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MEASURE-THEORETIC PRESSURE FOR AMENABLE GROUP ACTIONS

BY

YUN ZHAO (Suzhou)

Abstract. This paper defines measure-theoretic pressure for an amenable group action by using spanning sets, and shows that the measure-theoretic pressure of an ergodic measure can be described in terms of metric entropy and an integral of the observable associated with the ergodic measure. Using the theory of Carathéodory structure, we give an equivalent definition of measure-theoretic pressure for amenable group actions, and obtain an inverse variational principle, i.e., the topological pressure on a certain set is exactly the measure-theoretic pressure of an ergodic measure.

1. Introduction. The main ingredients of thermodynamic formalism are topological pressure and measure-theoretic pressure (free energy). Topological pressure is a certain functional defined on the space of observables that encodes several important quantities of the underlying dynamical system. The relation between topological pressure and free energy is established by the variational principle. Invariant measures maximizing free energy are called equilibrium states. The study of the variational principle and equilibrium states (existence, uniqueness and properties) plays a fundamental role in statistical mechanics, ergodic theory and dynamical systems; we refer to the books [Bo75, K98, Ru04, Wa82] for references and details.

Ruelle [Ru73] introduced topological pressure for $\mathbb{Z}^n$-actions and established the variational principle for topological pressure when the action is expansive and satisfies the specification property. Walters [Wa75] generalized the variational principle to general $\mathbb{Z}_+$-actions. Misiurewicz [M] gave a short and elegant proof of the variational principle for $\mathbb{Z}_+^n$-actions. See [OP, O, ST, T84, T92] for the variational principle of topological pressure for an amenable group action. Using the ideas from [DZ] and [HYZ11], Liang and Yan [LY12] defined subadditive topological pressure for amenable group actions, and established the variational principle. Recently, Bowen [Bo10] in-

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troduced the notion of entropy for measure preserving actions of countable sofic groups admitting a generating measurable partition with finite entropy, and in [Bo12] he further proved that sofic entropy equals classical entropy in the case where the acting group is amenable; then Kerr and Li [KL11] developed an operator-algebraic approach to actions of countable sofic groups not only on a standard probability space but also on a compact metric space, and established the variational principle concerning metric and topological entropy in the sofic context. Zhang [Z] localized these two kinds of entropy and proved a local version of the global variational principle. Later, Chung [Ch13] introduced topological pressure for countable sofic group actions on a compact metric space, and established the variational principle for it. Chung and Zhang [CZ15] introduced the notions of h-expansive and asymptotically h-expansive actions of sofic groups on a compact metric space and proved that metric entropy for an asymptotically h-expansive action is upper semicontinuous, and hence the dynamical system admits a measure with maximal entropy.

Using the theory of Carathéodory structure (see [Pe97] for details), Pesin and Pitskel' [PP] defined topological pressure on non-compact sets and established the variational principle under some supplementary conditions. Ren [Re] extended Pesin–Pitskel' topological pressure to amenable group actions. In particular, he proved the variational principle for Pesin–Pitskel' pressure on the whole space for amenable group actions.

Cao, Hu and Zhao [CHZ] defined measure-theoretic pressure for $\mathbb{Z}_+$-actions on compact metric spaces with respect to non-additive observables in two ways: by using spanning sets and by using Carathéodory structure as described in [Pe97]. It was proved that these two definitions are equivalent, and both can be expressed as the sum of the metric entropy and the integral of the limit of the average value of the non-additive observable.

Both topological pressure and measure-theoretic pressure have applications in dimension theory. In fact, all known equations that are used to compute or estimate the dimensions of conformal or non-conformal repellers are appropriate versions of an equation introduced by Bowen that involves topological pressure. See the books [Ba08, Ba11, Pe97] and the surveys [BG11, CP] for a detailed description of many of these relations. For any ergodic measure on a C^1 non-conformal repeller, Wang, Cao and Zhao [WCZ] showed that the zeros of measure-theoretic pressure give the lower and upper bounds of the dimension.

The main aim of this paper is to study measure-theoretic pressure for amenable group actions. More precisely, for an ergodic measure, measure-theoretic pressure defined by using Carathéodory structure can be described as the sum of the metric entropy and the integral of the observable. Consequently, this paper gives equivalence to an alternative definition of measure-theoretic pressure by considering spanning sets. The proofs of our results use ergodic theorems for amenable group actions.

The paper is organized in the following manner. The main results, as well as the definitions of measure-theoretic pressure and lower and upper measure-theoretic pressures for amenable group actions are given in Section 1. Some properties of Pesin–Pitskel' topological pressure for amenable group actions are presented in Section 2; they are used in the proof of our main results. We prove Theorems 2.5 and 2.8 in Sections 3 and 4 respectively.

2. Definitions and statements of the main results. Let (X,G) be a topological dynamical system, that is, X is a compact metric space with a metric d, and G a topological group acting on X. In this paper we assume that G is a discrete countable amenable group. Recall that a countable discrete group G is *amenable* if there exists a sequence $\{F_n\}_{n\ge1}$ of finite subsets of G such that

$$\lim_{n\to\infty}\frac{|F_n\bigtriangleup gF_n|}{|F_n|}=0$$

for all $g\in G$, where $|\cdot|$ denotes the counting measure, $gF=\{gf: f\in F\}$ and $F\bigtriangleup gF=(F\setminus gF)\cup(gF\setminus F)$. Such a sequence $\{F_n\}_{n\ge1}$ is called a *Følner sequence*. See Ornstein and Weiss' pioneering work [OW] for a detailed description of amenable group actions.

2.1. Definition of metric entropy. Denote by $\mathcal{M}(X,G)$ and $\mathcal{E}(X,G)$ the set of all G-invariant Borel probability measures on X and the set of ergodic measures respectively. Let ξ be a finite measurable partition of X. For a finite subset F of G, let $\xi_F:=\bigvee_{f\in F}f^{-1}\xi$. The *metric entropy* of $\mu\in\mathcal{M}(X,G)$ *with respect to the partition* ξ is defined by

$$h_\mu(G,\xi):=\lim_{n\to\infty}\frac{1}{|F_n|}H_\mu(\xi_{F_n})$$

where $\{F_n\}_{n\ge1}$ is a Følner sequence in G and $H_\mu(\xi)=\sum_{A\in\xi}-\mu(A)\log\mu(A)$. It is well-known that the above limit exists and the quantity $h_\mu(G,\xi)$ does not depend on the choice of the Følner sequence $\{F_n\}_{n\ge1}$ (see e.g. [OW]). Finally, the *metric entropy* of $\mu\in\mathcal{M}(X,G)$ is given by

$$h_\mu(G):=\sup h_\mu(G,\xi)$$

where the supremum is taken over all finite measurable partitions ξ of X.

2.2. Definition of measure-theoretic pressure using spanning sets. Given a finite subset $F\subset G$, let $d_F(x,y)=\max_{g\in F}d(gx,gy)$ for $x,y\in X$. The *dynamical ball* centered at x with radius ϵ associated with F is defined by

$$B_F(x,\epsilon)=\{y\in X: d_F(x,y)<\epsilon\}.$$

Given an invariant measure $\mu \in \mathcal{M}(X,G)$ and a Følner sequence $\{F_n\}_{n\geq 1}$ in G, for any $0<\delta<1$, $n\geq 1$ and $\epsilon>0$, a subset $T\subseteq X$ is an (n,ϵ,δ)-*spanning set* if $\bigcup_{x\in T} B_{F_n}(x,\epsilon)$ has μ-measure at least $1-\delta$.

Zheng et al. [ZCY] extended Katok's entropy formula [BK] to amenable group actions. More precisely, for a Følner sequence $\{F_n\}_{n\geq 1}$ satisfying some mild conditions, they showed that the metric entropy of an amenable group action can be regarded as the growth rate of the minimal number of ϵ-balls in the d_{F_n} metric that cover a set of measure at least $1-\delta$. Huang et al. [HYZ11, Theorem 4.19] proved a local version of Katok's entropy formula for amenable group actions (here local means an open cover is fixed when one defines the entropy). Motivated by this observation, the following definition can be given.

Given a Følner sequence $\{F_n\}_{n\geq 1}$ in G and a continuous function $\varphi : X\to\mathbb{R}$, for $\mu\in\mathcal{E}(X,G)$, $0<\delta<1$, $N\geq 1$ and $\epsilon>0$, set

$$P_\mu(\varphi,\{F_n\},N,\epsilon,\delta) = \inf\Big\{\sum_{x\in T}\exp[S_{F_N}\varphi(x)] : T \text{ is an } (N,\epsilon,\delta)\text{-spanning set}\Big\},$$

$$P_\mu(\varphi,\{F_n\},\epsilon,\delta) = \limsup_{N\to\infty}\frac{1}{|F_N|}\log P_\mu(\varphi,\{F_n\},N,\epsilon,\delta),$$

where $S_{F_N}\varphi(x) := \sum_{g\in F_N}\varphi(gx)$. It is easy to see that $P_\mu(\varphi,\{F_n\},\epsilon,\delta)$ is monotone in ϵ, and we let

$$P_\mu(\varphi,\{F_n\},\delta) = \lim_{\epsilon\to 0} P_\mu(\varphi,\{F_n\},\epsilon,\delta).$$

Definition 2.1. The quantity $P_\mu(\varphi,\{F_n\}) := \lim_{\delta\to 0} P_\mu(\varphi,\{F_n\},\delta)$ is called the *measure-theoretic pressure* of φ with respect to the Følner sequence $\{F_n\}_{n\geq 1}$.

Remark 2.2. It is easy to see that $P_\mu(\varphi,\{F_n\},\delta)$ increases as $\delta\searrow 0$, so the limit in the definition exists. In fact, inequalities (4.4) and (4.7) in the proof of Theorem 2.5 below show that $P_\mu(\varphi,\{F_n\},\delta)$ is independent of δ. Hence, the limit as $\delta\to 0$ is redundant in the definition. Furthermore, the proof of Theorem 2.5 tells us that when we define $P_\mu(\varphi,\{F_n\})$, there is no difference between the upper and lower limits in the definition of $P_\mu(\varphi,\{F_n\},\epsilon,\delta)$. The same phenomenon can also be seen for metric entropy (see [ZCY, Theorem 3.1]).

2.3. Definition of measure-theoretic pressure using Carathéodory structure. An alternative definition of measure-theoretic pressure for amenable group actions can be given by using the theory of Carathéodory structure as described in [Pe97]. In this section, we always assume that the cardinality of the amenable group G is infinite.

Let Z be a subset of X, which does not have to be compact or G-invariant. Fix $\epsilon > 0$ and a Følner sequence $\{F_n\}_{n\geq 1}$ in G. We call $\Gamma = \{B_{F_{n_i}}(x_i,\epsilon)\}_i$ a *cover of* Z if $Z \subseteq \bigcup_i B_{F_{n_i}}(x_i,\epsilon)$.

For $\alpha \in \mathbb{R}$, set

$$M(Z,\varphi,\{F_n\},\alpha,N,\epsilon) = \inf_{\Gamma} \sum_i \exp\Big(-\alpha|F_{n_i}| + \sup_{y\in B_{F_{n_i}}(x_i,\epsilon)} S_{F_{n_i}}\varphi(y)\Big) \tag{2.1}$$

where the infimum is taken over all countable covers Γ of Z with $n_i \geq N$ for every i. It is easy to show that $M(Z,\varphi,\{F_n\},\alpha,N,\epsilon)$ is monotone in N, and we let

$$m(Z,\varphi,\{F_n\},\alpha,\epsilon) := \lim_{N\to\infty} M(Z,\varphi,\{F_n\},\alpha,N,\epsilon).$$

Note that as G is infinite, $\lim_{n\to\infty}|F_n| = \infty$. It is clear that there is a jump-up value

$$\begin{aligned} P_Z(\varphi,\{F_n\},\epsilon) &= \inf\{\alpha : m(Z,\varphi,\{F_n\},\alpha,\epsilon) = 0\} \\ &= \sup\{\alpha : m(Z,\varphi,\{F_n\},\alpha,\epsilon) = \infty\}. \end{aligned}$$

Since $P_Z(\varphi,\{F_n\},\epsilon)$ is increasing as $\epsilon \to 0$, we call

$$P_Z(\varphi,\{F_n\}) := \lim_{\epsilon\to 0} P_Z(\varphi,\{F_n\},\epsilon)$$

the *topological pressure* of φ on the set Z (with respect to $\{F_n\}$).

Further, for $\mu \in \mathcal{M}(X,G)$, set

$$P^*_\mu(\varphi,\{F_n\},\epsilon) = \inf\{P_Z(\varphi,\{F_n\},\epsilon) : \mu(Z) = 1\}.$$

It is easy to prove that

$$P^*_\mu(\varphi,\{F_n\},\epsilon) = \lim_{\delta\to 0}\ \inf\{P_Z(\varphi,\{F_n\},\epsilon) : \mu(Z) \geq 1-\delta\}.$$

See [Pe97] for the general theory of Carathéodory structure.

Definition 2.3. We call $P^*_\mu(\varphi,\{F_n\}) = \lim_{\epsilon\to 0} P^*_\mu(\varphi,\{F_n\},\epsilon)$ the *measure-theoretic pressure* of φ with respect to μ.

Similarly, we define lower and upper capacity measure-theoretic pressure. Given a Følner sequence $\{F_n\}_{n\geq 1}$, $\alpha \in \mathbb{R}$ and a subset $Z \subset X$, define

$$R(Z,\varphi,\{F_n\},\alpha,N,\epsilon) = \inf_{\Gamma} \sum_i \exp\Big(-\alpha|F_N| + \sup_{y\in B_{F_N}(x_i,\epsilon)} S_{F_N}\varphi(y)\Big) \tag{2.2}$$

where the infimum is taken over all covers Γ of Z with $n_i = N$ for all i. We set

$$\begin{aligned} \underline{r}(Z,\varphi,\{F_n\},\alpha,\epsilon) &= \liminf_{N\to\infty} R(Z,\varphi,\{F_n\},\alpha,N,\epsilon), \\ \overline{r}(Z,\varphi,\{F_n\},\alpha,\epsilon) &= \limsup_{N\to\infty} R(Z,\varphi,\{F_n\},\alpha,N,\epsilon), \end{aligned}$$

and define the jump-up points of $\underline{r}(Z,\varphi,\{F_n\},\alpha,\epsilon)$ and $\overline{r}(Z,\varphi,\{F_n\},\alpha,\epsilon)$ as

$$\begin{aligned}\underline{CP}_Z(\varphi,\{F_n\},\epsilon) &= \inf\{\alpha : \underline{r}(Z,\varphi,\{F_n\},\alpha,\epsilon)=0\}\\ &= \sup\{\alpha : \underline{r}(Z,\varphi,\{F_n\},\alpha,\epsilon)=\infty\},\\ \overline{CP}_Z(\varphi,\{F_n\},\epsilon) &= \inf\{\alpha : \overline{r}(Z,\varphi,\{F_n\},\alpha,\epsilon)=0\}\\ &= \sup\{\alpha : \overline{r}(Z,\varphi,\{F_n\},\alpha,\epsilon)=\infty\}\end{aligned}$$

respectively. Similarly, we call the quantities

$$\begin{aligned}\underline{CP}_Z(\varphi,\{F_n\}) &:= \lim_{\epsilon\to 0}\underline{CP}_Z(\varphi,\{F_n\},\epsilon),\\ \overline{CP}_Z(\varphi,\{F_n\}) &:= \lim_{\epsilon\to 0}\overline{CP}_Z(\varphi,\{F_n\},\epsilon)\end{aligned}$$

the *lower* and *upper topological pressures* of φ on Z (with respect to $\{F_n\}_{n\geq 1}$).

For $\mu\in\mathcal{M}(X,G)$, define

$$\begin{aligned}\underline{CP}^*_\mu(\varphi,\{F_n\},\epsilon) &= \lim_{\delta\to 0}\inf\{\underline{CP}_Z(\varphi,\{F_n\},\epsilon) : \mu(Z)\geq 1-\delta\},\\ \overline{CP}^*_\mu(\varphi,\{F_n\},\epsilon) &= \lim_{\delta\to 0}\inf\{\overline{CP}_Z(\varphi,\{F_n\},\epsilon) : \mu(Z)\geq 1-\delta\}.\end{aligned}$$

DEFINITION 2.4. The *lower* and *upper measure-theoretic pressure* of φ with respect to the measure μ are defined by

$$\begin{aligned}\underline{CP}^*_\mu(\varphi,\{F_n\}) &= \lim_{\epsilon\to 0}\underline{CP}^*_\mu(\varphi,\{F_n\},\epsilon),\\ \overline{CP}^*_\mu(\varphi,\{F_n\}) &= \lim_{\epsilon\to 0}\overline{CP}^*_\mu(\varphi,\{F_n\},\epsilon).\end{aligned}$$

2.4. Statements of the main results. To state the main results of this paper, we recall that a Følner sequence $\{F_n\}_{n\geq 1}$ in G is *tempered* if there exists a constant C independent of n such that

$$\Big|\bigcup_{k<n} F_k^{-1}F_n\Big| \leq C|F_n|, \quad \forall n.$$

In fact, Lindenstrauss [L] proved that every Følner sequence has a tempered subsequence.

THEOREM 2.5. *Let X be a compact metric space and G a countable discrete amenable group acting on X. Let $\{F_n\}_{n\geq 1}$ be a tempered Følner sequence in G with $\lim_{n\to\infty}(\log n)/|F_n| = 0$. Then for any $\mu\in\mathcal{E}(X,G)$ and any continuous function φ on X, we have*

$$\begin{aligned}P^*_\mu(\varphi,\{F_n\}) &= \underline{CP}^*_\mu(\varphi,\{F_n\}) = \overline{CP}^*_\mu(\varphi,\{F_n\})\\ &= P_\mu(\varphi,\{F_n\}) = h_\mu(G) + \int\varphi\,d\mu.\end{aligned}$$

Under the conditions of Theorem 2.5, Ren [Re] proved the variational principle for topological pressure for amenable group actions. More precisely, for any continuous function φ on X we have $P_X(\varphi,\{F_n\}) = \sup\{h_\mu(G) +$

$\int \varphi\, d\mu : \mu \in \mathcal{M}(X, G)\}$. Due to this observation, using the ergodic decomposition theorem for amenable group actions (see [W]) we obtain the following corollary.

COROLLARY 2.6. *Let X be a compact metric space and G a countable discrete amenable group acting on X. Let $\{F_n\}_{n\geq 1}$ be a tempered Følner sequence in G with* $\lim_{n\to\infty} (\log n)/|F_n| = 0$. *Then for any continuous function φ on X, we have*

$$P_X(\varphi, \{F_n\}) = \sup\{P_\mu(\varphi, \{F_n\}) : \mu \in \mathcal{E}(X, G)\}.$$

REMARK. By Theorem 2.5, it is easy to see that the quantities $P_\mu(\varphi, \{F_n\})$, $P_\mu^*(\varphi, \{F_n\})$, $\underline{CP}_\mu^*(\varphi, \{F_n\})$ and $\overline{CP}_\mu^*(\varphi, \{F_n\})$ do not depend on the choice of $\{F_n\}_{n\geq 1}$. So we will suppress the dependence on $\{F_n\}_{n\geq 1}$ in notation if there is no confusion, e.g., we will simply write $M(Z, \varphi, \{F_n\}, \alpha, N, \epsilon)$ as $M(Z, \varphi, \alpha, N, \epsilon)$ in the following sections.

REMARK. Under the conditions of Theorem 2.5, if $\varphi \equiv 0$, by Theorem 2.5 and Remark 2.2 we conclude that

$$\begin{aligned} h_\mu(G) &= \lim_{\epsilon\to 0} \liminf_{n\to\infty} \frac{1}{|F_n|} \log \aleph(n, \epsilon, \delta, \{F_n\}) \\ &= \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{|F_n|} \log \aleph(n, \epsilon, \delta, \{F_n\}) \end{aligned}$$

where $\aleph(n, \epsilon, \delta, \{F_n\})$ is the minimal number of dynamical balls $B_{F_n}(x, \epsilon)$ that cover a set of measure larger than or equal to $1 - \delta$. This result was proved in [ZCY]. Our method used to prove Theorem 2.5 can be regarded as a different proof of the above formula.

COROLLARY 2.7. *Let X be a compact metric space and G a countable discrete amenable group acting on X. Let $\{F_n\}_{n\geq 1}$ be a tempered Følner sequence in G with* $\lim_{n\to\infty} \log n/|F_n| = 0$. *Then for any ergodic measure $\mu \in \mathcal{E}(X, G)$ and any continuous function φ on X we have*

$$h_\mu(G) + \int \varphi\, d\mu = \inf\{P_Z(\varphi, \{F_n\}) : \mu(Z) = 1\}. \tag{2.3}$$

As in [Pe97], we call formula (2.3) the *inverse variational principle* for measure-theoretic pressure for amenable group actions.

The next theorem says that the infimum in the inverse variational principle can be attained on a certain set.

THEOREM 2.8. *Let X be a compact metric space and G a countable discrete amenable group acting on X. Let $\{F_n\}_{n\geq 1}$ be a tempered Følner sequence in G with* $\lim_{n\to\infty} (\log n)/|F_n| = 0$. *For any $\mu \in \mathcal{E}(X, G)$ and any* *continuous function* φ *on* X, *let*

$$K = \Big\{ x \in X : \lim_{\epsilon \to 0} \limsup_{n\to\infty} \frac{-\log \mu(B_{F_n}(x,\epsilon))}{|F_n|} = h_\mu(G) \text{ and } \lim_{n\to\infty} \frac{1}{|F_n|} S_{F_n}\varphi(x) = \int \varphi \, d\mu \Big\}.$$

Then

$$P_\mu(\varphi, \{F_n\}) = P_K(\varphi, \{F_n\}) = h_\mu(G) + \int \varphi \, d\mu.$$

3. Properties of topological pressures. We present some properties of topological pressures for amenable group actions which will be used in the proof of the main results.

PROPOSITION 3.1. *Let* X *be a compact metric space,* G *an infinite countable discrete amenable group acting on* X, *and* $\{F_n\}_{n\ge1}$ *a Følner sequence in* G. *Then:*

(i) $\mathcal{P}_{Z_1}(\varphi,\epsilon) \le \mathcal{P}_{Z_2}(\varphi,\epsilon)$ *for any* $\epsilon > 0$ *and* $\mathcal{P}_{Z_1}(\varphi) \le \mathcal{P}_{Z_2}(\varphi)$ *if* $Z_1 \subset Z_2$, *where* $\mathcal{P}$ *denotes either* P *or* $\underline{CP}$ *or* $\overline{CP}$;

(ii) $P_Z(\varphi) = \sup_{i\ge1} P_{Z_i}(\varphi)$ *and* $\mathcal{P}_Z(\varphi) \ge \sup_{i\ge1} \mathcal{P}_{Z_i}(\varphi)$, *where* $Z = \bigcup_{i\ge1} Z_i$, *and* $\mathcal{P}$ *is* $\underline{CP}$ *or* $\overline{CP}$;

(iii) $P_Z(\varphi,\epsilon) \le \underline{CP}_Z(\varphi,\epsilon) \le \overline{CP}_Z(\varphi,\epsilon)$ *for any subset* $Z \subset X$ *and any* $\epsilon > 0$, *and* $P_Z(\varphi) \le \underline{CP}_Z(\varphi) \le \overline{CP}_Z(\varphi)$ *for any subset* $Z \subset X$;

(iv) $P^*_\mu(\varphi,\epsilon) \le \underline{CP}^*_\mu(\varphi,\epsilon) \le \overline{CP}^*_\mu(\varphi,\epsilon)$ *and* $P^*_\mu(\varphi) \le \underline{CP}^*_\mu(\varphi) \le \overline{CP}^*_\mu(\varphi)$ *for any* $\mu \in \mathcal{M}(X,G)$.

Proof. (i) and (ii) follow directly from the definitions. The third statement is immediate by similar arguments to those for [Ba96, Theorem 1.4(a)]. The last result follows from (iii) immediately by the definition. ∎

For each $B_{F_n}(x,\epsilon)$, we can replace $\sup_{y\in B_{F_n}(x,\epsilon)} S_{F_n}\varphi(y)$ by $S_{F_n}\varphi(x)$ in (2.1) and (2.2) to define new functions m', $\underline{r}'$ and $\overline{r}'$. For each subset $Z \subset X$, we denote the respective critical values by $P'_Z(\varphi,\epsilon)$, $\underline{CP}'_Z(\varphi,\epsilon)$ and $\overline{CP}'_Z(\varphi,\epsilon)$.

PROPOSITION 3.2. *For any subset* $Z \subset X$ *and any Følner sequence* $\{F_n\}_{n\ge1}$ *in an infinite countable discrete amenable group* G, *we have*

$$\mathcal{P}_Z(\varphi) = \lim_{\epsilon\to0} \mathcal{P}'_Z(\varphi,\epsilon)$$

where $\mathcal{P}$ *denotes either* P *or* $\underline{CP}$ *or* $\overline{CP}$.

Proof. Given $\epsilon > 0$, let

$$\beta(\epsilon) = \sup\{|\varphi(x) - \varphi(y)| : d(x,y) \le \epsilon\},$$

and observe that since φ is continuous and X is compact, φ is in fact uniformly continuous, hence $\lim_{\epsilon\to0} \beta(\epsilon) = 0$.

Given a cover $\Gamma = \{B_{F_{n_i}}(x_i, \epsilon)\}$ of Z with $n_i \geq N$ for any i, we have

$$\sum_i \exp\Big(-\alpha|F_{n_i}| + \sup_{y \in B_{F_{n_i}}(x_i,\epsilon)} S_{F_{n_i}}\varphi(y)\Big) \leq \sum_i \exp\big(-(\alpha - \beta(\epsilon))|F_{n_i}| + S_{F_{n_i}}\varphi(x_i)\big).$$

This yields

$$m(Z, \varphi, \alpha, \epsilon) \leq m'(Z, \varphi, \alpha - \beta(\epsilon), \epsilon).$$

Hence $P_Z(\varphi, \epsilon) - \beta(\epsilon) \leq P'_Z(\varphi, \epsilon)$. Similarly, $P_Z(\varphi, \epsilon) + \beta(\epsilon) \geq P'_Z(\varphi, \epsilon)$. Combining these inequalities yields the desired result.

The other two formulas can be proved in a similar fashion. ■

Remark. The same arguments show that the above proposition remains true if we replace $\sup_{y\in B_{F_n}(x,\epsilon)} S_{F_n}\varphi(y)$ by any number in the interval

$$\Big[\inf_{y\in B_{F_n}(x,\epsilon)} S_{F_n}\varphi(y), \sup_{y\in B_{F_n}(x,\epsilon)} S_{F_n}\varphi(y)\Big].$$

For any subset $Z \subset X$ and any Følner sequence $\{F_n\}_{n\geq 1}$ in G, set

$$\Lambda(Z, \varphi, N, \epsilon) = \inf_{\Gamma} \sum_i \exp\Big(\sup_{y\in B_{F_N}(x_i,\epsilon)} S_{F_N}\varphi(y)\Big),$$

where the infimum is taken over all covers Γ of Z with $n_i = N$ for all i. We have the following equivalent definition of lower and upper topological pressure (the proof is similar to that of [Pe97, Theorem 2.2]).

Proposition 3.3. *For any subset $Z \subset X$ and any Følner sequence $\{F_n\}_{n\geq 1}$ in an infinite countable discrete amenable group G, we have*

$$\underline{CP}_Z(\varphi) = \lim_{\epsilon\to 0} \liminf_{N\to\infty} \frac{1}{|F_N|} \log \Lambda(Z, \varphi, N, \epsilon),$$
$$\overline{CP}_Z(\varphi) = \lim_{\epsilon\to 0} \limsup_{N\to\infty} \frac{1}{|F_N|} \log \Lambda(Z, \varphi, N, \epsilon).$$

Proof. We will prove the first equality; the second one can be proved in a similar fashion. Denote

$$A = \underline{CP}_Z(\varphi, \epsilon), \qquad B = \liminf_{N\to\infty} \frac{1}{|F_N|} \log \Lambda(Z, \varphi, N, \epsilon).$$

Given $\beta > 0$, one can choose a sequence $N_k \to \infty$ such that

$$0 = \underline{r}(Z, \varphi, A + \beta, \epsilon) = \lim_{k\to\infty} R(Z, \varphi, A + \beta, N_k, \epsilon).$$

It follows that $R(Z, \varphi, A+\beta, N_k, \epsilon) \leq 1$ for all sufficiently large k. Therefore, for such k,

$$\exp[-(A + \beta)|F_{N_k}|]\Lambda(Z, \varphi, N_k, \epsilon) \leq 1.$$

This yields

$$A+\beta \geq \liminf_{k\to\infty} \frac{1}{|F_{N_k}|} \log \Lambda(Z,\varphi,N_k,\epsilon),$$

and hence $A+\beta \geq B$.

To prove the reverse inequality, we choose a sequence $N'_k \to \infty$ such that

$$B = \lim_{k\to\infty} \frac{1}{|F_{N'_k}|} \log \Lambda(Z,\varphi,N'_k,\epsilon).$$

On the other hand, we have

$$\liminf_{k\to\infty} R(Z,\varphi,A-\beta,N'_k,\epsilon) \geq \underline{r}(Z,\varphi,A-\beta,\epsilon) = \infty.$$

This implies that $R(Z,\varphi,A-\beta,N'_k,\epsilon) \geq 1$ for all sufficiently large k. Therefore, for such k,

$$\exp[-(A-\beta)|F_{N'_k}|]\Lambda(Z,\varphi,N'_k,\epsilon) \geq 1.$$

Hence,

$$A-\beta \leq \lim_{k\to\infty} \frac{1}{|F_{N'_k}|} \log \Lambda(Z,\varphi,N'_k,\epsilon) = B.$$

Since β is arbitrary, this implies that $A = B$. Consequently,

$$\underline{CP}_Z(\varphi) = \lim_{\epsilon\to 0} \liminf_{N\to\infty} \frac{1}{|F_N|} \log \Lambda(Z,\varphi,N,\epsilon). \ \blacksquare$$

4. Proof of Theorem 2.5. For clarity, we divide the proof into four lemmas.

Recall Brin–Katok's entropy formula for amenable group actions (see [ZC]), which says that if $\mu \in \mathcal{E}(X,G)$ and $\{F_n\}_{n\geq 1}$ is a tempered Følner sequence in G with $\lim_{n\to\infty} (\log n)/|F_n| = 0$, then for μ-almost every $x \in X$,

$$h_\mu(G) = \lim_{\epsilon\to 0} \liminf_{n\to\infty} -\frac{1}{|F_n|} \log \mu(B_{F_n}(x,\epsilon)) \tag{4.1}$$
$$= \lim_{\epsilon\to 0} \limsup_{n\to\infty} -\frac{1}{|F_n|} \log \mu(B_{F_n}(x,\epsilon)).$$

LEMMA 4.1. *Under the conditions of Theorem* 2.5, *for any* $\mu \in \mathcal{E}(X,G)$ *and any continuous function* φ *on* X *we have*

$$P_\mu(\varphi) \leq h_\mu(G) + \int \varphi \, d\mu.$$

Proof. If $h_\mu(G)$ is infinite, the result follows immediately. Hence, we assume that $h_\mu(G)$ is finite and set $h := h_\mu(G)$.

Take $0<\delta<1$ and a small number $\eta > 0$.

Take $\epsilon_\eta > 0$ and a subset $\widetilde{K} \subseteq X$ with $\mu(\widetilde{K}) > 1 - \delta/2$ such that if $\epsilon \in (0, \epsilon_\eta]$ and $x \in \widetilde{K}$, then

$$\begin{aligned} h - \eta/2 &\le \liminf_{n\to\infty} -\frac{1}{|F_n|}\log\mu(B_{F_n}(x,\epsilon)) \\ &\le \limsup_{n\to\infty} -\frac{1}{|F_n|}\log\mu(B_{F_n}(x,\epsilon)) \le h + \eta/2. \end{aligned}$$

This is possible because of (4.1). Take $0 < \epsilon \le \epsilon_\eta$. Hence, for every $x \in \widetilde{K}$, there exists $N_1(x) > 0$ such that for any $n \ge N_1(x)$,

$$\left|\frac{1}{|F_n|}\log\mu(B_{F_n}(x,\epsilon/2)) + h\right| \le \eta. \tag{4.2}$$

On the other hand, by the Birkhoff ergodic theorem for amenable group actions (see [L] or [W]), for μ-almost every $x \in X$ there exists $N_2(x) > 0$ such that for any $n \ge N_2(x)$,

$$\left|\frac{1}{|F_n|}S_{F_n}\varphi(x) - \int\varphi\,d\mu\right| \le \eta. \tag{4.3}$$

For each N, let $K_N := \{x \in \widetilde{K} : N_1(x), N_2(x) \le N\}$. It follows that $K_N \subset K_{N+1}$, and $\bigcup_{N\ge0} K_N$ is a set of μ-measure larger than or equal to $1-\delta/2$. Then take $N_0 > 0$ such that $\mu(K_N) > 1 - 2\delta/3$ for any $N \ge N_0$. Take a compact subset $\widetilde{K}_N$ of K_N with $\mu(\widetilde{K}_N) > 1-\delta$. Fix such an N, and for any $n \ge N$, let T_n be a maximal $(n, \epsilon/2)$-separated subset of $\widetilde{K}_N$, i.e., for any distinct $x, y \in T_n$, $d_{F_n}(x,y) \ge \epsilon/2$. Then $\widetilde{K}_N \subseteq \bigcup_{x\in T_n} B_{F_n}(x,\epsilon)$. This means that T_n is an (n,ϵ,δ)-spanning set.

By (4.2), if $x \in \widetilde{K}_N$, then $\mu\big(B_{F_n}(x,\epsilon/2)\big) \ge \exp[-|F_n|(h+\eta)]$. Hence, T_n contains at most $\exp[|F_n|(h+\eta)]$ elements. By (4.3), if $x \in \widetilde{K}_N$, then $S_{F_n}\varphi(x) \le |F_n|(\int\varphi\,d\mu + \eta)$. Now we get

$$\begin{aligned} \sum_{x\in T_n}\exp[S_{F_n}\varphi(x)] &\le \sum_{x\in T_n}\exp\Big[|F_n|\Big(\int\varphi\,d\mu+\eta\Big)\Big] \\ &\le \exp[|F_n|(h+\eta)]\cdot\exp\Big[|F_n|\Big(\int\varphi\,d\mu+\eta\Big)\Big] \\ &= \exp\Big[|F_n|\Big(h+\int\varphi\,d\mu+2\eta\Big)\Big]. \end{aligned}$$

Since T_n is an (n,ϵ,δ)-spanning set, we have

$$P_\mu(\varphi,n,\epsilon,\delta) \le \exp\Big[|F_n|\Big(h+\int\varphi\,d\mu+2\eta\Big)\Big].$$

Consequently,

$$P_\mu(\varphi,\epsilon,\delta) \le h + \int\varphi\,d\mu + 2\eta.$$

Since ϵ and η are arbitrary, we have

$$P_\mu(\varphi,\delta) \le h + \int \varphi\, d\mu, \tag{4.4}$$

which yields the desired inequality. ∎

LEMMA 4.2. *Under the conditions of Theorem* 2.5, *for any* $\mu \in \mathcal{E}(X,G)$ *and any continuous function* φ *on* X *we have*

$$P_\mu(\varphi) \ge h_\mu(G) + \int \varphi\, d\mu.$$

Proof. The method used here is similar to the proof of [Pe97, Theorem A2.1]. Fix $0 < \delta < 1$. For each $\eta > 0$, there exists $0 < \gamma \le \eta$, a finite measurable partition $\xi = \{C_1,\dots,C_m\}$ and a finite open cover $\mathcal{U} = \{U_1,\dots,U_k\}$ of X, where $k \ge m$, such that the following properties hold (by the regularity of the measure μ, see [PP]):

(1) $|U_i| \le \eta$ and $|C_j| \le \eta$, $1 \le i \le k$, $1 \le j \le m$, where $|\cdot|$ denotes diameter;
(2) $\overline{U_i} \subset C_i$, $1 \le i \le m$, where the bar denotes closure;
(3) $\mu(C_i \setminus U_i) \le \gamma$, $1 \le i \le m$, and $\mu(\bigcup_{i=m+1}^k U_i) \le \gamma$;
(4) $2\gamma \log m \le \eta$.

Next, fix a small number η so that $1-\delta > \eta > 0$ and take the corresponding $\gamma > 0$, partition ξ and open cover $\mathcal{U}$ of X. Fix a subset $Z \subset X$ with $\mu(Z) > 1-\delta$ and set $t_n(x) := \sharp\{g \in F_n : gx \in \bigcup_{i=m+1}^k U_i\}$. Let $\xi_{F_n}(x)$ denote the element of the partition $\xi_{F_n} := \bigvee_{g\in F_n} g^{-1}\xi$ that contains x.

CLAIM. *There exist* $A \subset Z$ *and* $N > 0$ *with* $\mu(A) \ge \mu(Z) - \gamma$ *such that for every* $x \in A$ *and* $n \ge N$, *we have*

(i) $t_n(x) \le 2\gamma|F_n|$;
(ii) $\mu(\xi_{F_n}(x)) \le \exp[-(h_\mu(G,\xi)-\gamma)|F_n|]$;
(iii) $\int \varphi\, d\mu - \gamma \le \frac{1}{|F_n|} S_{F_n}\varphi(x) \le \int \varphi\, d\mu + \gamma$.

Proof of the Claim. Let $f := \chi_{\bigcup_{i=m+1}^k U_i}$ be the characteristic function of $\bigcup_{i=m+1}^k U_i$. Then $t_n(x) = \sum_{g\in F_n} f(gx)$. According to the Birkhoff ergodic theorem for amenable group actions and the Egorov theorem, we can find a set $A_1 \subset Z$ with $\mu(A_1) \ge \mu(Z) - \gamma/3$ such that

$$\lim_{n\to\infty} \frac{1}{|F_n|} t_n(x) = \lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g\in F_n} f(gx) = \int f\, d\mu = \mu\Big(\bigcup_{i=m+1}^k U_i\Big) \le \gamma$$

uniformly on A_1. Therefore, we can choose N_1 such that if $n \ge N_1$ and $x \in A_1$, then $t_n(x) \le 2\gamma|F_n|$. Using the Shannon–McMillan–Breiman theorem for amenable group actions (see [L] or [W]) and the Egorov theorem, we can find a set $A_2 \subset Z$ with $\mu(A_2) \ge \mu(Z) - \gamma/3$ and choose N_2 such that if

$n \geq N_2$ and $x \in A_2$, then $\mu(\xi_{F_n}(x)) \leq \exp[-(h_\mu(G,\xi)-\gamma)|F_n|]$. Then, using the Egorov theorem and the fact that

$$\lim_{n\to\infty} \frac{1}{|F_n|} S_{F_n}\varphi(x) = \int \varphi\, d\mu \qquad \mu\text{-a.e. } x \in X,$$

we can find a set $A_3 \subset Z$ with $\mu(A_3) \geq \mu(Z) - \gamma/3$ and choose N_3 such that if $n \geq N_3$ and $x \in A_3$, then $\int \varphi\, d\mu - \gamma \leq |F_n|^{-1} S_{F_n}\varphi(x) \leq \int \varphi\, d\mu + \gamma$. Setting $A = A_1 \cap A_2 \cap A_3$ and $N = \max\{N_1, N_2, N_3\}$ proves the Claim.

Set $\xi^*_{F_n} := \{\xi_{F_n}(x) \in \xi_{F_n} : \xi_{F_n}(x) \cap A \neq \emptyset\}$. Using (ii) of the Claim shows that

$$\begin{aligned} \sharp\xi^*_{F_n} &\geq \sum_{\xi_{F_n}(x)\in\xi^*_{F_n}} \mu(\xi_{F_n}(x)) \exp[(h_\mu(G,\xi)-\gamma)|F_n|] \\ &\geq \mu(A) \exp[(h_\mu(G,\xi)-\gamma)|F_n|] \end{aligned} \tag{4.5}$$

for all $n \geq N$. Let 2ϵ be the Lebesgue number of the open cover $\mathcal{U}$ and let T be an (n,ϵ)-spanning set for Z, i.e., $Z \subset \bigcup_{x\in T} B_{F_n}(x,\epsilon)$. Let $T' \subset T$ be such that $B_{F_n}(x,\epsilon) \cap A \neq \emptyset$ for each $x \in T'$. Fix $x \in T'$ and $B := B_{F_n}(x,\epsilon)$, and let $p(B,\xi_{F_n}) := \sharp\{C \in \xi_{F_n} : C \cap A \cap B \neq \emptyset\}$.

We now estimate the number $p(B,\xi_{F_n})$. For any $g \in F_n$, note that $B(gx,\epsilon) \subset U_{i_l}$ for some $U_{i_l} \in \mathcal{U}$, since 2ϵ is the Lebesgue number of $\mathcal{U}$. If $i_l \in \{1,\dots,m\}$ then $g^{-1}U_{i_l} \subset g^{-1}C_{i_l}$. If $i_l \in \{m+1,\dots,k\}$, then at most m sets of the form $g^{-1}C_{i_l}$ may have non-empty intersection with $g^{-1}U_{i_l}$. Using (i) of the claim shows that

$$p(B,\xi_{F_n}) \leq m^{2\gamma|F_n|} = \exp(2\gamma|F_n|\log m).$$

This yields

$$\sharp\xi^*_{F_n} \leq \sum_{x\in T'} p(B_{F_n}(x,\epsilon),\xi_{F_n}) \leq \sharp T' \exp(2\gamma|F_n|\log m). \tag{4.6}$$

Furthermore, note that $d(x,y) < \epsilon$ implies that $d(\varphi(x),\varphi(y)) < \beta$ for some small $\beta > 0$. Take $y \in B \cap A$. By (iii) of the claim,

$$S_{F_n}\varphi(x) \geq S_{F_n}\varphi(y) - |F_n|\beta \geq |F_n|\Big(\int \varphi\, d\mu - \beta - \gamma\Big).$$

Hence,

$$\sum_{x\in T} \exp[S_{F_n}\varphi(x)] \geq \sum_{x\in T'} \exp[S_{F_n}\varphi(x)] \geq \sharp T' \exp\Big[|F_n|\Big(\int \varphi\, d\mu - \beta - \gamma\Big)\Big].$$

Combining (4.5) and (4.6) yields

$$\sum_{x\in T} \exp[S_{F_n}\varphi(x)] \geq \mu(A) \exp\Big[|F_n|\Big(h_\mu(G,\xi) + \int \varphi\, d\mu - 2\gamma - \beta - 2\gamma \log m\Big)\Big].$$

This implies that

$$\frac{1}{|F_n|}\log P_\mu(\varphi,n,\epsilon,\delta)\ge\frac{1}{|F_n|}\log\mu(A)+h_\mu(G,\xi)+\int\varphi\,d\mu-2\gamma-\beta-2\gamma\log m.$$

Since $\gamma<\eta$, $2\gamma\log m<\eta$, $\beta\to 0$ as $\epsilon\to 0$, $|\xi|:=\max_{1\le i\le m}|C_i|\le\eta$, and η is arbitrary, we have

$$P_\mu(\varphi,\delta)\ge h_\mu(G)+\int\varphi\,d\mu. \tag{4.7}$$

This yields the desired result. ■

Remark. Lemmas 4.1 and 4.2 extend the main result in [HLZ] to amenable group actions. However, the methods used here are quite different from that in [HLZ]. In fact, we prove Lemma 4.1 by modifying the arguments in the proof of [CHZ, Proposition 4.2], and Lemma 4.2 is proved in a similar way to [Pe97, Theorem A2.1].

Lemma 4.3. *Under the conditions of Theorem* 2.5*, for any* $\mu\in\mathcal{E}(X,G)$ *and any continuous function* φ *on* X *we have*

$$\overline{CP}^*_\mu(\varphi)\le h_\mu(G)+\int\varphi\,d\mu.$$

Proof. If $h_\mu(G)=\infty$, the result follows immediately. In the following, we assume $h_\mu(G)$ is finite and set $h=h_\mu(G)\ge 0$.

Take $0<\delta<1$ and a small number $\eta>0$. For each N, take K_N as in the proof of Lemma 4.1. Then one can find $N_0>0$ such that $\mu(K_N)>1-2\delta/3$ for any $N\ge N_0$. Fix such an N and a compact subset $\widetilde{K}_N$ as in the proof of Lemma 4.1. For any $n\ge N$, let T_n be a maximal (n,ϵ)-separated subset of $\widetilde{K}_N$. Then $\widetilde{K}_N\subseteq\bigcup_{x\in T_n}B_{F_n}(x,\epsilon)$. Furthermore, the balls $\{B_{F_n}(x,\epsilon/2):x\in T_n\}$ are pairwise disjoint and the cardinality of T_n is less than or equal to $\exp[|F_n|(h+\eta)]$. Set

$$\Lambda'(\widetilde{K}_N,\varphi,n,\epsilon)=\inf_{\Gamma}\sum_i\exp(S_{F_n}\varphi(x_i))$$

where the infimum is taken over all covers Γ of $\widetilde{K}_N$ with $n_i=n$ for all i. Therefore,

$$\begin{aligned}\Lambda'(\widetilde{K}_N,\varphi,n,\epsilon)&\le\sum_{x\in T_n}\exp(S_{F_n}\varphi(x))\\&\le\exp[|F_n|(h+\eta)]\cdot\exp\Big[|F_n|\Big(\int\varphi\,d\mu+\eta\Big)\Big]\\&=\exp\Big[|F_n|\Big(h+\int\varphi\,d\mu+2\eta\Big)\Big].\end{aligned}$$

By Propositions 3.2 and 3.3, we have

$$\overline{CP}_{\widetilde{K}_N}(\varphi,\epsilon)\le h+\int\varphi\,d\mu+2\eta.$$

By the definition of $\overline{CP}^*_\mu(\varphi)$ we obtain

$$\begin{aligned}\overline{CP}^*_\mu(\varphi) &= \sup_{\epsilon>0}\sup_{\delta>0}\inf\{\overline{CP}_Z(\varphi,\epsilon) : \mu(Z) \geq 1-\delta\}\\ &= \sup_{\delta>0}\sup_{\epsilon>0}\inf\{\overline{CP}_Z(\varphi,\epsilon) : \mu(Z) \geq 1-\delta\}.\end{aligned}$$

Since $\mu(\widetilde{K}_N) \geq 1-\delta$, we have

$$\sup_{\epsilon>0}\inf\{\overline{CP}_Z(\varphi,\epsilon) : \mu(Z) \geq 1-\delta\} \leq h + \int\varphi\,d\mu + 2\eta.$$

Since δ and η are arbitrary, we have

$$\overline{CP}^*_\mu(\varphi) \leq h + \int\varphi\,d\mu. \ \blacksquare$$

LEMMA 4.4. *Under the conditions of Theorem* 2.5, *for any* $\mu \in \mathcal{E}(X,G)$ *and any continuous function* φ *on* X *we have*

$$P^*_\mu(\varphi) \geq h_\mu(G) + \int\varphi\,d\mu.$$

Proof. It is sufficient to prove that for each $\eta > 0$ there exists $\epsilon_\eta > 0$ such that for all $0 < \epsilon \leq \epsilon_\eta$,

$$P_Z(\varphi,\epsilon) \geq h_\mu(G) + \int\varphi\,d\mu - 2\eta$$

for any subset $Z \subseteq X$ of full μ-measure. We first assume that $h_\mu(G)$ is finite and set $h = h_\mu(G)$.

Take a small number $\eta > 0$, and denote $\lambda = h + \int\varphi\,d\mu - 2\eta$.

Let

$$K = \Big\{x \in X : \lim_{\epsilon\to0}\limsup_{n\to\infty}\frac{-\log\mu(B_{F_n}(x,\epsilon))}{|F_n|} = h_\mu(G) \\ \text{and } \lim_{n\to\infty}\frac{1}{|F_n|}S_{F_n}\varphi(x) = \int\varphi\,d\mu\Big\}.$$

By Brin–Katok's theorem (see [ZC]) for amenable group actions, there exist $\epsilon_\eta > 0$, a subset $K_1 \subset K$ with $\mu(K_1) > 3/4$, and $N_1 > 0$ such that

$$\begin{aligned}\mu(B_{F_n}(x,2\epsilon)) &\leq \mu(B_{F_n}(x,2\epsilon_\eta))\\ &\leq \exp(-|F_n|(h-\eta)), \quad \forall x \in K_1,\ \forall n \geq N_1,\ \forall 0 < \epsilon \leq \epsilon_\eta.\end{aligned}$$

By the Birkhoff ergodic theorem for amenable group actions (see [L] or [W]), there exist a subset $K_2 \subset K$ with $\mu(K_2) > 3/4$ and $N_2 > 0$ such that for any $x \in K_2$ and $n \geq N_2$, we have

$$\left|\frac{1}{|F_n|}S_{F_n}\varphi(x) - \int\varphi\,d\mu\right| < \eta.$$

Set $N = \max\{N_1,N_2\}$. Given any measurable subset $Z \subseteq X$ with $\mu(Z) = 1$, it is clear that there exists a compact subset $\widetilde{K} \subset K_1\cap K_2\cap Z$ with $\mu(\widetilde{K}) > 1/2$.

Fix $\epsilon \in (0, \epsilon_\eta]$ and take a cover $\Gamma = \{B_{F_{n_i}}(x_i, \epsilon)\}_i$ of $\widetilde{K}$ with $n_i \geq n > N$ for any i. Since $\widetilde{K}$ is compact, we may assume that the cover is finite and consists of $B_{F_{n_1}}(x_1, \epsilon), \ldots, B_{F_{n_l}}(x_l, \epsilon)$. For each $i = 1, \ldots, l$, we choose $y_i \in \widetilde{K} \cap B_{F_{n_i}}(x_i, \epsilon)$. Hence, $B_{F_{n_i}}(x_i, \epsilon) \subset B_{F_{n_i}}(y_i, 2\epsilon)$ and $\{B_{F_{n_i}}(y_i, 2\epsilon)\}_i$ form a cover of $\widetilde{K}$ as well. This yields

$$\begin{aligned}\sum_{B_{F_{n_i}}(x_i,\epsilon)\in\Gamma} &\exp\Big(-|F_{n_i}|\lambda + \sup_{y\in B_{F_{n_i}}(x_i,\epsilon)} S_{F_{n_i}}\varphi(y)\Big)\\ &\geq \sum_{i=1}^{l} \exp(-|F_{n_i}|\lambda + S_{F_{n_i}}\varphi(y_i))\\ &\geq \sum_{i=1}^{l} \exp\Big(-|F_{n_i}|(\lambda - \textstyle\int \varphi\, d\mu + \eta)\Big)\\ &= \sum_{i=1}^{l} \exp(-|F_{n_i}|(h-\eta)) \geq \sum_{i=1}^{l} \mu(B_{F_{n_i}}(y_i, 2\epsilon)) \geq \frac{1}{2}.\end{aligned}$$

Note that the inequality holds for any cover $\Gamma = \{B_{n_i}(x_i, \epsilon)\}_i$ of $\widetilde{K}$ with $n_i \geq n$ for each i. Hence,

$$M(\widetilde{K}, \varphi, \lambda, n, \epsilon) \geq 1/2, \qquad \forall n > N.$$

Thus $m(\widetilde{K}, \varphi, \lambda, \epsilon) \geq 1/2$. This yields

$$P_{\widetilde{K}}(\varphi, \epsilon) \geq \lambda = h + \textstyle\int \varphi\, d\mu - 2\eta.$$

Using Proposition 3.1, we have

$$P_Z(\varphi, \epsilon) \geq h + \textstyle\int \varphi\, d\mu - 2\eta.$$

When $h_\mu(G)$ is infinite, by modifying subtly the above arguments, we can easily deduce that $P_\mu^*(\varphi) = \infty$. This finishes the proof of the lemma. ∎

Remark. Lemmas 4.3 and 4.4 extend [Pe97, Theorem 11.6] to amenable group actions. We modify subtly the arguments in [Pe97, Theorem 11.6] to get the desired results.

Proof of Theorem 2.5. By Lemmas 4.1 and 4.2, we have $P_\mu(\varphi) = h_\mu(G) + \int \varphi\, d\mu$. By Lemmas 4.3 and 4.4 and Proposition 3.1 we obtain $P_\mu^*(\varphi) = \underline{CP}_\mu^*(\varphi) = \overline{CP}_\mu^*(\varphi) = h_\mu(G) + \int \varphi\, d\mu$. This completes the proof of Theorem 2.5. ∎

Proof of Corollary 2.7. Set

$$A = \sup_{\epsilon>0} \inf_Z \{P_Z(\varphi, \epsilon) : \mu(Z) = 1\} \quad \text{and} \quad B = \inf_Z \sup_{\epsilon>0} \{P_Z(\varphi, \epsilon) : \mu(Z) = 1\}.$$

Clearly, $A \le B$. By the definition of $P^*_\mu(\varphi)$ and Theorem 2.5, we have $A = P^*_\mu(\varphi) = h_\mu(G) + \int \varphi\, d\mu$.

On the other hand, take a sequence $\{\epsilon_n\}_{n\ge1}$ of positive numbers such that $\epsilon_n \to 0$ $(n \to \infty)$. Fix a small $\beta > 0$. For each $n \ge 1$, there exists a subset $Z_n \subseteq X$ with $\mu(Z_n) = 1$ such that

$$A \ge \inf_Z \{P_Z(\varphi,\epsilon_n) : \mu(Z) = 1\} \ge P_{Z_n}(\varphi,\epsilon_n) - \beta \ge P_{\widetilde{Z}}(\varphi,\epsilon_n) - \beta$$

where $\widetilde{Z} = \bigcap_{n\ge1} Z_n$. Letting $n \to \infty$, since $\mu(\widetilde{Z}) = 1$,

$$A \ge P_{\widetilde{Z}}(\varphi) - \beta = \sup_{\epsilon>0} P_{\widetilde{Z}}(\varphi,\epsilon) - \beta \ge B - \beta.$$

Hence, $A \ge B$. Consequently,

$$h_\mu(G) + \int \varphi\, d\mu = B = \inf_Z \{P_Z(\varphi) : \mu(Z) = 1\}. \ \blacksquare$$

5. Proof of Theorem 2.8. First note that the inequality $P_K(\varphi) \ge h_\mu(G) + \int \varphi\, d\mu$ can be deduced by Lemma 4.4, as $P^*_\mu(\varphi) \le P_K(\varphi)$ from the assumption of $\mu(K) = 1$.

Next we prove that $P_K(\varphi) \le h_\mu(G) + \int \varphi\, d\mu$. If $h_\mu(G) = \infty$, the result follows immediately. In the following we assume that $h_\mu(G)$ is finite and set $h = h_\mu(G) \ge 0$. We modify subtly the arguments used in the proof of Lemma 4.1.

Take a small number $\eta > 0$. For each $x \in X$ and $\epsilon > 0$, set

$$\underline{h}(x,\epsilon) := \liminf_{n\to\infty} -\frac{1}{|F_n|} \log \mu(B_{F_n}(x,\epsilon)),$$
$$\overline{h}(x,\epsilon) := \limsup_{n\to\infty} -\frac{1}{|F_n|} \log \mu(B_{F_n}(x,\epsilon)).$$

For each positive integer M, define

$$L_M = \{x \in K : h - \eta/2 \le \underline{h}(x,\epsilon/2) \le \overline{h}(x,\epsilon/2) \le h + \eta/2,\ \forall 0 < \epsilon < 1/M\}.$$

Clearly, $L_{M_2} \subseteq L_{M_1}$ if $M_1 > M_2$ and $K = \bigcup_{M=1}^\infty L_M$. Given a positive integer M, let

$$L_{M,N} = \Big\{x \in L_M : \exp[|F_n|(-h-\eta)] \le \mu(B_{F_n}(x,\epsilon/2)) \le \exp[|F_n|(-h+\eta)],$$
$$\left|\frac{1}{|F_n|} S_{F_n}\varphi(x) - \int \varphi\, d\mu\right| < \eta,\ \forall n \ge N\Big\}.$$

It is easy to see that $L_{M,N_2} \subseteq L_{M,N_1}$ if $N_1 > N_2$, and $L_M = \bigcup_{N=1}^\infty L_{M,N}$.

Take two sufficiently large positive integers M, N so that $\mu(L_{M,N}) > 0$. For any $n \ge N$ and $0 < \epsilon < 1/M$, let T_n be a maximal $(n,\epsilon/2)$-separated subset of $L_{M,N}$, i.e., $d_{F_n}(x,y) \ge \epsilon/2$ for any distinct $x, y \in T_n$. Thus, $L_{M,N} \subseteq \bigcup_{x\in T_n} B_{F_n}(x,\epsilon/2)$.

Since $\mu(B_{F_n}(x,\epsilon/2)) \geq \exp[-|F_n|(h+\eta)]$ for any $x \in L_{M,N}$, T_n contains at most $\exp[|F_n|(h+\eta)]$ elements. Hence,

$$\begin{aligned}\sum_{x\in T_n}\exp[S_{F_n}\varphi(x)] &\leq \sum_{x\in T_n}\exp\Big[|F_n|\Big(\int\varphi\,d\mu+\eta\Big)\Big]\\ &\leq \exp[|F_n|(h+\eta)]\cdot\exp\Big[|F_n|\Big(\int\varphi\,d\mu+\eta\Big)\Big]\\ &= \exp\Big[|F_n|\Big(h+\int\varphi\,d\mu+2\eta\Big)\Big].\end{aligned}$$

This yields

$$\overline{CP}'_{L_{M,N}}(\varphi,\epsilon/2) \leq h + \int\varphi\,d\mu + 2\eta.$$

Letting $\epsilon \to 0$, since η is arbitrary, by Propositions 3.1(iii) and 3.2 we get

$$P_{L_{M,N}}(\varphi) \leq \overline{CP}_{L_{M,N}}(\varphi) \leq h + \int\varphi\,d\mu.$$

Letting $N \to \infty$, by Proposition 3.1(ii) we obtain $P_{L_M}(\varphi) \leq h + \int\varphi\,d\mu$. Consequently, $P_K(\varphi) \leq h + \int\varphi\,d\mu$ by letting $M \to \infty$. This completes the proof of the theorem.

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Yun Zhao
Department of Mathematics
Soochow University
Suzhou 215006, China
E-mail: zhaoyun@suda.edu.cn