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Small product sets in compact groups

by

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Abstract. We show that a subcritical pair (A, B) of sufficiently "spread-out" Borel sets in a compact and second countable group K with an *abelian* identity component must reduce to a Sturmian pair in either $\mathbb{T}$ or $\mathbb{T} \rtimes \{-1, 1\}$. This extends a classical result of Kneser.

1. The main result. Let K be a compact and second countable Hausdorff group with Haar probability measure m_K. Given two subsets $A, B \subset K$, we define their *product set* AB by

$$AB = \{ab : a \in A,\, b \in B\}.$$

If A and B are both Borel measurable subsets, then AB is always measurable with respect to the m_K*-completion* of the Borel σ-algebra on K, but it might fail to be Borel measurable. This technical point will not play a major role in this paper.

Before we proceed, we stress that all groups that we shall consider here are assumed to be Hausdorff.

Definition 1.1 (Critical and subcritical pairs). Suppose that $A, B \subset K$ are Borel sets. We say that (A, B) is *critical* if

$$m_K(A), m_K(B) > 0 \quad \text{and} \quad m_K(AB) < m_K(A) + m_K(B),$$

and *subcritical* if

$$m_K(A), m_K(B) > 0 \quad \text{and} \quad m_K(AB) = m_K(A) + m_K(B) < 1.$$

We warn the reader that the opposite nomenclature concerning these types of sets is sometimes adopted in the literature.

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Definition 1.2 (Reduction). Let M be a factor group of K and suppose that $I, J \subset M$ are Borel sets. We say that (A, B) *reduces* to (I, J) if

$$A \subset q^{-1}(I) \quad \text{and} \quad B \subset q^{-1}(J) \quad \text{and} \quad m_K(AB) = m_M(IJ),$$

where $q : K \to M$ denotes the canonical quotient map.

We denote by $\mathbb{T}$ the one-dimensional torus group $\mathbb{R}/\mathbb{Z}$, and by $\mathbb{T} \rtimes \{-1, 1\}$ the (non-abelian) semidirect product of $\mathbb{T}$ with the multiplicative group $\{-1, 1\}$.

Definition 1.3 (Sturmian pair). Let $I, J \subset \mathbb{T}$ be closed and symmetric intervals and assume that $m_{\mathbb{T}}(I) + m_{\mathbb{T}}(J) < 1$. Let M denote either $\mathbb{T}$ or $\mathbb{T} \rtimes \{-1, 1\}$. We say that a pair (A, B) of Borel sets in M is *Sturmian* if there exist $a, b \in M$ such that either

$$(A, B) = (aI, Jb) \quad \text{or} \quad (A, B) = (a(I \rtimes \{-1, 1\}), (J \rtimes \{-1, 1\})b).$$

One readily verifies that every Sturmian pair is subcritical in M.

In their very influential papers [6], [7], Kemperman and Kneser established:

Theorem 1.4 ([6, Theorem 1], [7, Satz 4]). *Suppose that $A, B \subset K$ are Borel sets.*

- *If (A, B) is critical, then there exists an open normal subgroup $U \lhd K$ such that $ABU = AB$. In particular, AB is clopen, and (A, B) reduces to a pair in a* finite *factor group of K.*
- *If K is* connected *and* abelian*, and (A, B) is subcritical, then it reduces to a Sturmian pair in $\mathbb{T}$.*

Remark 1.5. The first assertion effectively reduces the study of critical pairs in K to the study of critical pairs in finite groups, where powerful results of Kemperman [5], Vosper [11] and DeVos [2] are available.

Recently, Griesmer [3] was able to further advance the description of subcritical pairs in compact *abelian* groups. Motivated by his work, and by some recent applications in ergodic theory related to actions of countable and discrete amenable groups developed by the author and A. Fish [1], we now turn our attention to subcritical pairs in compact and second countable groups with an *abelian* identity component. The relevance of this class of groups in the setting of [1] stems from the observation that any compact group which contains a dense countable *amenable* subgroup must have an abelian identity component. For a proof of this observation, we refer the reader to the Appendix.

Definition 1.6 (Spread-out set). A Borel subset $B \subset K$ is *spread-out* if every conull subset of B projects onto every finite factor group of K. If

$A, B \subset K$ are Borel sets, we say that the pair (A, B) is *spread-out* if both A and B are spread-out.

REMARK 1.7. We stress that K is always a factor of itself, and thus a *proper* subset of a finite group can never be spread-out. Furthermore, if (A, B) is a critical pair in K and $m_K(AB) < 1$, then neither A nor B is spread-out by Theorem 1.4, since they project onto the *proper* subsets AU and BU respectively in the finite factor group G/U.

Our main result can now be formulated as follows:

THEOREM 1.8. *Let K be a compact and second countable group with an* abelian *identity component. Let $A, B \subset K$ be Borel sets and suppose that (A, B) is spread-out and subcritical. Then (A, B) reduces to a Sturmian pair.*

REMARK 1.9. We stress that this is a much weaker structural result than the previously mentioned results of Kneser, Kemperman, DeVos and Griesmer. For instance, our assumptions can never be satisfied when the identity component of K is trivial, e.g. if K is a finite group. Indeed, first note that if A is spread-out in K and $U \triangleleft K$ is an open normal subgroup, then we must have $A \cap xU \neq \emptyset$ for every $x \in K/U$, since otherwise A would not project onto the finite factor group K/U. If the identity component of K is trivial, then open normal subgroups form a neighborhood basis for the identity in K, and thus every spread-out set in K must be dense. It is not hard to see that the product set of any dense subset of K and a Borel subset of K of positive measure is conull. We conclude that if $A \subset K$ is spread-out and $B \subset K$ is any Borel set of positive measure, then AB is conull, so in particular, (A, B) cannot be subcritical.

2. An outline of the proof of Theorem 1.8. The aim of this section is to reduce the proof of Theorem 1.8 to two main propositions which will be proven in Sections 3 and 4 below.

Let K be a compact and second countable group with Haar probability measure m_K and identity component N. We recall that N is a closed normal subgroup of K, but we stress that it does not need to split K into a semidirect product. However, this is not far from the truth, as the following result by Lee shows:

THEOREM 2.1 ([8]). *There exists a closed and totally disconnected subgroup $L < K$ such that $NL = K$. In particular, the semidirect product group $G = N \rtimes L$, where L acts on N by conjugation, factors onto K, and we denote by p the canonical quotient map from G onto K.*

2.1. Subcriticality with respect to a subgroup. Let G, N, L and p be as in Theorem 2.1. We shall view N and L as closed subgroups of G. If

$U < G$ is a closed subgroup, we consider the Haar probability measure m_U on U as a Borel probability measure on G which is supported on U.

Definition 2.2 (Subcritical with respect to a subgroup). Let G be a compact and second countable group, and let $U \lhd G$ be a closed normal subgroup. We say that a pair (A, B) of Borel sets in G is *subcritical with respect to* U if it is a subcritical pair in G and there are conull Borel sets $X \subset G$ and $Y \subset X \times X$ such that

$$m_U(s^{-1}A \cap U) = m_G(A) \quad \text{and} \quad m_U(Bt^{-1} \cap U) = m_G(B) \tag{2.1}$$

for all $s, t \in X$, and

$$m_U((s^{-1}A \cap U)(Bt^{-1} \cap U)) = m_U(s^{-1}A \cap U) + m_U(Bt^{-1} \cap U), \tag{2.2}$$

$$m_U((s^{-1}A \cap U)(Bt^{-1} \cap U)) = m_U(s^{-1}ABt^{-1} \cap U) \tag{2.3}$$

for all $(s, t) \in Y$.

Remark 2.3. Since $m_G(A) + m_G(B) < 1$, we have

$$m_U((s^{-1}A \cap U)(Bt^{-1} \cap U)) = m_U(s^{-1}A \cap U) + m_U(Bt^{-1} \cap U) < 1.$$

In particular, $(s^{-1}A \cap U, Bt^{-1} \cap U)$ is a subcritical pair in U for every $(s, t) \in Y$, which at least partially motivates the terminology "subcritical with respect to U".

The proof of Theorem 1.8 breaks into two parts, guided by the following two propositions which will be established in Sections 3 and 4 respectively.

Proposition 2.4. *Let $A, B \subset G$ be Borel sets and suppose that (A, B) is spread-out and subcritical. Then (A, B) is subcritical with respect to N.*

Proposition 2.5. *Let $A, B \subset G$ be Borel sets and suppose that N is non-trivial and* abelian. *If (A, B) is spread-out and subcritical with respect to N, then there are conull Borel sets $A' \subset A$ and $B' \subset B$ such that (A', B') reduces to a Sturmian pair.*

2.2. Proof of Theorem 1.8. Let K be a compact and second countable group with a non-trivial *abelian* identity component N. Let $A_o, B_o \subset K$ be Borel sets of positive Haar measures, and suppose that (A_o, B_o) is spread-out and subcritical. Let G, N, L and p be as in Theorem 2.1. We define the Borel sets $A, B \subset G$ by

$$A = p^{-1}(A_o) \quad \text{and} \quad B = p^{-1}(B_o). \tag{2.4}$$

One readily verifies that (A, B) is subcritical in G.

Lemma 2.6. *(A, B) is spread-out in G.*

Proof. We argue by contradiction (the proof that B is spread-out is completely analogous). Suppose that we can find a conull subset $A' \subset p^{-1}(A_o)$

and a finite factor group Q of G such that $r(A') \neq Q$, where $r : G \to Q$ denotes the canonical factor map. Since

$$m_G(A't \cap p^{-1}(A_o)) = m_G(p^{-1}(A_o)) \quad \text{for all } t \in \ker p,$$

and $A' \subset A' \ker r$, we have

$$m_G((A' \ker r)t \cap p^{-1}(A_o)) = m_G(p^{-1}(A_o)) \quad \text{for all } t \in (\ker p) \ker r.$$

Since $Q = G/\ker r$ is a finite group, the (finite) intersection

$$A'' = \bigcap_{t \in (\ker p) \ker r} (A' \ker r)t \cap p^{-1}(A_o)$$

is a measurable conull subset of $p^{-1}(A_o)$ which is invariant under $\ker p$ on the right hand side. Since $A'' \subset A' \ker r$, we have $r(A'') \subset r(A')$, and since $r(A') \neq Q$ and $r(A'')r(\ker p) = r(A'')$, it follows that

$$r(\ker p) \neq Q \quad \text{and} \quad r(A'')/r(\ker p) \neq Q/r(\ker p).$$

In particular, if we let $s : K \to Q/r(\ker p)$ denote the canonical quotient map, then $s(p(A'')) = r(A'')/r(\ker p)$ is a proper subset of the finite factor group $Q/r(\ker p)$ of K. Since $A'' \subset p^{-1}(A_o)$ is conull, we deduce that $p(A'') \subset A_o$ is a conull subset. Since $p(A'')$ does not project onto $Q/r(\ker p)$ under s, we see that the set A_o is not spread-out in K, which contradicts our assumption about A_o. ∎

The assumptions of Proposition 2.4 are now satisfied and we conclude that (A, B) is subcritical with respect to N. By Proposition 2.5 there are conull subsets $A' \subset A$ and $B' \subset B$ such that (A', B') reduces to a Sturmian pair in either $M = \mathbb{T}$ or $M = \mathbb{T} \rtimes \{-1, 1\}$. We recall that this means that there exists a surjective continuous homomorphism $q : G \to M$ such that

$$A' \subset q^{-1}(C) \quad \text{and} \quad B' \subset q^{-1}(D), \tag{2.5}$$

where (C, D) is a Sturmian pair in M such that $m_M(CD) = m_G(A'B')$. In particular, we have

$$m_G(A) = m_G(A') \leq m_M(C) \quad \text{and} \quad m_G(B) = m_G(B') \leq m_M(D).$$

However, since both (A, B) and (C, D) are subcritical, we also have

$$m_G(A) + m_G(B) = m_G(AB) \geq m_G(A'B') = m_M(CD) = m_M(C) + m_M(D),$$

which now forces $m_G(A) = m_M(C)$ and $m_G(B) = m_M(D)$, and so in particular $m_G(AB) = m_M(CD)$.

We stress that it is not clear at this point whether (A, B) reduces to (C, D), that is to say, we do not yet know that the inclusions in (2.5) also hold when A' and B' are replaced with A and B respectively. Even if this were true, we would still need an argument to show that this implies that the pair (A_o, B_o) reduces to (C, D), which is what Theorem 1.8 asserts. In order to fill in these gaps, we shall utilize the notions of *stability* and *regularity* of pairs of Borel sets.

2.2.1. *Stability and regularity*

Definition 2.7 (Regular and stable pairs). We say that a closed set $A \subset G$ is *regular* if it is Jordan measurable with respect to m_G and equal to the closure of its interior, and we say that a pair (A, B) is *regular* if both A and B are regular sets. We say that (A, B) is *stable* if the inclusions

$$aB \subset AB \quad \text{and} \quad Ab \subset AB$$

imply that $a \in A$ and $b \in B$.

Remark 2.8. We leave it to the reader to verify that Sturmian pairs are always regular and stable, as is the the pair $(q^{-1}(C), q^{-1}(D))$ in (2.5) above.

If A_1 and A_2 are Borel sets, we write $A_1 \sim A_2$ if $m_G(A_1 \triangle A_2) = 0$, where $\triangle$ denotes the symmetric difference of sets.

Lemma 2.9. *Suppose that (A_1, B_1) and (A_2, B_2) are subcritical pairs in G such that*

$$A_1 \sim A_2 \quad \text{and} \quad B_1 \sim B_2.$$

If (A_2, B_2) is a regular and stable pair, then $A_1 \subset A_2$ and $B_1 \subset B_2$.

We recall from above that $A = p^{-1}(A_o)$ and $B = p^{-1}(B_o)$ and

$$p^{-1}(A_o) \sim q^{-1}(C) \quad \text{and} \quad p^{-1}(B_o) \sim q^{-1}(D),$$

and both

$$(p^{-1}(A_o), p^{-1}(B_o)) \quad \text{and} \quad (q^{-1}(C), q^{-1}(D))$$

are subcritical pairs in G. The latter pair is in addition both regular and stable. Lemma 2.9, applied to

$$A_1 = p^{-1}(A_o), \quad A_2 = q^{-1}(C), \quad B_1 = p^{-1}(B_o), \quad B_2 = q^{-1}(D),$$

now tells us that

$$p^{-1}(A_o) \subset q^{-1}(C) \quad \text{and} \quad p^{-1}(B_o) \subset q^{-1}(D). \tag{2.6}$$

We claim that these inclusions force $\ker p \subset \ker q$, which will finish the proof of Theorem 1.8. Indeed, if $\ker p \subset \ker q$, then the map $\pi : K \to M$ given by $\pi(k) = q(p^{-1}(k))$ is a well-defined homomorphism, and by (2.6), we have

$$A_o \subset \pi^{-1}(C) \quad \text{and} \quad B_o \subset \pi^{-1}(D).$$

Furthermore, since $m_K(A_o B_o) = m_G(AB) = m_M(CD)$, the pair (A_o, B_o) reduces to (C, D), and

$$m_K(A_o) = m_G(A) = m_M(C) \quad \text{and} \quad m_K(B_o) = m_G(B) = m_M(D).$$

2.2.2. *Proving* $\ker p \subset \ker q$*.* Let us return to (2.6). We first note that

$$p^{-1}(A_o) \subset q^{-1}(C)g \quad \text{for all } g \in \ker p,$$

whence

$$p^{-1}(A_o) \subset q^{-1}(C) \cap q^{-1}(C)g = q^{-1}(C \cap Cq(g))$$

for all $g \in \ker p$, and thus

$$m_K(A_o) \leq m_M(C \cap Cq(g)) \leq m_M(C) = m_K(A_o),$$

which implies that $m_M(C) = m_M(C \cap Cq(g))$ for all $g \in \ker p$. Hence $q(\ker p)$ is contained in the *right (essential) stabilizer* Q of C, where

$$Q = \{m \in M : m_M(C \cap Cm) = m_M(C)\} < M. \tag{2.7}$$

If we denote by r the right regular representation of M on $L^2(M)$, then we see that Q is the actual stabilizer of the indicator function χ_C in $L^2(M)$. Since M acts continuously through r on the Hilbert space $L^2(M)$, we conclude that Q is a *closed* subgroup of M.

If $M = \mathbb{T}$ and $C = aI$ for some $a \in \mathbb{T}$ and *proper* closed interval $I \subset \mathbb{T}$, then Q is clearly trivial, and thus $q(\ker p)$ is trivial as well. If $M = \mathbb{T} \rtimes \{-1, 1\}$ and $C = a(I \rtimes \{-1, 1\})$ for some $a \in \mathbb{T} \rtimes \{-1, 1\}$ and *proper* closed interval $I \subset \mathbb{T}$, then a tedious but straightforward calculation shows that $Q = \{0\} \rtimes \{-1, 1\}$, which is *not* a normal subgroup of M. However, since the image $q(\ker p)$ is always a normal subgroup of M, which must be contained in Q, we conclude that the subgroup $q(\ker p)$ is trivial.

2.3. Proof of Lemma 2.9. We assume that (A_1, B_1) and (A_2, B_2) are subcritical pairs in G, and $A_1 \sim A_2$ and $B_1 \sim B_2$, and (A_2, B_2) is a regular and stable pair. We define the sets

$$A_o = A_1 \cap A_2 \quad \text{and} \quad B_o = B_1 \cap B_2$$

and note that

$$m_G(A_o) = m_G(A_1) = m_G(A_2) \quad \text{and} \quad m_G(B_o) = m_G(B_1) = m_G(B_2),$$

and

$$\begin{aligned} m_G(A_oB_o) &\leq m_G(A_1B_1) = m_G(A_1) + m_G(B_1) \\ &= m_G(A_o) + m_G(B_o) < 1. \end{aligned} \tag{2.8}$$

If the inequality were strict, then (A_o, B_o) is critical, which by the first assertion of Theorem 1.4 implies that there exists an open *normal* subgroup $U \lhd G$ such that $A_oB_o = A_oB_oU$. We note that A_oU and B_oU are clopen sets, and thus

$$V_1 = A_2^o \setminus A_oU \quad \text{and} \quad V_2 = B_2^o \setminus B_oU \tag{2.9}$$

are open. Since A_2 and B_2 are Jordan measurable and $A_o \sim A_2$ and $B_o \sim B_2$, we have $A_o \sim A_2^o$ and $B_o \sim B_2^o$. Hence, V_1 and V_2 are open m_G-null sets, and thus empty. Furthermore, since A_2 and B_2 are regular sets and A_oU and B_oU are clopen, we conclude by (2.9) that

$$A_2 = \overline{A_2^o} \subset A_oU \quad \text{and} \quad B_2 = \overline{B_2^o} \subset B_oU,$$

and since U is normal,

$$A_2B_2 \subset A_oUB_o = A_oB_o \subset A_2B_2,$$

and thus

$$m_G(A_oB_o) = m_G(A_2B_2).$$

Since (A_2, B_2) is subcritical, we have

$$\begin{aligned} m_G(A_oB_o) &= m_G(A_2B_2) = m_G(A_2) + m_G(B_2) \\ &= m_G(A_o) + m_G(B_o) < 1, \end{aligned} \tag{2.10}$$

which contradicts our assumption that (A_o, B_o) is critical.

Going back to the chain of identities in (2.8) and (2.10), we see that we can henceforth assume that

$$m_G(A_oB_o) = m_G(A_1B_1) = m_G(A_2B_2).$$

We wish to prove that $A_1 \subset A_2$ and $B_1 \subset B_2$. Assume for the sake of contradiction that there exists an element $x \in A_1 \setminus A_2$. In particular, $x \notin A_2$ and $A_o \cup \{x\} \subset A_1$. Then, since $A_o \sim A_1 \sim A_2$ and $B_o \sim B_1 \sim B_2$ and $A_oB_o \sim A_2B_2$, we have

$$\begin{aligned} m_G(A_1) + m_G(B_1) &= m_G(A_1B_1) \\ &\geq m_G((A_o \cup \{x\})B_o) \\ &= m_G(A_oB_o \cup (xB_o \setminus A_oB_o)) \\ &= m_G(A_oB_o) + m_G(xB_o \setminus A_oB_o) \\ &= m_G(A_o) + m_G(B_o) + m_G(xB_2 \setminus A_2B_2) \\ &= m_G(A_1) + m_G(B_1) + m_G(xB_2 \setminus A_2B_2), \end{aligned}$$

which forces $xB_2 \setminus A_2B_2$ to be an m_G-null set. Since A_2 and B_2 are closed, so is A_2B_2. In particular, $xB_2^o \setminus A_2B_2$ is open and m_G-null, which forces it to be empty. Since B_2 is regular and A_2B_2 is closed, we have

$$x\overline{B_2^o} = xB_2 \subset A_2B_2.$$

By assumption, the pair (A_2, B_2) is stable, and thus $x \in A_2$, which is a contradiction. This shows that $A_1 \subset A_2$. The proof that $B_1 \subset B_2$ is the same.

3. Proof of Proposition 2.4. Let G be a compact and second countable group with identity component N. Since G is second countable, there exists a decreasing sequence (U_n) of *open* normal subgroups of G such that

N equals their intersection. The following two results immediately imply Proposition 2.4.

PROPOSITION 3.1. *Suppose that (A, B) is spread-out and subcritical in G. Then (A, B) is subcritical with respect to U_n for every n.*

PROPOSITION 3.2. *If (A, B) is subcritical with respect to U_n for every n, then (A, B) is subcritical with respect to N.*

3.1. Proof of Proposition 3.1. Fix an open normal subgroup U of G throughout this subsection, and suppose that (A, B) is spread-out and subcritical in G. We note that the Haar probability measure m_U on the clopen subgroup U, viewed as a Borel probability measure on G, is given by

$$m_U(D) = \frac{m_G(D \cap U)}{m_G(U)} \quad \text{for Borel sets } D \subset G. \tag{3.1}$$

We wish to prove that (A, B) is subcritical with respect to U. Given a Borel set $D \subset G$, we define

$$D_x = D \cap xU \quad \text{for } x \in G.$$

We note that D_x only depends on the right U-coset of x, so we may just as well view x as an element in G/U. Furthermore,

$$A_x = x(x^{-1}A \cap U) \quad \text{and} \quad B_y = (By^{-1} \cap U)y$$

and

$$A_x B_y = x(x^{-1}A \cap U)(By^{-1} \cap U)y \subset AB \cap Uxy = (AB)_{xy}$$

for all $x, y \in G$. Using (3.1), the left and right invariance of the Haar probability measure m_G and the identity $m_G(U) = 1/|G/U|$, we see that the conditions for subcriticality of (A, B) with respect to U (see Definition 2.2) can be equivalently rewritten as

$$m_G(A_x) = \frac{m_G(A)}{|G/U|} \quad \text{and} \quad m_G(B_y) = \frac{m_G(B)}{|G/U|} \tag{3.2}$$

and

$$m_G((AB)_{xy}) = m_G(A_x B_y) = m_G(A_x) + m_G(B_y) < m_G(U) \tag{3.3}$$

for all $x, y \in G/U$. We stress that the inequality in (3.3) follows from (3.2) and our assumption that (A, B) is subcritical.

In proving these identities, the following technical lemma will be useful.

LEMMA 3.3. *For all $x, y \in G/U$, the sets A_x and B_y are non-empty, and*

$$m_G((AB)_{xy}) \geq m_G(A_x) + m_G(B_y).$$

Before we prove this lemma, we show how to deduce (3.2)–(3.3) from it. We first note that

$$AB = \bigsqcup_{z \in G/U} (AB)_z = \bigsqcup_{z \in G/U} \bigcup_{xy=z} A_x B_y.$$

Pick $x_o, y_o \in G/U$ such that

$$m_G(A_{x_o}) = \max_{x \in G/U} m_G(A_x) \quad \text{and} \quad m_G(B_{y_o}) = \max_{y \in G/U} m_G(B_y),$$

and note that

$$m_G(A) \leq |G/U| m_G(A_{x_o}) \quad \text{and} \quad m_G(B) \leq |G/U| m_G(B_{y_o}). \tag{3.4}$$

By Lemma 3.3, we have

$$m_G((AB)_z) \geq m_G(A_{zy_o^{-1}}) + m_G(B_{y_o})$$

and

$$m_G((AB)_z) \geq m_G(A_{x_o}) + m_G(B_{x_o^{-1}z}),$$

for all $z \in G/U$. Hence,

$$m_G(AB) \geq \sum_{z \in G/U} (m_G(A_{zy_o^{-1}}) + m_G(B_{y_o})) = m_G(A) + |G/U| m_G(B_{y_o})$$

and

$$m_G(AB) \geq \sum_{z \in G/U} (m_G(A_{x_o}) + m_G(B_{x_o^{-1}z})) = |G/U| m_G(A_{x_o}) + m_G(B).$$

By (3.4) and subcriticality of (A, B), we see that

$$m_G(A_{x_o}) = \frac{m_G(A)}{|G/U|} \quad \text{and} \quad m_G(B_{y_o}) = \frac{m_G(B)}{|G/U|}.$$

This implies that

$$m_G(A) = \sum_{x \in G/U} m_G(A_x) \leq |G/U| m_G(A_{x_o}) = m_G(A),$$

and similarly for B. Hence,

$$m_G(A_x) = \frac{m_G(A)}{|G/U|} \quad \text{and} \quad m_G(B_y) = \frac{m_G(B)}{|G/U|} \quad \text{for all } x, y \in G/U.$$

This proves (3.2).

We can now replace x_o and y_o with arbitrary x and y in the inequalities above. Using Lemma 3.3, we conclude that for every fixed y, we have

$$\begin{aligned} m_G(AB) &\geq \sum_{z \in G/U} m_G((AB)_z) \geq \sum_{z \in G/U} (m_G(A_{zy^{-1}}) + m_G(B_y)) \\ &= m_G(A) + m_G(B). \end{aligned}$$

Since (A, B) is subcritical, these inequalities are in fact equalities, and we now see that

$$m_G((AB)_z) = m_G(A_{zy^{-1}}B_y) = m_G(A_{zy^{-1}}) + m_G(B_y) \quad \text{for all } y, z \in G/U.$$

This shows (3.3), and thus the proof of Proposition 3.1 is finished.

3.2. Proof of Lemma 3.3. First note that if A is spread-out in G, then A_x is non-empty for every $x \in G/U$. Indeed, if it were empty for some x, then $xU \notin q(A)$ where q denotes the canonical quotient map onto G/U, and thus A does not project onto the finite group G/U, which contradicts our assumption that A is spread-out.

If X is a subset of G, we denote by X^c the complement of X in G. Suppose that (A, B) is subcritical in G and define $C = (AB)^c$. Then $A^{-1}C \subset B^c$, and thus

$$m_G(A^{-1}C) \leq 1 - m_G(B) = m_G(A) + m_G(C) < 1,$$

since B is not a null set. We note that if the first inequality is strict, then the pair (A^{-1}, C) is critical in G, and thus A cannot be spread-out by the first assertion in Theorem 1.4. We conclude that (A^{-1}, C) is subcritical in G. Furthermore,

$$(A^{-1})_x C_z \subset (A^{-1}C)_{xz} \subset (B^c)_{xz} \quad \text{for all } x, z \in G/U. \tag{3.5}$$

Lemma 3.4. *For all $x, z \in G/U$, we have*

$$m_G((A^{-1})_x) + m_G(C_z) \leq m_G((A^{-1})_x C_z).$$

We claim that this lemma implies Lemma 3.3. Indeed, first note that for every Borel set $D \subset G$, we have

$$(D^{-1})_z = D^{-1}_{z^{-1}}, \quad m_G((D^c)_z) = m_G(U) - m_G(D_z) \quad \text{for all } z \in G/U. \tag{3.6}$$

In particular, by (3.5) and the definition of C,

$$m_G((A^{-1})_x C_z) \leq m_G(U) - m_G(B_{xz}) \quad \text{and} \quad m_G(C_z) = m_G(U) - m_G((AB)_z)$$

for all $x, z \in G/U$. Suppose that Lemma 3.4 holds. Then, using these relations, we conclude that

$$m_G((A^{-1})_x) + m_G(U) - m_G((AB)_z) \leq m_G((A^{-1})_x C_z) \leq m_G(U) - m_G(B_{xz}),$$

which readily translates to

$$m_G(A_{x^{-1}}) + m_G(B_{xz}) \leq m_G((AB)_z) \quad \text{for all } x, z \in G/U,$$

where we have used the relation $(A^{-1})_x = A^{-1}_{x^{-1}}$ from (3.6), and the fact that m_G is inversion-invariant (since G is compact, and thus unimodular). The proof of Lemma 3.3 is now complete.

3.2.1. *Proof of Lemma 3.4.* Suppose that (S,T) is a subcritical pair in G and

$$m_G(S_xT_y) < m_G(S_x) + m_G(T_y) \quad \text{for } \textit{some } x,y \in G/U. \tag{3.7}$$

This translates to the bound

$$m_U((x^{-1}S \cap U)(Ty^{-1} \cap U)) < m_U(x^{-1}S \cap U) + m_U(Ty^{-1} \cap U),$$

and thus we see that the pair $((x^{-1}S \cap U), (Ty^{-1} \cap U))$ is critical in U. There are now two cases to consider.

Case I: Suppose that

$$m_U((x^{-1}S \cap U)(Ty^{-1} \cap U)) < 1.$$

Then $((x^{-1}S \cap U), (Ty^{-1} \cap U))$ reduces to a pair of *proper* subsets of a finite quotient group U/Q for some open proper normal subgroup Q of U. In particular,

$$(x^{-1}S \cap U)Q \neq U.$$

We can also view Q as an open (but not necessarily normal) subgroup of G. However, since G/Q is finite, we see that $R = \bigcap_{g \in G/Q} gQg^{-1}$ is an open *normal* subgroup of G, which by construction is contained in Q and thus in U. In particular,

$$(x^{-1}S \cap U)R \neq U,$$

whence

$$x^{-1}SR = (x^{-1}S \cap U)R \cup (x^{-1}S \cap U^c)R \subset (x^{-1}S \cap U)R \cup U^c \neq G.$$

We see that S does not project onto the finite quotient group G/R, and thus S is *not* spread-out.

Case II: Suppose that

$$1 = m_U((x^{-1}S \cap U)(Ty^{-1} \cap U)) < m_U(x^{-1}S \cap U) + m_U(Ty^{-1} \cap U).$$

Then it is not hard to see that the product set equals U, and thus $S_xT_y = xyU \subset ST$.

Hence we have the following alternative: If (S,T) is subcritical and (3.7) holds for some $x,y \in G/U$, then either

- S is *not* spread-out, or
- ST contains a coset of U.

Let us now apply this observation to the pair $(S,T) = (A^{-1}, C)$ in Lemma 3.4. Since A is assumed to be spread-out, so is A^{-1}, and thus the assertion in Lemma 3.4 can only fail if ST contains a coset of U, i.e. if $A^{-1}C \supset zU$ for some $z \in G/U$. However, recall that $A^{-1}C \subset B^c$. We conclude that $B \cap zU = \emptyset$, and thus B is not spread-out, contrary to our assumption.

3.3. Proof of Proposition 3.2. Let (U_n) be a decreasing sequence of open and normal subgroups of G with intersection N. We recall that the Haar probability measures on U_n can be viewed as Borel probability measures on G via

$$m_{U_n}(C) = \frac{m_G(C \cap U_n)}{m_G(U_n)} \quad \text{for Borel sets } C \subset G.$$

By uniqueness of Haar probability measures on compact groups, we observe that $m_{U_n} \to m_N$ in the weak*-topology on the space of Borel probability measures on G. Suppose that (A, B) is a subcritical pair with respect to U_n for every n, and set $\mu_n = m_{U_n}$. We leave it as an exercise to show that since U_n is open, we can take the sets X and Y in Definition 2.2 to be G and $G \times G$.

We note that (2.1) can be rewritten as

$$m_G(A) = \int_G \chi_A(sy)\, d\mu_n(y) \quad \text{and} \quad m_G(B) = \int_G \chi_B(yt)\, d\mu_n(y) \tag{3.8}$$

for all $s, t \in G$, and if we combine (2.1) with (2.2) and (2.3), we have

$$1 > m_G(A) + m_G(B) = \int_G \chi_{AB}(syt)\, d\mu_n(y) \tag{3.9}$$

for all $s, t \in G$.

We claim that (3.8) and (3.9) still hold for a *conull* set of $(s, t) \in G \times G$ if μ_n is replaced with m_N. This will follow from Lemma 3.5 below. After the proof of this lemma, we will show how this can be used to finish the proof of Proposition 3.2.

LEMMA 3.5. *Let (μ_n) be a sequence of Borel probability measures on G which converges in the weak*-topology to a Borel probability measure μ on G. Then, for every bounded real-valued Haar measurable function f on G, there exist a subsequence (n_j) and conull subsets $X \subset G$ and $Y \subset X \times X$ such that*

$$\int_G f(sy)\, d\mu_{n_j}(y) \to \int_G f(sy)\, d\mu(y) \quad \textit{and} \quad \int_G f(yt)\, d\mu_{n_j}(y) \to \int_G f(yt)\, d\mu(y)$$

for all $s, t \in X$, and

$$\int_G f(syt)\, d\mu_{n_j}(y) \to \int_G f(syt)\, d\mu(y) \quad \textit{for all } (s,t) \in Y.$$

Proof. Since $\mu_n \to \mu$ in the weak*-topology, the lemma is trivial when f is continuous, and then no passage to a subsequence is necessary. By a standard approximation argument, combined with dominated convergence, the lemma also holds for every bounded Haar measurable function on G with respect to the *norm* topologies on $L^2(G)$ and $L^2(G \times G)$ respectively.

Since every L^2-convergent sequence admits an almost everywhere convergent subsequence, we are done. ■

Applying Lemma 3.5 to the sequence $\mu_n = m_{U_n}$ above and $\mu = m_N$, and using the relations (3.8) and (3.9) for the functions $f = \chi_A, \chi_B$ and χ_{AB}, we conclude that there are conull Borel sets $X \subset G$ and $Y \subset X \times X$ such that

$$m_G(A) = m_N(s^{-1}A \cap N) \quad \text{and} \quad m_G(B) = m_N(Bt^{-1} \cap N) \tag{3.10}$$

for all $s, t \in X$, and

$$1 > m_G(A) + m_G(B) = m_N(s^{-1}ABt^{-1} \cap N) \quad \text{for all } (s,t) \in Y. \tag{3.11}$$

Since N is connected and abelian, we know by the second assertion in Theorem 1.4 that the pairs $(s^{-1}A \cap N, Bt^{-1} \cap N)$ are never critical, and since

$$s^{-1}ABt^{-1} \supset (s^{-1}A \cap N)(Bt^{-1} \cap N)$$

we have

$$\begin{aligned} m_N(s^{-1}ABt^{-1} \cap N) &\geq m_N((s^{-1}A \cap N)(Bt^{-1} \cap N)) \\ &\geq m_N(s^{-1}A \cap N) + m_N(Bt^{-1} \cap N). \end{aligned}$$

Upon combining (3.10) and (3.11), we now see that

$$m_N(s^{-1}ABt^{-1} \cap N) = m_N((s^{-1}A \cap N)(Bt^{-1} \cap N)),$$
$$m_N((s^{-1}A \cap N)(Bt^{-1} \cap N)) = m_N(s^{-1}A \cap N) + m_N(Bt^{-1} \cap N)$$

for all $(s,t) \in Y$, which shows that (A, B) is subcritical with respect to N.

4. Proof of Proposition 2.5. Let G be a compact and second countable group with an *abelian* identity component N and a closed subgroup L such that $N \cap L = \{e\}$ and $NL = G$. Let $A, B \subset G$ be Borel sets and suppose that (A, B) is subcritical with respect to N. We recall that this means that there are conull subsets $X \subset G$ and $Y \subset X \times X$ such that

$$m_N(s^{-1}A \cap N) = m_G(A) \quad \text{and} \quad m_N(Bt^{-1} \cap N) = m_G(B) \tag{4.1}$$

for all $s, t \in X$, and

$$m_N((s^{-1}A \cap N)(Bt^{-1} \cap N)) = m_G(A) + m_G(B) < 1, \tag{4.2}$$

$$m_N(s^{-1}ABt^{-1} \cap N) = m_N((s^{-1}A \cap N)(Bt^{-1} \cap N)) \tag{4.3}$$

for all $(s,t) \in Y$. Since m_G is inversion-invariant, we may without loss of generality assume that X and Y are invariant under taking inverses, so in particular, the identities (4.1)–(4.3) above also hold with s^{-1} replaced with s. We shall henceforth assume that these replacements have been made.

4.1. A basic reduction. Fix $s, t \in G$. Since $N \cap L = \{e\}$ and $NL = G$, we can write $s = n_s l_s$ and $t = n_t l_t$ for unique elements $n_s, n_t \in N$ and $l_s, l_t \in L$. Hence, if $s, t \in X$ and $(s,t) \in Y$, then

$$sA \cap N = n_s(l_s A \cap N) \quad \text{and} \quad Bt^{-1} \cap N = (Bl_t^{-1} \cap N)n_t^{-1}$$

and

$$sABt^{-1} \cap N = n_s(l_s ABl_t^{-1} \cap N)n_t^{-1}.$$

Since m_N is left and right N-invariant, we conclude that the relations (4.1)–(4.3) (with s replaced with s^{-1}) only depend on l_s and l_t. In particular, the sets X and Y are respectively left N- and $N \times N$-invariant, so we conclude that there are conull sets $X_o \subset L$ and $Y_o \subset L \times L$ such that

$$m_N(sA \cap N) = m_G(A) \quad \text{and} \quad m_N(Bt^{-1} \cap N) = m_G(B) \tag{4.4}$$

for all $s, t \in X_o$, and

$$m_N((sA \cap N)(Bt^{-1} \cap N)) = m_G(A) + m_G(B) < 1, \tag{4.5}$$

$$m_N(sABt^{-1} \cap N) = m_N((sA \cap N)(Bt^{-1} \cap N)) \tag{4.6}$$

for all $(s,t) \in Y_o$. The main difference from (4.1)–(4.3) is that s and t are now elements of L.

4.2. Combing the dual group of N. Recall that N is assumed to be a compact and *connected* abelian group. Let $\widehat{N}$ denote the dual group of N, i.e. the group of all continuous homomorphisms from N to $\mathbb{T}$ with pointwise addition (which we write multiplicatively). It is a classical fact that $\widehat{N}$ is a countable and *torsion-free* group. In particular, the map $\xi \mapsto \check{\xi}$ on $\widehat{N}$ given by

$$\check{\xi}(n) = \xi(n)^{-1} \quad \text{for } n \in N,$$

has only one fixed point, namely the trivial homomorphism, here denoted by 1. Hence, we can inductively construct a set $S \subset \widehat{N} \backslash \{1\}$ with the property that

$$\widehat{N} \setminus \{1\} = S \cup \check{S} \quad \text{and} \quad S \cap \check{S} = \emptyset. \tag{4.7}$$

Suppose that we have fixed such a set S once and for all.

Note that (4.4) and (4.5) above show that

$$(sA \cap N, Bt^{-1} \cap N) \quad \text{for } y = (s,t) \in Y_o$$

are all subcritical pairs in N. Since N is abelian, Theorem 1.4 asserts they all reduce to Sturmian pairs in $\mathbb{T}$. Recall that this means that for every $y = (s,t) \in Y_o$, we can find

- a continuous homomorphism $\xi_y : N \to \mathbb{T}$;
- closed and symmetric intervals $I_y, J_y \subset \mathbb{T}$ such that

$$m_N((sA \cap N)(Bt^{-1} \cap N)) = m_{\mathbb{T}}(I_y J_y); \tag{4.8}$$

- $a(y), b(y) \in \mathbb{T}$ such that

$$sA \cap N \subset \xi_y^{-1}(I_y a(y)) \quad \text{and} \quad Bt^{-1} \cap N \subset \xi_y^{-1}(J_y b(y)).$$

In particular, by (4.4),

$$m_G(A) \leq m_{\mathbb{T}}(I_y) \quad \text{and} \quad m_G(B) \leq m_{\mathbb{T}}(J_y). \tag{4.9}$$

Remark 4.1. Since I_y and J_y are symmetric, we have

$$\xi_y^{-1}(I_y a(y)) = \check{\xi}_y^{-1}(I_y a(y)^{-1}), \qquad \xi_y^{-1}(I_y b(y)) = \check{\xi}_y^{-1}(I_y b(y)^{-1}) \quad \text{for all } y.$$

Hence we can, possibly upon changing $a(y)$ and $b(y)$ to $a(y)^{-1}$ and $b(y)^{-1}$ whenever necessary, assume that $\xi_y \in S$ for all $y \in Y_o$. We shall make this assumption throughout the rest of the paper.

Furthermore, since (I_y, J_y) is clearly subcritical in $\mathbb{T}$ for every $y \in Y_o$, by (4.5) and (4.8) we have

$$\begin{aligned} m_G(A) + m_G(B) &= m_N((sA \cap N)(Bt^{-1} \cap N)) = m_{\mathbb{T}}(I_y J_y) \\ &= m_{\mathbb{T}}(I_y) + m_{\mathbb{T}}(J_y) < 1. \end{aligned}$$

By (4.9), we conclude that $m_G(A) = m_{\mathbb{T}}(I_y)$ and $m_G(B) = m_{\mathbb{T}}(J_y)$. Since we have assumed that I_y and J_y are both closed and *symmetric* intervals, they are *uniquely* determined by their Haar measures. In particular, we conclude that I_y and J_y are independent of y, and we shall henceforth simply denote them by I and J.

To summarize: Let $I, J \subset \mathbb{T}$ denote the *unique* closed and symmetric intervals of Haar measures $m_G(A)$ and $m_G(B)$ respectively. Fix a set $S \subset \widehat{N} \setminus \{1\}$ as in (4.7). Then, for every $y = (s,t) \in Y_o$, there exist $\xi_y \in S$ and $a(y), b(y) \in \mathbb{T}$ such that

$$sA \cap N \subset \xi_y^{-1}(I a(y)) \quad \text{and} \quad Bt^{-1} \cap N \subset \xi_y^{-1}(J b(y)). \tag{4.10}$$

Furthermore, we have

$$m_N(sA \cap N) = m_{\mathbb{T}}(I) \quad \text{and} \quad m_N(Bt^{-1} \cap N) = m_{\mathbb{T}}(I). \tag{4.11}$$

4.3. Getting rid of dependences. If $E, F \subset N$ are Borel sets, we write $E \sim F$ if $m_N(E \triangle F) = 0$, where $\triangle$ denotes symmetric difference.

By (4.10) and (4.11), we see that whenever $y = (s,t)$ and $y' = (s,t')$ are elements having the same *first* coordinate, and which both belong to Y_o, then

$$\xi_y^{-1}(I a(y)) \sim \xi_{y'}^{-1}(I a(y')).$$

It is not hard to see (using the fact that I is regular and has trivial stabilizer in $\mathbb{T}$) that this forces either

$$\xi_y = \xi_{y'} \quad \text{and} \quad a(y) = a(y'), \quad \text{or} \quad \xi_y = \check{\xi}_{y'} \quad \text{and} \quad a(y) = a(y')^{-1}.$$

However, since $\xi_y, \xi_{y'} \in S$ for all $y, y' \in Y_o$, and $S \cap \check{S} = \emptyset$, the second possibility cannot occur. Similarly, if $z = (u, v)$ and $z' = (u', v)$ are elements having the same *second* coordinate, and which both belong to Y_o, then

$$\xi_z^{-1}(Jb(z)) \sim \xi_{z'}^{-1}(Jb(z')),$$

which forces either

$$\xi_z = \xi_{z'} \quad \text{and} \quad b(z) = b(z'), \quad \text{or} \quad \xi_z = \check{\xi}_{z'} \quad \text{and} \quad b(z) = b(z')^{-1}.$$

Since $\xi_z, \xi_{z'} \in S$ for all $z, z' \in Y_o$, the second possibility does not occur, and we conclude that

$$\xi_y = \xi_{y'} \quad \text{and} \quad \xi_z = \xi_{z'} \quad \text{and} \quad a(y) = a(y') \quad \text{and} \quad b(z) = b(z'). \tag{4.12}$$

We claim that the identities (4.12) imply that there exist $\xi \in S$ such that $\xi_y = \xi$ for a.e. y, and functions $\alpha, \beta : X_o \to \mathbb{T}$ (possibly upon shrinking X_o to a conull subset thereof) such that

$$a(y) = \alpha(s) \quad \text{and} \quad b(y) = \beta(t) \qquad \text{for a.e. } y = (s, t). \tag{4.13}$$

Indeed, first note that the continuous map $q : L^4 \to L^2$ defined by

$$q((s, t), (u, v)) = (s, v) \qquad \text{for } (s, t), (u, v) \in L^2$$

maps the Haar measure on L^4 to the Haar measure on L^2. In particular, since $Y_o \subset L^2$ is a conull Borel set, the set

$$F = q^{-1}(Y_o) \cap (Y_o \times Y_o) = \{((s, t), (u, v)) \in Y_o \times Y_o : (s, v) \in Y_o\}$$

is a conull Borel set of $Y_o \times Y_o$, so by Fubini's Theorem, there exists $(s, t) \in Y_o$ such that the section

$$F_{(s,t)} = \{(u, v) \in Y_o : ((s, t), (u, v)) \in F\}$$

is conull. Set $\xi = \xi_{(s,t)}$, and pick $(u, v) \in F_{(s,t)}$. By construction we have $(s, t), (s, v), (u, v) \in Y_o$, so by (4.12), we must have

$$\xi = \xi_{(s,t)} = \xi_{(s,v)} = \xi_{(u,v)}.$$

In other words, $\xi_{(u,v)} = \xi$ for almost every $(u, v) \in Y_o$, which proves the first assertion. To prove (4.13), we argue as follows. By [12, Theorem A.9], upon possibly replacing X_o with a conull Borel subset thereof, we can find Borel maps $q_1, q_2 : X_o \to L$ such that

$$(u, q_1(u)) \in Y_o \quad \text{and} \quad (q_2(u), u) \in Y_o \qquad \text{for all } u \in X_o.$$

If we now define the (not a priori Borel measurable) maps $\alpha, \beta : X_o \to \mathbb{T}$ by

$$\alpha(u) = a(u, q_1(u)) \quad \text{and} \quad \beta(u) = b(q_2(u), u) \qquad \text{for } u, v \in X_o,$$

then by (4.12), we see that for all $(u, v) \in Y_o \cap (X_o \times X_o)$,

$$a(u, v) = a(u, q_2(u)) = \alpha(u) \quad \text{and} \quad b(u, v) = b(q_2(v), v) = \beta(v),$$

and thus we have proved (4.13).

Remark 4.2. Since the Borel measurability of q_1 and q_2 is irrelevant at this point, some readers might wish *not* to invoke [12, Theorem A.9] to prove the existence of q_1 and q_2. Instead, one can first extract a common conull Borel subset of X_o (which we henceforth identify with X_o) of the projections of Y_o onto each coordinate axis, and then use the Axiom of Choice to produce right inverses q_1 and q_2 of the coordinate projections restricted to $Y_o \cap (X_o \times X_o)$.

To summarize: There are conull Borel subsets $X_o \subset L$ and $Y_o \subset X_o \times X_o$ (which may be different from the sets X_o and Y_o at the beginning of this subsection), maps $\alpha, \beta : X_o \to \mathbb{T}$ (with no obvious regularity whatsoever) and $\xi \in S$ such that

$$sA \cap N \subset \xi^{-1}(I\alpha(s)) \quad \text{and} \quad Bt^{-1} \cap N \subset \xi^{-1}(J\beta(t)) \tag{4.14}$$

for all $(s,t) \in Y_o$, where I and J denote the *unique* closed and symmetric intervals in $\mathbb{T}$ with Haar measures equal to $m_G(A)$ and $m(B)$ respectively. Furthermore,

$$m_N(sA \cap N) = m_{\mathbb{T}}(I) \quad \text{and} \quad m_N(Bt^{-1} \cap N) = m_{\mathbb{T}}(J).$$

It now follows from (4.6), (4.8) and (4.10), combined with the observation that $\xi_y = \xi$ almost everywhere and (4.13), that

$$sABt^{-1} \cap N \sim (sA \cap N)(Bt^{-1} \cap N) \sim \xi^{-1}(IJ\alpha(s)\beta(t)) \tag{4.15}$$

for all $(s,t) \in Y_o$. Note that we may without loss of generality (under further restrictions) assume that X_o is a symmetric subset of G (since m_G is inversion-invariant), and that the projections of Y_o onto each L-coordinate coincide with X_o. We shall henceforth make these assumptions.

Technical interludes: In what follows, we shall prove that $\xi \in S$ and the maps $\alpha, \beta : X_o \to \mathbb{T}$ can be used to construct a *continuous* homomorphism π from G into $\mathbb{T} \rtimes \{-1,1\}$ so that the sets A and B, modulo null sets, are contained in pre-images of a Sturmian pair in $\mathbb{T} \rtimes \{-1,1\}$ under this homomorphism. Note however that we have not yet established any regularity, or even measurability, for the maps α and β. The technical tools for this will be outlined in the next two subsections.

4.4. Interlude I: Borel measurability of α and β. We note that if M is a compact group with Haar probability measure m_M, and $D \subset M$ is a Borel set, then

$$\mathrm{Stab}_M(D) = \{m \in M : m_M(Dm \cap D) = m_M(D)\}$$

is a closed subgroup of M (see the discussion after (2.7)). We see that if $M = \mathbb{T}$ and I is a proper closed interval of $\mathbb{T}$, then $\mathrm{Stab}_{\mathbb{T}}(I)$ is the trivial subgroup.

Lemma 4.3. *Suppose that*

- *G and M are compact and second countable groups, and $L, N < G$ are closed subgroups;*
- *$\xi : N \to M$ is a surjective continuous homomorphism;*
- *$C \subset G$ and $I \subset M$ are Borel sets, and $\mathrm{Stab}_M(I)$ is trivial;*
- *$X \subset L$ is conull and there exists a map $\gamma : X \to M$ such that*

$$sC \cap N \sim \xi^{-1}(I\gamma(s)) \quad \text{for all } s \in X.$$

Then γ is Borel measurable.

We shall use this lemma as follows. Since I and J are symmetric, we deduce by (4.11) and (4.14) that

$$sA \cap N \sim \xi^{-1}(I\alpha(s)) \quad \text{and} \quad tB^{-1} \cap N \sim \xi^{-1}(J\beta(t)^{-1})$$

for all $s, t \in X_o$. Applied to G, N and L as in the previous subsections, and $X = X_o \subset L$ and $M = \mathbb{T}$ and

$$C = I \quad \text{and} \quad \gamma(s) = \alpha(s), \quad \text{or} \quad C = J \quad \text{and} \quad \gamma(s) = \beta(s)^{-1},$$

the lemma above implies that α and β are in fact Borel measurable as maps from $X_o \to \mathbb{T}$.

Proof of Lemma 4.3. Fix a countable basis (U_n) for the topology on M and note that, by assumption,

$$sC \cap N \cap \xi^{-1}(U_n) \sim \xi^{-1}(I\gamma(s) \cap U_n) \tag{4.16}$$

for all n and for all s in X. Define $\Psi : M \to [0,1]^{\mathbb{N}}$ and $\Phi : L \to [0,1]^{\mathbb{N}}$ by

$$\Psi(t)_n = m_M(I\,t \cap U_n) \quad \text{and} \quad \Phi(s)_n = m_N(sC \cap N \cap \xi^{-1}(U_n))$$

for $n \geq 1$ and $t \in M$ and $s \in L$. We claim that both Ψ and Φ are Borel measurable. It suffices to show that $\Psi(\cdot)_n$ and $\Phi(\cdot)_n$ are Borel measurable for every n. Note that M and L act jointly continuously on M and G respectively, and thus they also act jointly continuously on the space of Borel probability measures on M and G respectively, endowed with the weak*-topology. Hence, the functions

$$t \mapsto \int_{U_n} f_1(mt)\, dm_M(m) \quad \text{and} \quad s \mapsto \int_{\xi^{-1}(U_n)} f_2(s^{-1}x)\, dm_N(x), \tag{4.17}$$

where m_N is viewed as a Borel probability measure on G, are continuous on M and L respectively, for every fixed pair (f_1, f_2) of continuous functions on M and G respectively. If we instead were to plug in the (discontinuous) functions $f_1 = \chi_I$ and $f_2 = \chi_C$, then the corresponding maps in (4.17) coincide with $\Psi(\cdot)_n$ and $\Phi(\cdot)_n$ respectively. However, in this case, both f_1 and f_2 are pointwise limits of sequences of continuous functions, so by monotone convergence both $\Psi(\cdot)_n$ and $\Phi(\cdot)_n$ (for fixed n) are pointwise limits of sequences of continuous functions, and thus Borel measurable.

We further claim that Ψ is injective. First note that if $E, F \subset M$ are Borel sets and

$$m_M(E \cap U_n) = m_M(F \cap U_n) \quad \text{for all } n, \tag{4.18}$$

then $m_M(E \triangle F) = 0$. Indeed, if (4.18) holds, then

$$m_M((E \setminus F) \cap U_n) = m_M(E \cap U_n) - m_M(F \cap U_n) = 0,$$
$$m_M((F \setminus E) \cap U_n) = m_M(F \cap U_n) - m_M(E \cap U_n) = 0$$

for all n, and thus it suffices to show that if $D \subset M$ is a Borel set such that $m_M(D \cap U_n) = 0$ for all n, then $m_M(D) = 0$. However, since the union of all U_n covers M, we must have

$$m_M(D) = m_M\Big(D \cap \bigcup_n U_n\Big) \leq \sum_n m_M(D \cap U_n) = 0.$$

Hence, if $t_1, t_2 \in M$ are such that

$$m_M(It_1 \cap U_n) = m_M(It_2 \cap U_n) \quad \text{for all } n,$$

then $m_M(I \triangle It_2 t_1^{-1}) = 0$, and thus $t_2 t_1^{-1} \in \mathrm{Stab}_M(I)$, which forces $t_1 = t_2$, since $\mathrm{Stab}_M(I)$ is assumed to be trivial. This shows that Ψ is injective.

Let us now summarize the discussion so far. We have shown that the maps Ψ and Φ are Borel measurable from M and L respectively into $[0,1]^{\mathbb{N}}$. Furthermore, Ψ is injective, and by (4.16), we have

$$\Psi(\gamma(s)) = \Phi(s), \quad \forall s \in X.$$

Hence, $\gamma = \Psi^{-1} \circ \Phi$. By [12, Theorem A.4], Ψ^{-1} is Borel measurable, so we conclude that γ is Borel measurable as well. ■

4.5. Interlude II: Restrictions of homomorphisms. The following lemma will be used in 4.6.4 below.

Lemma 4.4. *Suppose that*

- *G and M are compact and second countable groups;*
- *$X \subset G$ and $Z \subset X \times X$ are conull;*
- *there are Borel measurable maps $\sigma, \tau : X \to M$ such that*

$$\sigma(x_1)\tau(y_1) = \sigma(x_2)\tau(y_2)$$

whenever (x_1, y_1) and (x_2, y_2) belong to Z and $x_1 y_1 = x_2 y_2$.

Then there exist a continuous homomorphism $\pi : G \to M$ and $a, b \in M$ such that

$$a\sigma(x) = \pi(x) \quad \text{and} \quad \tau(x)b = \pi(x) \quad \text{for a.e. } x \in G.$$

Proof. Since $Z \subset X \times X$ is conull, there exist, by Fubini's Theorem, an element $x_o \in X$ and a conull subset $X' \subset X$ such that $\{x_o\} \times X' \subset Z$. Since the multiplication map $(x,y)\mapsto xy$ pushes $m_G\otimes m_G$ onto m_G, the set

$$Z' := (x_o,e)^{-1}Z\cap\{(x,y)\in G\times G : xy\in X'\}$$

is conull in $G\times G$. We note that for any $(x,y)\in Z'$, we have

$$(x_o,xy)\in Z \quad\text{and}\quad (x_ox,y)\in Z,$$

and thus (since $x_o(xy)=(x_ox)y$)

$$\sigma(x_o)\tau(xy)=\sigma(x_ox)\tau(y) \quad \text{for all } (x,y)\in Z'. \tag{4.19}$$

By Fubini's Theorem, we can find an element $y_o\in X$ and a conull subset $X''\subset G$ such that $X''\times\{y_o\}\subset Z'$. Arguing as before, we see that the set

$$Z'' := Z'(e,y_o)^{-1}\cap\{(x,y)\in G\times G : xy\in X''\}$$

is conull in $G\times G$. We note that if $(x,y)\in Z''$, then

$$(x,yy_o)\in Z' \quad\text{and}\quad (xy,y_o)\in Z',$$

and thus, by (4.19),

$$\sigma(x_o)\tau(xyy_o)=\sigma(x_ox)\tau(yy_o), \tag{4.20}$$

$$\sigma(x_o)\tau(xyy_o)=\sigma(x_oxy)\tau(y_o) \tag{4.21}$$

for all $(x,y)\in Z''$.

Since X' and X'' are both conull, so is $Y := X''\cap X'y_o^{-1}$. It is straightforward to check that $(e,y)\in Z''$ for all $y\in Y$. Hence, by (4.21),

$$\sigma(x_o)\tau(yy_o)=\sigma(x_oy)\tau(y_o) \quad \text{for all } y\in Y,$$

and thus

$$\pi_o(y) := \sigma(x_o)^{-1}\sigma(x_oy)=\tau(yy_o)\tau(y_o)^{-1} \quad \text{for all } y\in Y. \tag{4.22}$$

Let us now define the conull set

$$W := \{(x,y)\in Z'' : xy\in Y\}\cap(Y\times Y).$$

Then, for all $(x,y)\in W$, by (4.20) and (4.22) we have

$$\pi_o(xy)=\tau(xyy_o)\tau(y_o)^{-1}=\sigma(x_o)^{-1}\sigma(x_ox)\tau(yy_o)\tau(y_o)^{-1}=\pi_o(x)\pi_o(y).$$

In other words, π_o satisfies the condition of being a homomorphism from G into M almost everywhere. By [12, Theorem B.2], we can now conclude that there exists a *continuous* homomorphism $\pi : G\to M$ whose restriction to W coincides with π_o. In particular, letting $a^{-1}=\sigma(x_o)\pi(x_o)^{-1}$ and $b^{-1}=\pi(y_o)^{-1}\tau(y_o)$, we see from (4.22) that

$$a\sigma(y)=\pi(y) \quad\text{and}\quad \tau(y)b=\pi(y) \quad \text{for a.e. } y\in G. \ \blacksquare$$

4.6. Relations between α and β. Let us go back to the setting of Subsection 4.3, and recall the summary at the end of that subsection. In particular, let $S\subset\widehat{N}\setminus\{1\}$ and $\xi\in S$ be as in that subsection. Since L acts continuously on N by conjugation, it also acts continuously on the dual $\widehat{N}$ via the adjoint representation,

$$(s \cdot \eta)(n) = \eta(s^{-1}ns) \quad \text{for } \eta \in \widehat{N} \text{ and } s \in L.$$

In what follows, we will set $\xi_s = s^{-1} \cdot \xi$. This should *not* be confused with the notation used in Subsections 4.2 and 4.3. By continuity of the L-action on $\widehat{N}$, each set of the form $\{s \in L : \xi_s = \eta\}$, where $\eta \in \widehat{N}$, is closed in L. Hence, since S is (at most) countable, the set

$$E = \{s \in L : \xi_s \in S\} \subset L$$

is a countable union of closed subsets of L, and thus Borel. In particular, if we define $\varepsilon : L \to \{-1, 1\}$ by

$$\varepsilon(s) = \begin{cases} +1 & \text{if } s \in E, \\ -1 & \text{if } s \notin E. \end{cases}$$

then ε is Borel measurable. Finally, since $\xi \in S$ we have $\varepsilon(e) = 1$.

Since $\xi^{-1} = \check{\xi}$ for every $\xi \in \widehat{N}$, we note that $\xi_s^{\varepsilon(s)} \in S$ for all $s \in L$. Furthermore, note that for every Borel set $D \subset \mathbb{T}$, we have

$$s^{-1}\xi^{-1}(D)s = \xi_s^{-1}(D) \quad \text{for all } s \in L. \tag{4.23}$$

4.6.1. *Bounding AB.* Since N is a normal subgroup of G, we have

$$sABt^{-1} \cap N = s(ABt^{-1}s \cap s^{-1}Ns)s^{-1} = s(AB(s^{-1}t)^{-1} \cap N)s^{-1}$$

for all $s, t \in L$. Hence, by (4.15) and (4.23),

$$AB(s^{-1}t)^{-1} \cap N \sim s^{-1}\xi^{-1}(IJ\alpha(s)\beta(t))s = \xi_s^{-1}(IJ\alpha(s)\beta(t))$$

for all $(s, t) \in Y_o$. We see that the left hand side only depends on $s^{-1}t$. In particular, for all pairs (s, t) and (u, v) in Y_o such that $s^{-1}t = u^{-1}v$, we must have

$$\xi_s^{-1}(IJ\alpha(s)\beta(t)) \sim \xi_u^{-1}(IJ\alpha(u)\beta(v)).$$

Since N is normal, we deduce that

$$\ker \xi_s = \ker \xi_u = \ker \xi,$$

and thus the composition $\xi_s \circ \xi_u^{-1}$ is a well-defined automorphism of $\mathbb{T}$. Since $\mathrm{Aut}(\mathbb{T}) = \{-1, 1\}$, we see that either

$$\xi_s = \xi_u \quad \text{and} \quad \alpha(s)\beta(t) = \alpha(u)\beta(v),$$

or

$$\xi_s = \check{\xi}_u \quad \text{and} \quad \alpha(s)\beta(t) = (\alpha(u)\beta(v))^{-1}.$$

Since $\xi_s^{\varepsilon(s)} \in S$ for all $s \in L$, we conclude that

$$\xi_s^{\varepsilon(s)} = \xi_u^{\varepsilon(u)} \quad \text{and} \quad (\alpha(s)\beta(t))^{\varepsilon(s)} = (\alpha(u)\beta(v))^{\varepsilon(u)} \tag{4.24}$$

whenever $(s,t),(u,v) \in Y_o$ with $s^{-1}t = u^{-1}v$. Since the first identity is independent of t and v, we see that the map $s \mapsto \xi_s^{\varepsilon(s)}$ is almost everywhere constant, say equal to $\eta \in \widehat{N} \setminus \{1\}$ on a conull subset of X_o (which we henceforth identify with X_o). We note that the sets

$$R_+ = \{s \in L : \xi_s = \eta\} \quad \text{and} \quad R_- = \{s \in L : \xi_s^{-1} = \eta\}$$

are *closed*, and by assumption $X_o \subset R_+ \cup R_-$. Since X_o is conull, and the union $R_+ \cup R_-$ is closed, we conclude that $R_+ \cup R_- = L$. Hence, $\xi_s^{\varepsilon(s)} = \eta$ for *all* $s \in L$. Since $\varepsilon(e) = 1$, we see that $\xi = \eta$, and thus

$$\xi(sns^{-1})^{\varepsilon(s)} = \xi(n) \quad \text{for all } s \in L \text{ and } n \in N. \tag{4.25}$$

We conclude that the L-action on $\widehat{N}$ preserves the two-element set $\{\xi, \check{\xi}\}$, and the corresponding homomorphism $L \to \mathrm{Aut}(\mathbb{T}) \cong \{-1,1\}$, coincides with ε. Since ε is a Borel measurable homomorphism between second countable groups, it must be continuous by [12, Theorem B.3].

To summarize: There exists a continuous homomorphism $\varepsilon : L \to \{-1,1\}$ such that (4.25) holds and

$$(\alpha(s)\beta(t))^{\varepsilon(s)} = (\alpha(u)\beta(v))^{\varepsilon(u)} \tag{4.26}$$

whenever $s^{-1}t = u^{-1}v$. In particular, for any Borel set $D \subset \mathbb{T}$, we have

$$s^{-1}\xi^{-1}(D)s = \xi_s^{-1}(D) = \xi^{-1}(D^{\varepsilon(s)}) \tag{4.27}$$

for all $s \in L$, where we adopt the convention that $D^1 = D$.

4.6.2. *Bounding A.* By (4.14), we have

$$sA \cap N = s(A \cap s^{-1}N) \subset \xi^{-1}(I\alpha(s)) \quad \text{for all } s \in X_o,$$

and thus, by (4.27) and our assumption that I is symmetric,

$$A \cap Ns^{-1} \subset \big(s^{-1}\xi^{-1}(I\alpha(s))s\big)s^{-1} = \xi^{-1}(I\alpha(s)^{\varepsilon(s)})s^{-1}$$

for all $s \in X_o$. Since $X_o^{-1} = X_o$ and $\varepsilon(s^{-1}) = \varepsilon(s)$ (indeed, $1 = \varepsilon(ss^{-1}) = \varepsilon(s)\varepsilon(s^{-1})$ for all s), we also have

$$A \cap Ns \subset \xi^{-1}(I\alpha(s^{-1})^{\varepsilon(s)})s \quad \text{for all } s \in X_o. \tag{4.28}$$

Let us define the map $\sigma : NX_o \to \mathbb{T} \rtimes \{-1,1\}$ by

$$\sigma(ms) = (\xi(m)\alpha(s^{-1})^{-\varepsilon(s)}, \varepsilon(s)). \tag{4.29}$$

Since both α and ε are Borel measurable, so is σ. We note that

$$\sigma^{-1}(I \rtimes \{-1,1\}) \cap Ns = \{ms : \xi(m)\alpha(s^{-1})^{-\varepsilon(s)} \in I\} = \xi^{-1}(I\alpha(s^{-1})^{\varepsilon(s)})s$$

for all $s \in X_o$, and thus, by (4.28),

$$A \cap Ns \subset \sigma^{-1}(I \rtimes \{-1,1\}) \cap Ns \quad \text{for all } s \in X_o.$$

We conclude that

$$A' := A \cap NX_o \subset \sigma^{-1}(I \rtimes \{-1,1\}). \tag{4.30}$$

Note that A' is a conull subset of A, since NX_o is a conull subset of G.

4.6.3. *Bounding B.* By (4.14), we have

$$Bt^{-1} \cap N = (B \cap Nt)t^{-1} \subset \xi^{-1}(J\beta(t))$$

for all $t \in X_o$, and thus

$$B \cap Nt \subset \xi^{-1}(J\beta(t))t \quad \text{for all } t \in X_o.$$

Define $\tau : NX_o \to \mathbb{T} \rtimes \{-1,1\}$ by

$$\tau(nt) = (\xi(n)\beta(t)^{-1}, \varepsilon(t)). \tag{4.31}$$

Since both β and ε are Borel measurable, so is τ. Note that

$$\tau^{-1}(J \rtimes \{-1,1\}) \cap Nt = \{nt : \xi(n)\beta(t)^{-1} \in J\} = \xi^{-1}(J\beta(t))t$$

for all $t \in X_o$, and thus

$$B \cap Nt \subset \tau^{-1}(J \rtimes \{-1,1\}) \cap Nt \quad \text{for all } t \in X_o.$$

We conclude that

$$B' := B \cap NX_o \subset \tau^{-1}(J \rtimes \{-1,1\}), \tag{4.32}$$

and $B' \subset B$ is conull, since NX_o is a conull subset of G.

4.6.4. *The pair (σ, τ).* We wish to verify that the pair (σ, τ) of Borel maps above satisfies the conditions in Lemma 4.4 with $M = \mathbb{T} \rtimes \{-1,1\}$ and the conull subsets

$$X = NX_o \subset G \quad \text{and} \quad Z = \{(ms, nt) : m, n \in N,\ (s,t) \in Y_o\} \subset X \times X.$$

The multiplication in $\mathbb{T} \rtimes \{-1,1\}$ of elements (r_1, δ_1) and (r_2, δ_2) will be written

$$(r_1, \delta_1)(r_2, \delta_2) = (r_1 r_2^{\delta_1}, \delta_1\delta_2). \tag{4.33}$$

Suppose that (ms, nt) and (pu, qv) belong to Z and

$$(ms)(nt) = m(sns^{-1})st = (pu)(qv) = p(uqu^{-1})uv.$$

Since $N \cap L = \{e\}$, this forces

$$m(sns^{-1}) = p(uqu^{-1}) \quad \text{and} \quad st = uv.$$

By (4.26) and our assumption that $X_o^{-1} = X_o$, we have

$$(\alpha(s^{-1})\beta(t))^{\varepsilon(s)} = (\alpha(u^{-1})\beta(v))^{\varepsilon(u)}$$

whenever (s,t) and (u,v) belong to Y_o and $st = uv$.

Recall (4.25) and the definitions of σ and τ from (4.29) and (4.31) respectively. Upon combining the relations above, and using the multiplication convention in $\mathbb{T} \rtimes \{-1,1\}$ explained in (4.33), we see that

$$\begin{aligned}\sigma(ms)\tau(nt) &= (\xi(m)\alpha(s^{-1})^{-\varepsilon(s)}, \varepsilon(s))(\xi(n)\beta(t)^{-1}, \varepsilon(t)) \\ &= \big(\xi(m)\xi(n)^{\varepsilon(s)}\alpha(s^{-1})^{-\varepsilon(s)}\beta(t)^{-\varepsilon(s)}, \varepsilon(s)\varepsilon(t)\big) \\ &= \big(\xi(m)\xi(sns^{-1})(\alpha(s^{-1})\beta(t))^{-\varepsilon(s)}, \varepsilon(st)\big) \\ &= \big(\xi(msns^{-1})(\alpha(s^{-1})\beta(t))^{-\varepsilon(s)}, \varepsilon(st)\big) \\ &= \big(\xi(puqu^{-1})(\alpha(u^{-1})\beta(v))^{-\varepsilon(u)}, \varepsilon(uv)\big) \\ &= \sigma(pu)\tau(qv).\end{aligned}$$

By Lemma 4.4 we conclude that there exist a continuous homomorphism $\pi : G \to \mathbb{T} \rtimes \{-1,1\}$ and $a, b \in M$ such that $\sigma(g) = a^{-1}\pi(g)$ and $\tau(g) = \pi(g)b^{-1}$ almost everywhere. Upon possibly passing to further conull subsets in (4.30) and (4.32), we conclude that

$$A' \subset \pi^{-1}(a(I \rtimes \{-1,1\})) \quad \text{and} \quad B' \subset \pi^{-1}((I \rtimes \{-1,1\})b), \tag{4.34}$$

where $A' \subset A$ and $B' \subset B$ are conull subsets.

4.6.5. *Determining possible images of* π. We recall that $M = \mathbb{T} \rtimes \{-1,1\}$ and π is a continuous homomorphism from $G = N \rtimes L$ into M. Since N is connected and abelian, $\pi(N)$ is a compact and connected abelian subgroup of M, and thus either trivial or equal to $\mathbb{T}$.

We claim that the first case cannot occur. Indeed, recall that our standing assumption in Proposition 2.5 is that (A, B) is subcritical with respect to N, which by (4.4) in particular implies that

$$m_N(sA \cap N) = m_G(A) \quad \text{for } m_L\text{-a.e. } s \in L.$$

Since $A' \subset A$ is conull, and

$$\int_L m_N(sA' \cap N)\, dm_L(s) = m_G(A') = m_G(A) = \int_L m_N(sA' \cap N)\, dm_L(s),$$

we see that $m_N(sA' \cap N) = m_G(A)$ for m_L-a.e. $s \in L$ as well.

By (4.34), the set A' is also a conull subset of $\pi^{-1}(a(I \rtimes \{-1,1\})$, so the same type of argument as above shows that

$$m_N\big(s\pi^{-1}(a(I \rtimes \{-1,1\}) \cap N)\big) = m_G(A) \quad \text{for } m_L\text{-a.e. } s \in L.$$

We now note that if $\pi(N) = \{e_M\}$, so that $N < \ker \pi$, then the left hand side is either 0 or 1, which contradicts our assumption that $0 < m_G(A) < 1$.

We conclude that $\pi(N) = \mathbb{T}$, so either $\pi(L) \supset \{0\} \rtimes \{-1,1\}$, in which case we must have $\pi(G) = \mathbb{T} \rtimes \{-1,1\}$, or $\pi(G) = \mathbb{T} \rtimes \{1\} \cong \mathbb{T}$.

4.6.6. *Finishing the proof of Proposition 2.5.* Let us briefly summarize the argument so far: We have produced conull subsets $A' \subset A$ and $B' \subset B$ such that

$$A' \subset \pi^{-1}(a(I \rtimes \{-1,1\})) \quad \text{and} \quad B' \subset \pi^{-1}((I \rtimes \{-1,1\})b),$$

where $I, J \subset \mathbb{T}$ are closed intervals with

$$m_G(A) = m_{\mathbb{T}}(I) \quad \text{and} \quad m_G(B) = m_{\mathbb{T}}(J).$$

By the previous subsection, $\pi(G)$ can be either $\mathbb{T}$ or $\mathbb{T} \rtimes \{-1, 1\}$, and thus in either case,

$$m_G\big(\pi^{-1}(a(I \rtimes \{-1,1\}))\big) = m_{\mathbb{T}}(I), \qquad m_G\big(\pi^{-1}((J \rtimes \{-1,1\})b)\big) = m_{\mathbb{T}}(J).$$

We want to prove that (A', B') reduces to a Sturmian pair in either $\mathbb{T}$ or $\mathbb{T} \rtimes \{-1, 1\}$. From the arguments above, it is clear that it remains to show that $m_G(A'B') = m_G(AB)$. We shall argue by contradiction: If $m_G(A'B') < m_G(AB)$, then, since $A' \subset A$ and $B' \subset B$ are conull subsets, we have

$$m_G(A'B') < m_G(AB) = m_G(A) + m_G(B) = m_G(A') + m_G(B') < 1,$$

and thus (A', B') is critical in G. By the first assertion in Theorem 1.4, this would imply that there exists a finite factor group of G such that neither A' nor B' projects onto it. This contradicts our assumption that (A, B) is spread-out in G, finishing the proof of Proposition 2.5.

Appendix. Compact groups with a dense amenable subgroup. A countable group Γ is *amenable* if there is sequence (F_n) of finite subsets of Γ such that

$$\overline{\lim_n} \frac{|F_n \triangle \gamma F_n|}{|F_n|} = 0 \qquad \text{for all } \gamma \in \Gamma.$$

It is well-known (see e.g. the book [9]) that every countable solvable (so in particular every abelian or nilpotent) group is amenable. Furthermore, subgroups and quotients of amenable groups are again amenable. On the other hand, countable groups with a free subgroup of rank at least two cannot be amenable. It was shown by Tits [10] that the absence of a free subgroup of rank two exactly characterizes amenability among linear groups.

Let us now prove a property of dense embeddings of countable amenbale groups into compact groups which motivated the study pursued in this paper.

PROPOSITION A.1. *If K is a compact and second countable Hausdorff group with a dense countable amenable subgroup Γ, then the identity component K^o of K is abelian.*

Proof. Towards a contradiction suppose that there exist $x, y \in K^o$ such that $xyx^{-1}y^{-1} \neq e_K$. By Peter–Weyl's Theorem, we can find a positive integer n and a representation π of K into $U(n)$ such that $\pi(x)\pi(y)\pi(x)^{-1}\pi(y)^{-1} \neq e_{\pi(K)}$. We note that $\pi(x)$ and $\pi(y)$ belong to the identity component of the (possibly disconnected) compact Lie group $\pi(K)$. Furthermore, $\pi(\Gamma)$ is a dense countable amenable subgroup of $\pi(K)$.

By [4, Theorem 6.5(iii)], $\pi(K)^o$ has finite index in $\pi(K)$, and a straightforward argument shows that $\Lambda := \pi(\Gamma) \cap \pi(K)^o$ is a dense countable amenable subgroup of $\pi(K)^o$. In particular, the commutator subgroup $[\Lambda, \Lambda]$ is a dense amenable subgroup of $[\pi(K)^o, \pi(K)^o]$. By [4, Theorem 6.18], the latter is a semisimple and connected compact Lie group. At this point, Tits' Alternative [10] can be applied: a non-trivial semisimple and connected compact Lie group cannot contain a dense amenable subgroup. We conclude that $\pi(K)^o$ is abelian, which contradicts $\pi(x)\pi(y)\pi(x)^{-1}\pi(y)^{-1} \neq e_{\pi(K)}$. ∎

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