\mathbb{F}_2}$. Since $\alpha_t > \beta_t$, it follows from (2.1) and (2.4) that $f_i \nmid \gcd(h+1_{\mathbb{F}_2}, f)$ and $f_i \nmid \gcd(x*h+1_{\mathbb{F}_2}, f)$ for each i in $[1,r] \setminus \{t\}$. This implies that $f_t \mid (h+1_{\mathbb{F}_2})$ and $f_t \mid (x*h+1_{\mathbb{F}_2})$, and thus $f_t \mid x*(h+1_{\mathbb{F}_2}) - (x*h+1_{\mathbb{F}_2}) = x+1_{\mathbb{F}_2}$, which is absurd. This proves that (2.5) or (2.6) holds.

Take $d \in \mathcal{S}_R$ with $\theta_d \equiv h+1_{\mathbb{F}_2} \pmod{f}$ or $\theta_d \equiv x*h+1_{\mathbb{F}_2} \pmod{f}$ according to whether (2.5) or (2.6) holds. It follows that $d \in \mathrm{U}(\mathcal{S}_R)$, and (2.3) and (2.4) imply that $\theta_a * \theta_d \equiv \theta_a \pmod{f}$ and $\theta_b * \theta_d \not\equiv \theta_b \pmod{f}$. That is, $d \in \mathrm{St}(a) \setminus \mathrm{St}(b)$, as desired. ∎

3. The proof of Theorem 1.1. By Lemma 2.2, it suffices to show that $\mathrm{D}(\mathcal{S}_R) \leq \mathrm{D}(\mathrm{U}(\mathcal{S}_R)) + \delta_f$. Let $T = a_1 \cdots a_\ell$ be an arbitrary sequence in $\mathcal{S}_R$ of length

$$\ell = \mathrm{D}(\mathrm{U}(\mathcal{S}_R)) + \delta_f. \tag{3.1}$$

We shall prove that T contains a *proper* subsequence T' with $\sigma(T') = \sigma(T)$.

Take a shortest subsequence V of T such that

$$\sigma(V)\ \mathcal{H}\ \sigma(T). \tag{3.2}$$

We may assume without loss of generality that

$$V = a_1 \cdots a_t \quad \text{where } t \in [0,\ell].$$

If $t=0$, then $V=\varepsilon$ and $\sigma(V) = 0_{\mathcal{S}_R}$, which implies $\sigma(T) \in \mathrm{U}(\mathcal{S}_R)$ by (3.2). It follows that T is a sequence in $\mathrm{U}(\mathcal{S}_R)$ and of length $|T| \geq \mathrm{D}(\mathrm{U}(\mathcal{S}_R))$, and thus T is reducible. Hence, we can assume that

$$t > 0.$$

By the minimality of $|V|$, we derive that

$$0_{\mathcal{S}_R} >_{\mathcal{H}} a_1 >_{\mathcal{H}} a_1 + a_2 >_{\mathcal{H}} \cdots >_{\mathcal{H}} \sum_{i=1}^{t} a_i.$$

Recall that an empty sum of elements of $\mathcal{S}_R$ is taken equal to $0_{\mathcal{S}_R}$. Denote $K_i = \mathrm{St}(\sum_{j=1}^{i} a_j)$ for each $i \in [0,t]$. Note that K_i is a subgroup of $\mathrm{U}(\mathcal{S}_R)$ for each $i \in [0,t]$. It follows from Lemma 2.3(i) that

$$K_0 \subseteq K_1 \subseteq \cdots \subseteq K_t.$$

Moreover, we have the following:

ASSERTION A. *There exists a subset $M \subseteq [0,t-1]$ with $|M| \geq t - \delta_f$ such that $K_i \subsetneq K_{i+1}$ for each $i \in M$.*

Proof. Let $v \in [0,t-1]$ with $K_v = K_{v+1}$. Set $a = \sum_{i=1}^{v+1} a_i$ and $b = \sum_{i=1}^{v} a_i$. Since $a <_{\mathcal{H}} b$, it follows from Lemma 2.3(i)&(ii) that $n_j \geq \alpha_j \geq \beta_j$

for each $j \in [1, r]$, and moveover $(\alpha_1 - \beta_1) + (\alpha_2 - \beta_2) + \sum_{i=3}^{r}(\alpha_i - \beta_i) > 0$. Since $K_v = K_{v+1}$, from Lemma 2.3(iii) we now have

$$(\alpha_1 - \beta_1)(2n_1 - 1 - \alpha_1 - \beta_1) + (\alpha_2 - \beta_2)(2n_2 - 1 - \alpha_2 - \beta_2) + \sum_{i=3}^{r}(\alpha_i - \beta_i) = 0.$$

Hence, either $\alpha_1 = n_1$ and $\beta_1 = n_1 - 1$, or $\alpha_2 = n_2$ and $\beta_2 = n_2 - 1$. By the arbitrariness of v, this completes the proof of Assertion A. ■

For $i \in M$, since

$$\frac{\mathrm{U}(\mathcal{S}_R)}{K_{i+1}} \cong \frac{\mathrm{U}(\mathcal{S}_R)/K_i}{K_{i+1}/K_i} \quad \text{and} \quad \mathrm{D}(K_{i+1}/K_i) \geq 2,$$

it follows from Lemma 2.1 that

$$\begin{aligned} \mathrm{D}(\mathrm{U}(\mathcal{S}_R)/K_{i+1}) &= \mathrm{D}\left(\frac{\mathrm{U}(\mathcal{S}_R)/K_i}{K_{i+1}/K_i}\right) \\ &\leq \mathrm{D}(\mathrm{U}(\mathcal{S}_R)/K_i) - (\mathrm{D}(K_{i+1}/K_i) - 1) \\ &\leq \mathrm{D}(\mathrm{U}(\mathcal{S}_R)/K_i) - 1. \end{aligned} \tag{3.3}$$

Combining (3.1), (3.3) and Assertion A, we conclude that

$$\begin{aligned} 1 \leq \mathrm{D}(\mathrm{U}(\mathcal{S}_R)/K_t) &\leq \mathrm{D}(\mathrm{U}(\mathcal{S}_R)/K_0) - |M| \leq \mathrm{D}(\mathrm{U}(\mathcal{S}_R)) - (t - \delta_f) \\ &= (\ell - \delta_f) - (t - \delta_f) = \ell - t = |TV^{[-1]}|. \end{aligned} \tag{3.4}$$

By Lemma 2.3(i) and (3.2), we have

$$\mathrm{pot}_{f_i}(\gcd(\theta_{\sigma(V)}, f)) = \mathrm{pot}_{f_i}(\gcd(\theta_{\sigma(T)}, f)) \tag{3.5}$$

for each $i \in [1, r]$. Let

$$\mathcal{J} = \{j \in [1, r] : f_j^{n_j} \mid \theta_{\sigma(T)}\}.$$

By (3.5),

$$f_i \nmid \theta_a \qquad \text{for each term } a \text{ of } TV^{[-1]} \text{ and each } i \in [1, r] \setminus \mathcal{J}, \tag{3.6}$$

$$f_j^{n_j} \mid \theta_{\sigma(V)} \qquad \text{for each } j \in \mathcal{J}. \tag{3.7}$$

For each term a of $TV^{[-1]}$, let $\tilde{a}$ be the element of $\mathcal{S}_R$ such that

$$\theta_{\tilde{a}} \equiv \theta_a \pmod{f_i^{n_i}} \qquad \text{for each } i \in [1, r] \setminus \mathcal{J}, \tag{3.8}$$

$$\theta_{\tilde{a}} \equiv 1_{\mathbb{F}_2} \pmod{f_j^{n_j}} \qquad \text{for each } j \in \mathcal{J}. \tag{3.9}$$

By (3.6), (3.8) and (3.9), we conclude that $\gcd(\theta_{\tilde{a}}, f) = 1_{\mathbb{F}_2}$, i.e.,

$$\tilde{a} \in \mathrm{U}(\mathcal{S}_R) \qquad \text{for each term } a \text{ of } TV^{[-1]}. \tag{3.10}$$

By (3.7) and (3.8), we conclude that

$$\sigma(V) + \tilde{a} = \sigma(V) + a \qquad \text{for each term } a \text{ of } TV^{[-1]}. \tag{3.11}$$

From (3.4) and (3.10), we see that $\prod_{a|TV^{[-1]}} \tilde{a}$ is a nonempty sequence of elements in $\mathrm{U}(\mathcal{S}_R)$ of length $|\prod_{a|TV^{[-1]}} \tilde{a}| = |TV^{[-1]}| \geq \mathrm{D}(\mathrm{U}(\mathcal{S}_R)/K_t)$. It follows that there exists a *nonempty* subsequence $W \mid TV^{[-1]}$ such that $\sigma(\coprod_{a|W} \tilde{a}) \in K_t$, which implies

$$\sigma(V) + \sigma\Big(\coprod_{a|W} \tilde{a}\Big) = \sigma(V). \tag{3.12}$$

By (3.11) and (3.12), we conclude that

$$\begin{aligned}\sigma(T) &= \sigma(TW^{[-1]}V^{[-1]}) + (\sigma(V) + \sigma(W))\\ &= \sigma(TW^{[-1]}V^{[-1]}) + \Big(\sigma(V) + \sigma\Big(\coprod_{a|W} \tilde{a}\Big)\Big)\\ &= \sigma(TW^{[-1]}V^{[-1]}) + \sigma(V) = \sigma(TW^{[-1]}),\end{aligned}$$

and $T' = TW^{[-1]}$ is the desired proper subsequence of T. This completes the proof of the theorem. ∎

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REFERENCES

[1] S. D. Adhikari, W. D. Gao and G. Q. Wang, *Erdős–Ginzburg–Ziv theorem for finite commutative semigroups*, Semigroup Forum 88 (2014), 555–568.

[2] W. D. Gao and A. Geroldinger, *Zero-sum problems in finite abelian groups: A survey*, Expo. Math. 24 (2006), 337–369.

[3] A. Geroldinger and F. Halter-Koch, *Non-Unique Factorizations. Algebraic, Combinatorial and Analytic Theory*, Pure Appl. Math. 278, Chapman & Hall/CRC, 2006.

[4] K. Rogers, *A combinatorial problem in Abelian groups*, Math. Proc. Cambridge Philos. Soc. 59 (1963), 559–562.

[5] G. Q. Wang, *Davenport constant for semigroups II*, J. Number Theory 153 (2015), 124–134.

[6] G. Q. Wang, *Additively irreducible sequences in commutative semigroups*, arXiv:1504.06818 (2015).

[7] G. Q. Wang, *Structure of the largest idempotent-product free sequences in semigroups*, arXiv:1405.6278 (2014).

[8] G. Q. Wang and W. D. Gao, *Davenport constant for semigroups*, Semigroup Forum 76 (2008), 234–238.

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