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On the generalized approximate weak Chebyshev greedy algorithm

by

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Abstract. The Weak Chebyshev Greedy Algorithm (WCGA) is defined for any Banach space X and a dictionary $\mathcal{D}$, and provides nonlinear n-term approximation for a given element $f \in X$ with respect to $\mathcal{D}$. In this paper we study the generalized Approximate Weak Chebyshev Greedy Algorithm (gAWCGA), a modification of the WCGA in which we are allowed to calculate n-term approximation with relative and absolute errors in computing a norming functional, an element of best approximation, and an approximant. This is natural for numerical applications and simplifies realization of the algorithm. We obtain conditions that are sufficient for the convergence of the gAWCGA for any element of a uniformly smooth Banach space, and show that they are necessary in the class of uniformly smooth Banach spaces with modulus of smoothness of nontrivial power type (e.g. L_p spaces for $1 < p < \infty$). In particular, we show that if all the errors are in ℓ_1 then the conditions for the convergence of the gAWCGA are the same as for the WCGA. We also construct an example of a smooth Banach space in which the algorithm diverges for a dictionary and an element, thus showing that the smoothness of the space is not sufficient for the convergence of the WCGA.

1. Introduction. This paper is devoted to the problem of greedy approximation in Banach spaces. We consider the Weak Chebyshev Greedy Algorithm (WCGA), which was studied by V. N. Temlyakov (see, for instance, [11], [15]). The WCGA is defined for any Banach space, and provides nonlinear n-term approximations of a given element of the space with respect to a fixed set of elements. For numerical applications it seems logical to allow the steps of the WCGA to be calculated not exactly, but with some inaccuracies. Such approach was used for other types of greedy algorithms (e.g. see [8] and [7]). For more information about other types of

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greedy approximation, the reader may refer to the survey papers [9], [13], and [14].

The modification of the WCGA with relative errors in computing the steps of the algorithm, the Approximate Weak Chebyshev Algorithm (AWCGA), was studied in [12] and [2]. In this paper we study another modification of the WCGA, in which we are allowed to make both absolute and relative errors at every step of the algorithm. Similar to the terminology proposed in [7], we call this modification the generalized Approximate Weak Chebyshev Algorithm (gAWCGA).

Recall that a *dictionary* is a set $\mathcal{D}$ of elements of a real Banach space X such that $\overline{\operatorname{span}}\,\mathcal{D} = X$ and the elements of $\mathcal{D}$ are *normalized*, i.e. $\|g\| = 1$ for any $g \in \mathcal{D}$. For convenience we assume that all dictionaries are *symmetric*, i.e. if $g \in \mathcal{D}$ then $-g \in \mathcal{D}$. We set

$$A_1(\mathcal{D}) = \overline{\operatorname{conv}}\,\mathcal{D}, \qquad A_0(\mathcal{D}) = \operatorname{span}\mathcal{D}.$$

We define the following classes of sequences, which represent inaccuracies in calculating the steps of the algorithm. A *weakness sequence* is a sequence $\{(t_n, t'_n)\}_{n=1}^{\infty}$ of pairs of real numbers such that $0 \le t_n \le 1$ and $t'_n \ge 0$ for all $n \ge 1$. A *perturbation sequence* $\{(\delta_n, \delta'_n)\}_{n=0}^{\infty}$ is such that $\delta_n \ge 0$ and $\delta'_n \ge 0$ for all $n \ge 0$. An *error sequence* $\{(\eta_n, \eta'_n)\}_{n=1}^{\infty}$ is such that $\eta_n \ge 0$ and $\eta'_n \ge 0$ for all $n \ge 1$. We set $\eta_0 = \sup_{n\ge1} \eta_n$ and $\eta'_0 = \sup_{n\ge1} \eta'_n$.

For a Banach space X, a dictionary $\mathcal{D}$, and an element $f \in X$, the generalized Approximate Weak Chebyshev Greedy Algorithm with a weakness sequence $\{(t_n, t'_n)\}_{n=1}^{\infty}$, a perturbation sequence $\{(\delta_n, \delta'_n)\}_{n=0}^{\infty}$, and an error sequence $\{(\eta_n, \eta'_n)\}_{n=1}^{\infty}$ is defined as follows.

Definition (gAWCGA). Set $f_0 = f$ and for each $n \ge 1$,

- take any functional F_{n-1} on X satisfying

$$\|F_{n-1}\| \le 1 \quad \text{and} \quad F_{n-1}(f_{n-1}) \ge (1 - \delta_{n-1})\|f_{n-1}\| - \delta'_{n-1}, \tag{1.1}$$

- choose $\phi_n \in \mathcal{D}$ such that

$$F_{n-1}(\phi_n) \ge t_n \sup_{g\in\mathcal{D}} F_{n-1}(g) - t'_n, \tag{1.2}$$

- for $\Phi_n = \operatorname{span}\{\phi_j\}_{j=1}^{n}$ denote $E_n = \inf_{G\in\Phi_n} \|f - G\|$ and find $G_n \in \Phi_n$ satisfying

$$\|f - G_n\| \le (1 + \eta_n)E_n + \eta'_n, \tag{1.3}$$

- call G_n the nth *approximation* of f and $f_n = f - G_n$ the nth *remainder.*

Note that if for every $n \ge 1$ either $t_n < 1$ or $t'_n > 0$ then for any Banach space X, any dictionary $\mathcal{D}$, and any $f \in X$ the algorithm is feasible. We say that the gAWCGA for f *converges* if for every realization of the algorithm the sequence $\{G_n\}_{n=1}^{\infty}$ converges to f. Otherwise, we say that the gAWCGA *diverges*.

Note also that if $t'_n = \delta_{n-1} = \delta'_{n-1} = \eta_n = \eta'_n = 0$ for all $n \geq 1$ then the gAWCGA coincides with the WCGA studied in [11] and [3]. In the case $t'_n = \delta'_{n-1} = \eta'_n = 0$ the gAWCGA coincides with the AWCGA studied in [12] and [2].

One of the goals of this paper is to investigate the behavior of the gAWCGA in a uniformly smooth Banach space X and to obtain conditions on the weakness, perturbation, and error sequences that guarantee the convergence of the gAWCGA for all dictionaries $\mathcal{D} \subset X$ and all $f \in X$. In Section 2 we state sufficient conditions for the convergence of the gAWCGA with arbitrary sequences $\{t'_n\}_{n=1}^{\infty}$, $\{\delta'_n\}_{n=0}^{\infty}$, and $\{\eta'_n\}_{n=1}^{\infty}$ in a uniformly smooth Banach space X, and show that they are also necessary if X has modulus of smoothness of nontrivial power type.

We understand the necessity of the conditions in the following way: if at least one of the conditions does not hold, one can find a uniformly smooth Banach space X, a dictionary $\mathcal{D}$, and an element $f \in X$ such that the gAWCGA for f with the given weakness, perturbation, and error sequences diverges. We note that in our case such a Banach space and dictionary need not be complicated. In fact, we give an example of a divergent gAWCGA in ℓ_p with the canonical basis as a dictionary.

In Section 3 we prove theorems stated in Section 2. We note that while we are interested in the strong convergence of the WCGA and its modifications, a more general setting was considered in [3].

Another goal of this paper is to discuss restrictions on a Banach space X that are required for the convergence of the WCGA. It is known (see [11]) that the WCGA with a constant weakness sequence $0 < t \leq 1$ (denoted further as WCGA(t)) converges in all uniformly smooth Banach spaces for all dictionaries and all elements of the space. However, uniform smoothness is not necessary: it is shown in [3] that every separable reflexive Banach space X admits an equivalent norm for which the WCGA(t) converges for any dictionary $\mathcal{D}$ and any $f \in X$. Furthermore, one can find a separable reflexive Banach space that does not admit an equivalent uniformly smooth norm (see e.g. [1]). Thus, the condition of uniform smoothness can be weakened. In particular, it is shown in [3] that if a reflexive Banach space X has the Kadec–Klee property and Fréchet differentiable norm, then the WCGA(t) converges for any dictionary $\mathcal{D}$ and any $f \in X$.

On the other hand, it is shown in [6] that the smoothness of the space is equivalent to the norms of the remainders of the WCGA being decreasing for any dictionary $\mathcal{D}$ and any $f \in X$. Thus, the smoothness of the space is necessary for the convergence of the algorithm and it would be natural to expect that it is also sufficient. In Section 4 we refute this hypothesis by exhibiting a smooth Banach space, a dictionary, and an element for which the WCGA diverges. To construct the desired Banach space, we adopt the technique used in [4] to prove the necessity of smoothness of the space for the convergence of incremental approximation. Namely, we renorm ℓ_1 by introducing a sequence $\{\vartheta_n\}_{n=1}^{\infty}$ of recursively defined seminorms, each of which is the ℓ_{p_n}-norm of the previously calculated seminorm ϑ_{n-1} and the nth coordinate of the element, where the sequence $\{p_n\}_{n=1}^{\infty}$ decreases to 1 sufficiently fast. The reason for such a complicated approach is that the resulting space has to be smooth but not uniformly smooth, which is already a nontrivial task. We note that an analogous space was used in [10] to prove the insufficiency of smoothness of the space for the convergence of the X-Greedy Algorithm.

2. Convergence of the gAWCGA in uniformly smooth Banach spaces. We begin by recalling a few definitions. A functional F on a Banach space X is a *norming functional* of a nonzero element $x \in X$ if $\|F\| = 1$ and $F(x) = \|x\|$. A Banach space X is *smooth* if for any nonzero $x \in X$ there exists a unique norming functional F_x of x.

For a Banach space X the *modulus of smoothness* ρ is defined by

$$\rho(u) = \sup_{\|x\|=\|y\|=1} \frac{\|x+uy\| + \|x-uy\|}{2} - 1. \tag{2.1}$$

Note that ρ is an even and convex function, and therefore it is nondecreasing on $(0, \infty)$. A Banach space is *uniformly smooth* if $\rho(u) = o(u)$ as $u \to 0$. We say that ρ is *of power type* $1 \le q \le 2$ if $\rho(u) \le \gamma u^q$ for some $\gamma > 0$. It is easy to see that any Banach space has modulus of smoothness of power type 1, and any Hilbert space has modulus of smoothness of power type 2. Denote by $\mathcal{P}_q$ the class of all uniformly smooth Banach spaces with modulus of smoothness of nontrivial power type $1 < q \le 2$. In particular (see [4, Lemma B.1]) the modulus of smoothness ρ_p of L_p satisfies

$$\rho_p(u) \le \begin{cases} \frac{1}{p}\, u^p, & 1 < p \le 2, \\ \frac{p-1}{2}\, u^2, & 2 \le p < \infty, \end{cases}$$

hence $L_p \in \mathcal{P}_q$ with $q = \min\{p, 2\}$ for any $1 < p < \infty$.

For a weakness sequence $\{t_n\}_{n=1}^{\infty}$ and a number $0 < \theta \le 1/2$ let $\{\xi_n\}_{n=1}^{\infty}$ be a sequence of positive numbers which satisfy $\rho(\xi_n) = \theta t_n \xi_n$ for each $n \ge 1$. It is shown in [11] that if a Banach space is uniformly smooth then for any $0 < \theta \le 1/2$ the sequence $\{\xi_n\}_{n=1}^{\infty}$ exists and is uniquely determined by $\{t_n\}_{n=1}^{\infty}$.

We now state some known results concerning the convergence of the WCGA and its modifications in arbitrary uniformly smooth Banach spaces. The first result gives sufficient conditions for the convergence of the WCGA (see [11, Theorem 2.1]).

Theorem A. *The WCGA with a weakness sequence* $\{t_n\}_{n=1}^\infty$ *converges for any uniformly smooth Banach space* X*, any dictionary* $\mathcal{D}$*, and any* $f \in X$ *if for any* $0 < \theta \le 1/2$.

$$\sum_{n=1}^\infty t_n \xi_n = \infty.$$

The next theorem gives sufficient conditions for the convergence of the AWCGA (see [12, Theorem 2.2]).

Theorem B. *The AWCGA with a weakness sequence* $\{t_n\}_{n=1}^\infty$*, a perturbation sequence* $\{\delta_n\}_{n=0}^\infty$*, and an error sequence* $\{\eta_n\}_{n=1}^\infty$ *converges for any uniformly smooth Banach space* X*, any dictionary* $\mathcal{D}$*, and any* $f \in X$ *if*

$$\eta_0 = \sup_{n\ge 1} \eta_n < \infty$$

and if for any $0 < \theta \le 1/2$ *the following conditions hold:*

$$\sum_{n=1}^\infty t_n \xi_n = \infty, \qquad \delta_n = o(t_n \xi_n), \qquad \eta_n = o(t_n \xi_n).$$

We will prove a similar result for the convergence of the gAWCGA with somewhat weaker restrictions on the approximation parameters. Specifically, we require the parameters to be sufficiently small only along some increasing sequence $\{n_k\}_{k=1}^\infty$ of natural numbers.

Theorem 1. *The gAWCGA with a weakness sequence* $\{(t_n, t'_n)\}_{n=1}^\infty$*, a perturbation sequence* $\{(\delta_n, \delta'_n)\}_{n=0}^\infty$*, and an error sequence* $\{(\eta_n, \eta'_n)\}_{n=1}^\infty$ *converges for any uniformly smooth Banach space* X*, any dictionary* $\mathcal{D}$*, and any* $f \in X$ *if*

$$\eta_0 = \sup_{n\ge 1} \eta_n < \infty, \qquad \lim_{n\to\infty} \eta'_n = 0 \tag{2.2}$$

and if there exists a subsequence $\{n_k\}_{k=1}^\infty$ *such that, for any* $0 < \theta \le 1/2$,

$$\sum_{k=1}^\infty t_{n_k+1} \xi_{n_k+1} = \infty, \tag{2.3}$$

$$t'_{n_k+1} = o(t_{n_k+1}), \tag{2.4}$$

$$\delta_{n_k} = o(t_{n_k+1} \xi_{n_k+1}), \tag{2.5}$$

$$\delta'_{n_k} = o(t_{n_k+1} \xi_{n_k+1}), \tag{2.6}$$

$$\eta_{n_k} = o(t_{n_k+1} \xi_{n_k+1}), \tag{2.7}$$

$$\eta'_{n_k} = o(t_{n_k+1} \xi_{n_k+1}). \tag{2.8}$$

If the modulus of smoothness of the space is of the nontrivial power type, the previous theorems can be rewritten in form of necessary and sufficient conditions for convergence. The following result is [11, Corollary 2.1].

Theorem C. *The WCGA with a weakness sequence* $\{t_n\}_{n=1}^{\infty}$ *converges for any uniformly smooth Banach space* $X \in \mathcal{P}_q$, *any dictionary* $\mathcal{D}$, *and any* $f \in X$ *if and only if*

$$\sum_{n=1}^{\infty} t_n^p = \infty \qquad \text{where } p = q/(q-1).$$

The next theorem gives necessary and sufficient conditions for the convergence of the AWCGA (see [2, Theorem 1]).

Theorem D. *The AWCGA with a weakness sequence* $\{t_n\}_{n=1}^{\infty}$, *a perturbation sequence* $\{\delta_n\}_{n=0}^{\infty}$, *and an error sequence* $\{\eta_n\}_{n=1}^{\infty}$ *converges for any uniformly smooth Banach space* $X \in \mathcal{P}_q$, *any dictionary* $\mathcal{D}$, *and any* $f \in X$ *if and only if*

$$\eta_0 = \sup_{n \geq 1} \eta_n < \infty$$

and if there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ *such that*

$$\sum_{k=1}^{\infty} t_{n_k+1}^p = \infty, \qquad \delta_{n_k} = o(t_{n_k+1}^p), \qquad \eta_{n_k} = o(t_{n_k+1}^p),$$

where $p = q/(q-1)$.

We will prove the following necessary and sufficient conditions for the convergence of the gAWCGA.

Theorem 2. *The gAWCGA with a weakness sequence* $\{(t_n, t'_n)\}_{n=1}^{\infty}$, *a perturbation sequence* $\{(\delta_n, \delta'_n)\}_{n=0}^{\infty}$, *and an error sequence* $\{(\eta_n, \eta'_n)\}_{n=1}^{\infty}$ *converges for any uniformly smooth Banach space* $X \in \mathcal{P}_q$, *any dictionary* $\mathcal{D}$, *and any* $f \in X$ *if and only if*

$$\eta_0 = \sup_{n \geq 1} \eta_n < \infty, \qquad \lim_{n \to \infty} \eta'_n = 0 \tag{2.9}$$

and if there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ *such that*

$$\sum_{k=1}^{\infty} t_{n_k+1}^p = \infty, \tag{2.10}$$

$$t'_{n_k+1} = o(t_{n_k+1}), \tag{2.11}$$

$$\delta_{n_k} = o(t_{n_k+1}^p), \tag{2.12}$$

$$\delta'_{n_k} = o(t_{n_k+1}^p), \tag{2.13}$$

$$\eta_{n_k} = o(t_{n_k+1}^p), \tag{2.14}$$

$$\eta'_{n_k} = o(t_{n_k+1}^p), \tag{2.15}$$

where $p = q/(q-1)$.

The following corollary states that if the weakness parameter $\{t_n\}_{n=1}^{\infty}$ is separated from zero (e.g. $t_n = t > 0$ for all n) then the gAWCGA converges as long as η'_n goes to zero and other inaccuracy parameters go to zero along the same subsequence.

COROLLARY 2.1. *Suppose* $\liminf_{n\to\infty} t_n > 0$. *Then the gAWCGA with a weakness sequence* $\{(t_n, t'_n)\}_{n=1}^{\infty}$, *a perturbation sequence* $\{(\delta_n, \delta'_n)\}_{n=0}^{\infty}$, *and a bounded error sequence* $\{(\eta_n, \eta'_n)\}_{n=1}^{\infty}$ *with* $\lim_{n\to\infty} \eta'_n = 0$ *converges for any uniformly smooth Banach space* $X \in \mathcal{P}_q$, *any dictionary* $\mathcal{D}$, *and any* $f \in X$ *if and only if*

$$\liminf_{n\to\infty}(t'_{n+1} + \delta_n + \delta'_n + \eta_n) = 0.$$

The last two corollaries state that the conditions for the convergence of the gAWCGA are the same as for the WCGA if the inaccuracy sequences are in ℓ_1.

COROLLARY 2.2. *Let* $\{t'_n\}_{n=1}^{\infty} \in \ell_1$, $\{\delta_n\}_{n=0}^{\infty} \in \ell_1$, $\{\delta'_n\}_{n=0}^{\infty} \in \ell_1$, $\{\eta_n\}_{n=1}^{\infty} \in \ell_1$, *and* $\{\eta'_n\}_{n=1}^{\infty} \in \ell_1$. *Then the gAWCGA with a weakness sequence* $\{(t_n, t'_n)\}_{n=1}^{\infty}$, *the perturbation sequence* $\{(\delta_n, \delta'_n)\}_{n=0}^{\infty}$, *and the error sequence* $\{(\eta_n, \eta'_n)\}_{n=1}^{\infty}$ *converges for any uniformly smooth Banach space* X, *any dictionary* $\mathcal{D}$, *and any* $f \in X$ *if for any* $0 < \theta \le 1/2$,

$$\sum_{n=1}^{\infty} t_n \xi_n = \infty.$$

COROLLARY 2.3. *Let* $\{t'_n\}_{n=1}^{\infty} \in \ell_1$, $\{\delta_n\}_{n=0}^{\infty} \in \ell_1$, $\{\delta'_n\}_{n=0}^{\infty} \in \ell_1$, $\{\eta_n\}_{n=1}^{\infty} \in \ell_1$, *and* $\{\eta'_n\}_{n=1}^{\infty} \in \ell_1$. *Then the gAWCGA with a weakness sequence* $\{(t_n, t'_n)\}_{n=1}^{\infty}$, *the perturbation sequence* $\{(\delta_n, \delta'_n)\}_{n=0}^{\infty}$, *and the error sequence* $\{(\eta_n, \eta'_n)\}_{n=1}^{\infty}$ *converges for any uniformly smooth Banach space* $X \in \mathcal{P}_q$, *any dictionary* $\mathcal{D}$, *and any* $f \in X$ *if and only if*

$$\sum_{n=1}^{\infty} t_n^p = \infty.$$

We note that in the last corollary the sequence $\{t'_n\}_{n=1}^{\infty}$ might be in ℓ_p as well; we take it from ℓ_1 for the simplicity of formulation. Corollaries 2.2 and 2.3 are obtained by using Theorems 1 and 2, and the following simple fact (see [2, Lemma 2]).

LEMMA E. *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be any nonnegative sequences such that*

$$\sum_{n=1}^{\infty} a_n < \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} b_n = \infty.$$

Then there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ *such that*

$$\sum_{k=1}^{\infty} b_{n_k} = \infty \quad \text{and} \quad a_{n_k} = o(b_{n_k}).$$

3. Proofs of Theorems 1 and 2. We will need several technical results. The following is [11, Lemma 2.2].

LEMMA F. *For any bounded linear functional F on X and any dictionary $\mathcal{D}$,*

$$\sup_{g\in\mathcal{D}} |F(g)| = \sup_{g\in A_1(\mathcal{D})} |F(g)|.$$

We use the following lemmas from [12] rewritten for the gAWCGA.

LEMMA 1. *Let X be a Banach space with modulus of smoothness ρ. Then for any $\phi \in \Phi_n$,*

$$|F_n(\phi)| \le \beta_n(\phi) := \inf_{\lambda>0} \frac{1}{\lambda}\bigg(\delta_n + \eta_n + \frac{\delta'_n + \eta'_n}{\|f_n\|} + 2\rho(\lambda\|\phi\|)\bigg).$$

Proof. Take any $\phi \in \Phi_n$. By the definition (2.1) of the modulus of smoothness, for any $\lambda > 0$,

$$\|f_n - \lambda\phi\| + \|f_n + \lambda\phi\| \le 2\|f_n\|\bigg(1 + \rho\bigg(\frac{\lambda\|\phi\|}{\|f_n\|}\bigg)\bigg).$$

Assume that $F_n(\phi) \ge 0$ (the case $F_n(\phi) < 0$ is handled similarly). Then, using (1.1), we obtain

$$\|f_n + \lambda\phi\| \ge F_n(f_n + \lambda\phi) \ge (1 - \delta_n)\|f_n\| - \delta'_n + \lambda F_n(\phi),$$

thus

$$\|f_n - \lambda\phi\| \le \|f_n\|\bigg(1 + \delta_n + 2\rho\bigg(\frac{\lambda\|\phi\|}{\|f_n\|}\bigg)\bigg) + \delta'_n - \lambda F_n(\phi).$$

On the other hand, by (1.3),

$$\|f_n - \lambda\phi\| \ge E_n \ge (1 + \eta_n)^{-1}(\|f_n\| - \eta'_n) \ge (1 - \eta_n)\|f_n\| - \eta'_n.$$

Therefore

$$\lambda F_n(\phi) \le \|f_n\|\bigg(\delta_n + \eta_n + 2\rho\bigg(\frac{\lambda\|\phi\|}{\|f_n\|}\bigg)\bigg) + \delta'_n + \eta'_n,$$

and since the inequality holds for any $\lambda > 0$,

$$F_n(\phi) \le \inf_{\lambda>0} \frac{1}{\lambda}\bigg(\delta_n + \eta_n + \frac{\delta'_n + \eta'_n}{\|f_n\|} + 2\rho(\lambda\|\phi\|)\bigg) = \beta_n(\phi). \ \blacksquare$$

LEMMA 2. *Let X be a Banach space with modulus of smoothness ρ. Take a number $\epsilon \ge 0$ and elements $f, h \in X$ such that $\|f - h\| \le \epsilon$ and $h/A \in A_1(\mathcal{D})$ with some number $A = A(\epsilon) > 0$. Then*

$$|F_n(\phi_{n+1})| \ge t_{n+1}A^{-1}\big((1 - \delta_n)\|f_n\| - \delta'_n - \beta_n(G_n) - \epsilon\big) - t'_{n+1}.$$

Proof. Condition (1.2) and Lemma F imply that

$$|F_n(\phi_{n+1})| \ge t_{n+1} \sup_{g\in\mathcal{D}} |F_n(g)| - t'_{n+1} = t_{n+1} \sup_{g\in A_1(\mathcal{D})} |F_n(g)| - t'_{n+1}.$$

Taking $g = h/A \in A_1(\mathcal{D})$ we obtain

$$\sup_{g\in A_1(\mathcal{D})} |F_n(g)| \ge A^{-1}|F_n(h)| \ge A^{-1}(|F_n(f)| - \epsilon)$$
$$\ge A^{-1}(|F_n(f_n)| - |F_n(G_n)| - \epsilon).$$

Hence condition (1.1) and Lemma 1 yield

$$|F_n(\phi_{n+1})| \ge t_{n+1}A^{-1}\big((1-\delta_n)\|f_n\| - \delta'_n - \beta_n(G_n) - \epsilon\big) - t'_{n+1}. \ \blacksquare$$

LEMMA 3. *Let X be a Banach space with modulus of smoothness ρ. Take $\epsilon \ge 0$ and $f, h \in X$ such that $\|f - h\| \le \epsilon$ and $h/A \in A_1(\mathcal{D})$ with some $A = A(\epsilon) > 0$. Then for any $m > n$,*

$$E_m \le \inf_{\mu\ge 0} \|f_n\| \bigg[1 + \delta_n + \frac{\delta'_n}{\|f_n\|} + 2\rho\bigg(\frac{\mu}{\|f_n\|}\bigg) - \frac{\mu t_{n+1}}{A\|f_n\|}\big((1-\delta_n)\|f_n\| - \delta'_n - \beta_n(G_n) - \epsilon\big)\bigg] + \mu t'_{n+1}.$$

Proof. By (2.1), for any $\mu \ge 0$,

$$\|f_n - \mu\phi_{n+1}\| + \|f_n + \mu\phi_{n+1}\| \le 2\|f_n\|\bigg(1 + \rho\bigg(\frac{\mu}{\|f_n\|}\bigg)\bigg).$$

Assume that $F_n(\phi_{n+1}) \ge 0$ (the case $F_n(\phi_{n+1}) < 0$ is handled similarly). Then, using (1.1) and Lemma 2, we get

$$\begin{aligned}\|f_n + \mu\phi_{n+1}\| &\ge F_n(f_n + \mu\phi_{n+1}) \ge (1-\delta_n)\|f_n\| - \delta'_n + \mu|F_n(\phi_{n+1})| \\ &\ge (1-\delta_n)\|f_n\| - \delta'_n \\ &\quad + \mu t_{n+1}A^{-1}\big((1-\delta_n)\|f_n\| - \delta'_n - \beta_n(G_n) - \epsilon\big) - \mu t'_{n+1}.\end{aligned}$$

Thus

$$\|f_n - \mu\phi_{n+1}\| \le \|f_n\|\bigg(1 + \delta_n + \frac{\delta'_n}{\|f_n\|} + 2\rho\bigg(\frac{\mu}{\|f_n\|}\bigg)\bigg) - \mu t_{n+1}A^{-1}\big((1-\delta_n)\|f_n\| - \delta'_n - \beta_n(G_n) - \epsilon\big) + \mu t'_{n+1}.$$

On the other hand, since $E_m \le E_{n+1} \le \|f_n - \mu\phi_{n+1}\|$ for any $\mu \ge 0$,

$$E_m \le \|f_n\| \bigg[1 + \delta_n + \frac{\delta'_n}{\|f_n\|} + 2\rho\bigg(\frac{\mu}{\|f_n\|}\bigg) - \frac{\mu t_{n+1}}{A\|f_n\|}\big((1-\delta_n)\|f_n\| - \delta'_n - \beta_n(G_n) - \epsilon\big)\bigg] + \mu t'_{n+1}.$$

Taking the infimum over all $\mu \ge 0$ completes the proof. $\blacksquare$

We are now ready to prove Theorem 1.

Proof of Theorem 1. Assume that for some $f \in X$ the gAWCGA does not converge. Then the monotone sequence $\{E_n\}_{n=1}^{\infty}$ does not converge to 0 since otherwise (2.2) would imply

$$\lim_{n\to\infty} \|f_n\| \le \lim_{n\to\infty} ((1+\eta_0)E_n + \eta_n') = 0.$$

Thus there exists $\alpha > 0$ such that for any $n \ge 1$,

$$\|f_n\| \ge E_n \ge \alpha. \tag{3.1}$$

Denote $C_f = (2+\eta_0)\|f\| + \eta_0' < \infty$, where $\eta_0 = \sup_{n\ge1} \eta_n$ and $\eta_0' = \sup_{n\ge1} \eta_n'$. Then inequality (1.3) gives, for any $n \ge 1$,

$$\begin{aligned} \|f_n\| &\le (1+\eta_0)\|f\| + \eta_0' \le C_f, \\ \|G_n\| &\le \|f_n\| + \|f\| \le C_f. \end{aligned} \tag{3.2}$$

Let $\{n_k\}_{k=1}^{\infty}$ be a subsequence for which the assumptions of the theorem hold. Then

$$\begin{aligned} \beta_{n_k}(G_{n_k}) &= \inf_{\lambda>0} \frac{1}{\lambda}\bigg(\delta_{n_k} + \eta_{n_k} + \frac{\delta_{n_k}' + \eta_{n_k}'}{\|f_{n_k}\|} + 2\rho(\lambda\|G_{n_k}\|)\bigg) \\ &\le \inf_{\lambda>0} \frac{1}{\lambda}\bigg(\delta_{n_k} + \eta_{n_k} + \frac{\delta_{n_k}' + \eta_{n_k}'}{\alpha} + 2\rho(\lambda C_f)\bigg) \end{aligned}$$

and, by (2.4)–(2.8) and the inequality $0 \le \theta t_n \xi_n \le 1$, there exists $K \ge 1$ such that for any $k \ge K$ the following estimates hold with $\theta = \frac{\alpha^2}{24AC_f}$:

$$(1/2 - \delta_{n_k})\alpha - \delta_{n_k}' - \beta_{n_k}(G_{n_k}) \ge \alpha/4, \tag{3.3}$$

$$\delta_{n_k} + \delta_{n_k}'/\alpha \le \theta\xi_{n_k+1}t_{n_k+1}, \tag{3.4}$$

$$(1+\eta_{n_k})(1 - 3\theta\xi_{n_k+1}t_{n_k+1}) \le 1 - 2\theta\xi_{n_k+1}t_{n_k+1}, \tag{3.5}$$

$$\eta_{n_k}' + \alpha\xi_{n_k+1}t_{n_k+1}' \le \alpha\theta\xi_{n_k+1}t_{n_k+1}. \tag{3.6}$$

Take $\epsilon = \alpha/2$ and find $h \in X$ such that $\|f - h\| \le \epsilon$ and $h/A \in A_1(\mathcal{D})$ for some $A > 0$. Then Lemma 3, assumption (3.1), and estimates (3.2) and (3.3) yield, for any $k \ge K$,

$$\begin{aligned} E_{n_{k+1}} \le \inf_{\mu\ge0} \|f_{n_k}\| \bigg[&1 + \delta_{n_k} + \frac{\delta_{n_k}'}{\alpha} + 2\rho\Big(\frac{\mu}{\alpha}\Big) \\ &- \frac{\mu t_{n_k+1}}{AC_f}\bigg(\Big(\frac{1}{2} - \delta_{n_k}\Big)\alpha - \delta_{n_k}' - \beta_{n_k}(G_{n_k})\bigg)\bigg] + \mu t_{n_k+1}' \\ \le \inf_{\mu\ge0} \|f_{n_k}\| \bigg[&1 + \delta_{n_k} + \frac{\delta_{n_k}'}{\alpha} + 2\rho\Big(\frac{\mu}{\alpha}\Big) - \frac{\alpha\mu t_{n_k+1}}{4AC_f}\bigg] + \mu t_{n_k+1}'. \end{aligned}$$

By taking $\mu = \alpha\xi_{n_k+1}$, and using estimates (3.4)–(3.6) and condition (1.3), we obtain

$$
\begin{aligned}
(3.7)\qquad E_{n_{k+1}} &\le \|f_{n_k}\|(1+\delta_{n_k}+\delta'_{n_k}/\alpha-4\theta\xi_{n_k+1}t_{n_k+1})+\alpha\xi_{n_k+1}t'_{n_k+1}\\
&\le \|f_{n_k}\|(1-3\theta\xi_{n_k+1}t_{n_k+1})+\alpha\xi_{n_k+1}t'_{n_k+1}\\
&\le E_{n_k}(1-2\theta\xi_{n_k+1}t_{n_k+1})+\eta'_{n_k}+\alpha\xi_{n_k+1}t'_{n_k+1}\\
&\le E_{n_k}(1-\theta\xi_{n_k+1}t_{n_k+1}).
\end{aligned}
$$

Note that (2.3) implies that the infinite product $\prod_{k=1}^{\infty}(1-\theta\xi_{n_k+1}t_{n_k+1})$ diverges to 0. Then, recursively applying estimate (3.7), we obtain, for sufficiently large $N\ge K$,

$$
E_{n_{N+1}} \le E_{n_K}\prod_{k=K}^{N}(1-\theta\xi_{n_k+1}t_{n_k+1}) \le \|f\|\prod_{k=K}^{N}(1-\theta\xi_{n_k+1}t_{n_k+1}) < \alpha,
$$

which contradicts assumption (3.1). Therefore $\lim_{n\to\infty}E_n = 0$, i.e. the gAWCGA of f converges to f. ∎

To prove Theorem 2 we will use the following simple lemma.

Lemma 4. *Let $q>1$, $a\ge 0$ and $b\ge 1$. Then*

$$
(a+b^q)^{1/q}\le a+b.
$$

Proof. From the convexity of $(1+x)^q$ we have, for any $x\ge 0$,

$$
(1+x)^q \ge 1+qx.
$$

Then by taking $x=a/b$ we get

$$
(a+b)^q = b^q(1+x)^q \ge b^q(1+qx) = b^q + aqb^{q-1} \ge a+b^q. \ ∎
$$

Proof of Theorem 2. We start with the proof of sufficiency. Assume that conditions (2.10)–(2.15) hold for some subsequence $\{n_k\}_{k=1}^{\infty}$. Choose any $0<\theta\le 1/2$ and find the corresponding sequence $\{\xi_n\}_{n=1}^{\infty}$. Then using the definition $\rho(\xi_n)=\theta t_n\xi_n$ and the estimate $\rho(u)\le\gamma u^q$, we derive

$$
\xi_n \ge \left(\frac{\theta}{\gamma}t_n\right)^{p-1}.
$$

Thus for any $n\ge 1$,

$$
t_n^p \le \left(\frac{\gamma}{\theta}\right)^{p-1} t_n\xi_n,
$$

and conditions (2.10)–(2.15) imply (2.3)–(2.8) for the subsequence $\{n_k\}_{k=1}^{\infty}$ and any $0<\theta\le 1/2$. Therefore Theorem 1 guarantees the convergence of the gAWCGA for any dictionary $\mathcal{D}$ and any $f\in X$.

Now assuming that at least one of (2.9)–(2.15) fails, we will give an example of a Banach space $X\in\mathcal{P}_q$, a dictionary $\mathcal{D}$, and an element $f\in\mathcal{D}$ such that the gAWCGA of f diverges.

Let $X = \ell_q \in \mathcal{P}_q$ and $\mathcal{D} = \{\pm e_n\}_{n=0}^\infty$, where $\{e_n\}_{n=0}^\infty$ is the canonical basis in ℓ_q.

Assume that (2.9) fails, i.e. there exist a subsequence $\{n_k\}_{k=1}^\infty$ and $\alpha > 0$ such that for any $k \ge 1$,

$$\eta_{n_k} \ge \alpha k \quad \text{or} \quad \eta'_{n_k} \ge \alpha.$$

Take a positive nonincreasing sequence $\{a_j\}_{j=1}^\infty \in \ell_q$ such that

$$a_1 \ge \alpha \quad \text{and} \quad \Big(\sum_{j=n_k+1}^\infty a_j^q\Big)^{1/q} \ge k^{-1}$$

for any $k \ge 1$. Denote $f = \sum_{j=1}^\infty a_j e_j \in \ell_q$ and consider the following realization of the gAWCGA of f:

For $n \notin \{n_k\}_{k=1}^\infty$ choose F_{n-1} to be the norming functional for f_{n-1}, $\phi_n = e_n$, and $G_n = \sum_{j=1}^n a_j e_j$.

For $n \in \{n_k\}_{k=1}^\infty$ choose F_{n-1} to be the norming functional for f_{n-1}, $\phi_n = e_n$ and $G_n = \alpha e_1 + \sum_{j=1}^n a_j e_j$, which is possible since

$$\|f_{n_k}\|_q = \Big(\alpha^q + \sum_{j=n_k+1}^\infty a_j^q\Big)^{1/q} \le \alpha + E_{n_k},$$

and either

$$\|f_{n_k}\|_q \le E_{n_k} + \eta'_{n_k} \quad \text{or} \quad \|f_{n_k}\|_q \le (1+\alpha k)E_{n_k} \le (1+\eta_{n_k})E_{n_k}.$$

Then for any $k \ge 1$ we have $\|f_{n_k}\|_q \ge \alpha$, hence $\|f_n\|_q \nrightarrow 0$ and the gAWCGA for f diverges.

Assume now that conditions (2.10)–(2.15) do not all hold, i.e. for any subsequence $\{n_k\}_{k=1}^\infty$ at least one of the following statements *fails*:

$$\sum_{k=1}^\infty t_{n_k+1}^p = \infty,$$
$$t'_{n_k+1} = o(t_{n_k+1}),$$
$$\delta_{n_k} = o(t_{n_k+1}^p),$$
$$\delta'_{n_k} = o(t_{n_k+1}^p),$$
$$\eta_{n_k} = o(t_{n_k+1}^p),$$
$$\eta'_{n_k} = o(t_{n_k+1}^p).$$

For $\alpha > 0$ define

$$\Lambda_1 = \{n > 1 : \delta_{n-1} + \delta'_{n-1} \ge \alpha t_n^p \text{ or } \eta_{n-1} + \eta'_{n-1} \ge \alpha t_n^p \text{ or } t'_n \ge \alpha^{1/p} t_n\}$$

and $\Lambda_2 = \mathbb{N} \setminus \Lambda_1$. We claim that there exists an $\alpha > 0$ such that

$$\sum_{j \in \Lambda_2} t_j^p < \infty. \tag{3.8}$$

Indeed, if $\sum_{j\in\Lambda_2} t_j^p = \infty$ for any $\alpha > 0$ then for every $k \geq 1$ consider $\alpha(k) = 1/k$, and choose a sequence $\{\Gamma_k\}_{k=1}^{\infty}$ of disjoint finite sets with $\Gamma_k \subset \Lambda_2(k)$ and $\sum_{j\in\Gamma_k} t_j^p \geq 1$. Hence by considering the union $\bigcup_{k=1}^{\infty}(\Gamma_k + \{-1\})$ (where $+$ denotes Minkowski addition), we obtain a subsequence for which conditions (2.10)–(2.15) hold, contrary to assumption. Fix an $\alpha > 0$ for which claim (3.8) holds, and consider the corresponding sets Λ_1 and Λ_2.

If $|\Lambda_1| < \infty$ then $\sum_{j=1}^{\infty} t_j^p < \infty$. Take $f = e_0 + \sum_{j=1}^{\infty} t_j^{p/q} e_j$ and consider the following realization of the gAWCGA of f:

For each $n \geq 1$ choose F_{n-1} to be the norming functional for f_{n-1}, $\phi_n = e_n$, and $G_n = \sum_{j=1}^{n} t_j^{p/q} e_j$. Then for any $n \geq 1$ we have $\|f_n\|_q \geq 1$, hence the gAWCGA for f diverges.

Consider now the case $|\Lambda_1| = \infty$. Take any nonnegative sequence $\{a_j\}_{j\in\Lambda_1}$ such that $a_j \leq 1$ for any $j \geq 1$, $\sum_{j\in\Lambda_1} a_j^q \geq 1/\alpha$ and $\sum_{j\in\Lambda_1} a_j^p < \infty$. Denote

$$f = \alpha^{1/q}\beta\Big(\sum_{j\in\Lambda_1} a_j e_j + \sum_{j\in\Lambda_2} t_j^{p/q} e_j\Big),$$

where

$$\beta = \Big(\eta_0 + \eta_0' + \alpha\Big(\sum_{j\in\Lambda_1} a_j^q + \sum_{j\in\Lambda_2} t_j^p\Big)\Big)^{-1/q} \leq 1.$$

We claim that for some realization of the gAWCGA for f the indices from Λ_1 will not be chosen. Namely, we show that there exists a realization such that for any $n \geq 1$ the set Γ_n of indices of e_j chosen at the first n steps of the algorithm and the nth remainder f_n satisfy

$$\begin{aligned} &\Gamma_n \cap \Lambda_1 = \emptyset, \\ &f_n = \beta(\eta_n + \eta_n')^{1/q} e_1 + \alpha^{1/q}\beta\Big(\sum_{j\in\Lambda_1} a_j e_j + \sum_{j\in\Lambda_2^{(n)}} t_j^{p/q} e_j\Big), \end{aligned} \tag{3.9}$$

where $\Lambda_2^{(n)} = \Lambda_2 \setminus \Gamma_n$. Consider the following realization of the gAWCGA for f:

For $n = 1$ choose

$$F_0(x) = F_f(x) = \frac{\sum_{j\in\Lambda_1} a_j^{q/p} x_j + \sum_{j\in\Lambda_2} t_j x_j}{(\alpha\beta^q)^{-1/p}\|f\|_q^{q/p}}.$$

Then, since $a_j \leq 1$, we get

$$\begin{aligned} &F_0(e_0) = 0, \\ &F_0(e_j) \leq (\alpha\beta^q)^{1/p}\|f\|_q^{-q/p} \qquad \text{for any } j \in \Lambda_1, \\ &F_0(e_j) = t_j(\alpha\beta^q)^{1/p}\|f\|_q^{-q/p} \quad \text{for any } j \in \Lambda_2, \end{aligned}$$

and the choice $\phi_1 = e_1$ satisfies (1.2) since $1 \in \Lambda_2$. Thus $\Gamma_1 = \{1\}$, and the element

$$f_1=\beta(\eta_1+\eta_1')^{1/q}e_1+\alpha^{1/q}\beta\Big(\sum_{j\in\Lambda_1}a_je_j+\sum_{j\in\Lambda_2^{(1)}}t_j^{p/q}e_j\Big)$$

satisfies (1.3) since the estimate

$$\beta\le E_1=\alpha^{1/q}\beta\Big(\sum_{j\in\Lambda_1}a_j^q+\sum_{j\in\Lambda_2^{(1)}}t_j^p\Big)^{1/q}\le 1$$

and Lemma 4 imply that

$$\begin{aligned}\|f_1\|_q&=\beta\Big(\eta_1+\eta_1'+\alpha\Big(\sum_{j\in\Lambda_1}a_j^q+\sum_{j\in\Lambda_2^{(1)}}t_j^p\Big)\Big)^{1/q}\\&\le\beta(\eta_1+\eta_1')+E_1\le(1+\eta_1)E_1+\eta_1'.\end{aligned}$$

Hence for $n=1$ claim (3.9) holds.

For $n\ge 1$, if

$$f_n=\beta(\eta_n+\eta_n')^{1/q}e_1+\alpha^{1/q}\beta\Big(\sum_{j\in\Lambda_1}a_je_j+\sum_{j\in\Lambda_2^{(n)}}t_j^{p/q}e_j\Big),$$

then the function

$$F_n(x)=\frac{(\delta_n+\delta_n')^{1/p}x_0+(\eta_n+\eta_n')^{1/p}x_1+\alpha^{1/p}\big(\sum_{j\in\Lambda_1}a_j^{q/p}x_j+\sum_{j\in\Lambda_2^{(n)}}t_jx_j\big)}{\big(\beta^{-q}(1+\delta_n+\delta_n')\|f_n\|_q^q\big)^{1/p}}$$

satisfies (1.1) since the estimate

$$\beta\le\|f_n\|_q=\beta\Big(\eta_n+\eta_n'+\alpha\Big(\sum_{j\in\Lambda_1}a_j^q+\sum_{j\in\Lambda_2^{(n)}}t_j^p\Big)\Big)^{1/q}\le 1$$

and Hölder's inequality imply that

$$|F_n(x)|\le\frac{(\delta_n+\delta_n'+\beta^{-q}\|f_n\|_q^q)^{1/p}(\sum_{j=0}^\infty x_j^q)^{1/q}}{\big(\beta^{-q}(1+\delta_n+\delta_n')\|f_n\|_q^q\big)^{1/p}}\le\|x\|_q$$

and

$$F_n(f_n)=\frac{\|f_n\|_q^q}{(1+\delta_n+\delta_n')^{1/p}\|f_n\|_q^{q/p}}\ge(1-\delta_n)\|f_n\|_q-\delta_n',$$

where the last inequality holds since $\|f_n\|_q\le 1$ and

$$\begin{aligned}(1+\delta_n+\delta_n')^{1/p}((1-\delta_n)\|f_n\|_q-\delta_n')&\le(1+\delta_n+\delta_n')^{1/p}(1-\delta_n-\delta_n')\|f_n\|_q\\&=(1-(\delta_n+\delta_n')^2)^{1/p}(1-\delta_n-\delta_n')^{1/q}\|f_n\|_q\le\|f_n\|_q.\end{aligned}$$

Hence such a choice of functional is admissible. Let

$$A_n = \big(\beta^{-q}(1+\delta_n+\delta'_n)\|f_n\|_q^q\big)^{-1/p}.$$

Then, since $a_j \le 1$, we get

$$\begin{aligned} F_n(e_0) &= (\delta_n+\delta'_n)^{1/p}A_n, \\ F_n(e_1) &= (\eta_n+\eta'_n)^{1/p}A_n, \\ F_n(e_j) &\le \alpha^{1/p}A_n \qquad \text{for any } j\in\Lambda_1, \\ F_n(e_j) &= t_j\alpha^{1/p}A_n \quad \text{for any } j\in\Lambda_2^{(n)}, \\ F_n(e_j) &= 0 \qquad\qquad \text{for any } j\in\Gamma_n\setminus\{0,1\}. \end{aligned}$$

If $n+1\in\Lambda_2$ we choose $\phi_{n+1}=e_{n+1}$. Otherwise $n+1\in\Lambda_1$, and by definition of that set at least one of the following inequalities holds:

$$\begin{aligned} &F_n(e_0) \ge t_{n+1}\alpha^{1/p}A_n \ge t_{n+1}\alpha^{1/p}A_n - t'_{n+1}, \\ &F_n(e_1) \ge t_{n+1}\alpha^{1/p}A_n \ge t_{n+1}\alpha^{1/p}A_n - t'_{n+1}, \\ &t_{n+1}\sup_{g\in\mathcal{D}}F_n(g) - t'_{n+1} \le t_{n+1}\alpha^{1/p}A_n - \alpha^{1/p}t_{n+1} \le 0. \end{aligned}$$

Then we choose $\phi_{n+1}=e_0$ or $\phi_{n+1}=e_1$. In either case $\Gamma_{n+1}\cap\Lambda_1=\emptyset$ and the element

$$f_{n+1} = \beta(\eta_{n+1}+\eta'_{n+1})^{1/q}e_1 + \alpha^{1/q}\beta\Big(\sum_{j\in\Lambda_1}a_je_j + \sum_{j\in\Lambda_2^{(n+1)}}t_j^{p/q}e_j\Big)$$

satisfies (1.3) since the estimate

$$\beta \le E_{n+1} = \alpha^{1/q}\beta\Big(\sum_{j\in\Lambda_1}a_j^q + \sum_{j\in\Lambda_2^{(n+1)}}t_j^p\Big)^{1/q} \le 1$$

and Lemma 4 yield

$$\begin{aligned} \|f_{n+1}\|_q &= \beta\Big(\eta_{n+1}+\eta'_{n+1} + \alpha\Big(\sum_{j\in\Lambda_1}a_j^q + \sum_{j\in\Lambda_2^{(n+1)}}t_j^p\Big)\Big)^{1/q} \\ &\le \beta(\eta_{n+1}+\eta'_{n+1}) + E_{n+1} \le (1+\eta_{n+1})E_{n+1} + \eta'_{n+1}. \end{aligned}$$

Hence claim (3.9) holds for any $n\ge1$. Thus $\|f_n\|\ge\beta\not\to0$ and the gAWCGA of f diverges. ■

4. Nonsufficiency of smoothness of the space for the convergence of the WCGA. In this section we demonstrate that smoothness of the space is not sufficient for the convergence of the WCGA. Specifically, we will construct an example of a smooth Banach space X, a dictionary $\mathcal{D}$, and an element $f \in X$ such that the WCGA for f with any weakness sequence $\{t_n\}_{n=1}^{\infty}$ diverges. The space we construct was used in [4] and, in a special case, in [10].

In order to obtain the desired space we renorm ℓ_1. Take a nonincreasing sequence $\{p_n\}_{n=1}^{\infty}$ of numbers with $p_n > 1$ for any $n \geq 1$ and with

$$\sum_{n=1}^{\infty}\left(1 - \frac{1}{p_n}\right) < \infty. \tag{4.1}$$

Let $\{e_n\}_{n=1}^{\infty}$ be the canonical basis of ℓ_1. Consider a sequence $\{\vartheta_n\}_{n=0}^{\infty}$ of nonlinear functionals defined as follows: for any $x = \sum_{n=1}^{\infty} x_n e_n \in \ell_1$,

$$\vartheta_0(x) = 0, \qquad \vartheta_n(x) = (\vartheta_{n-1}^{p_n}(x) + |x_n|^{p_n})^{1/p_n} \qquad \text{for } n \geq 1.$$

In particular,

$$\begin{aligned}
\vartheta_1(x) &= |x_1|,\\
\vartheta_2(x) &= (|x_1|^{p_2} + |x_2|^{p_2})^{1/p_2},\\
\vartheta_3(x) &= \big((|x_1|^{p_2} + |x_2|^{p_2})^{p_3/p_2} + |x_3|^{p_3}\big)^{1/p_3}.
\end{aligned}$$

We claim that ϑ_n is a norm on ℓ_1^n. Indeed, for any $x \in \ell_1^n$,

$$\begin{aligned}
\vartheta_n(x) = 0 \quad &\text{if and only if} \quad x = 0,\\
\vartheta_n(\lambda x) = |\lambda|\vartheta_n(x) \quad &\text{for any } \lambda \in \mathbb{R}.
\end{aligned}$$

We prove the triangle inequality for ϑ_n using induction on n. The base case $n = 1$ is obvious. Then, using Minkowski's inequality, for any $n > 1$ and any $x, y \in \ell_1^n$ we obtain

$$\begin{aligned}
\vartheta_n(x+y) &= \big(\vartheta_{n-1}^{p_n}(x+y) + |x_n + y_n|^{p_n}\big)^{1/p_n}\\
&\leq \big((\vartheta_{n-1}(x) + \vartheta_{n-1}(y))^{p_n} + (|x_n| + |y_n|)^{p_n}\big)^{1/p_n}\\
&\leq (\vartheta_{n-1}^{p_n}(x) + |x_n|^{p_n})^{1/p_n} + (\vartheta_{n-1}^{p_n}(y) + |y_n|^{p_n})^{1/p_n}\\
&= \vartheta_n(x) + \vartheta_n(y).
\end{aligned}$$

Define

$$X = \Big\{x \in \ell_1 : \lim_{n\to\infty} \vartheta_n(x) < \infty\Big\}, \qquad \|x\|_X = \lim_{n\to\infty} \vartheta_n(x).$$

Since for any $x \in \ell_1$ the sequence $\{\vartheta_n(x)\}_{n=0}^{\infty}$ is nondecreasing, the limit always exists. Moreover, for any $n \geq 1$,

$$\vartheta_n(x) \leq \vartheta_{n-1}(x) + |x_n| \leq \sum_{k=1}^{n} |x_k|,$$

and, by Hölder's inequality,

$$\begin{aligned}\sum_{k=1}^{n}|x_k| &\le 2^{1-1/p_2}\vartheta_2(x)+\sum_{k=3}^{n}|x_k| \le 2^{1-1/p_2}\Big(\vartheta_2(x)+\sum_{k=3}^{n}|x_k|\Big)\\ &\le 2^{1-1/p_2}\Big(2^{1-1/p_3}\vartheta_3(x)+\sum_{k=4}^{n}|x_k|\Big) \le 2^{\sum_{k=2}^{3}(1-1/p_k)}\Big(\vartheta_3(x)+\sum_{k=4}^{n}|x_k|\Big)\\ &\dots\\ &\le 2^{\sum_{k=2}^{n-1}(1-1/p_k)}(\vartheta_{n-1}(x)+|x_n|) \le 2^{\sum_{k=2}^{n}(1-1/p_k)}\vartheta_n(x).\end{aligned}$$

Therefore, by letting $n\to\infty$, we obtain, for any $x\in X$,

$$\rho\|x\|_1 \le \|x\|_X \le \|x\|_1, \tag{4.2}$$

where $\rho = 2^{-\sum_{k=1}^{\infty}(1-1/p_k)} > 0$ by the choice (4.1) of $\{p_n\}_{n=1}^{\infty}$. Hence, the $\|\cdot\|_X$-norm is equivalent to the $\|\cdot\|_1$-norm, and $X=(\ell_1,\|\cdot\|_X)$ is a Banach space. We note that while we impose condition (4.1) to obtain the equivalence of norms, weaker restrictions on the rate of decay of $\{p_n\}_{n=1}^{\infty}$ might be used (see [5, Proposition 1]).

Next, we show that the space X is smooth, that is for any $x\in X$ there is a unique norming functional F_x.

First, we find the dual of X. Let $\{e_n^*\}_{n=1}^{\infty}$ be the canonical basis in ℓ_∞. Consider the sequence $\{q_n\}_{n=1}^{\infty}$ of numbers given by

$$q_n = \frac{p_n}{p_n-1}.$$

Similarly, we define the sequence $\{\nu_n\}_{n=0}^{\infty}$ of functionals as follows: for any sequence $a=\sum_{n=1}^{\infty}a_ne_n^*\in\ell_\infty$,

$$\nu_0(a)=0,\qquad \nu_n(a)=(\nu_{n-1}^{q_n}(a)+|a_n|^{q_n})^{1/q_n}\qquad \text{for } n\ge 1.$$

Define

$$X^*=\Big\{a\in\ell_\infty : \lim_{n\to\infty}\nu_n(a)<\infty\Big\},\qquad \|a\|_{X^*}=\lim_{n\to\infty}\nu_n(a).$$

In the same way as above we show that the $\|\cdot\|_{X^*}$-norm and the $\|\cdot\|_\infty$-norm are equivalent. For any $n\ge 1$,

$$\nu_n(a)\ge \sup_{k\le n}|a_k|,$$

and

$$\begin{aligned}\nu_n(a)=(\nu_{n-1}^{q_n}(a)+|a_n|^{q_n})^{1/q_n} &\le 2^{1/q_n}\max\{\nu_{n-1}(a),|a_n|\}\\ &\le 2^{1/q_{n-1}+1/q_n}\max\{\nu_{n-2}(a),|a_{n-1}|,|a_n|\}\le\cdots\\ &\le 2^{\sum_{k=3}^{n}1/q_k}\max\{\nu_2(a),|a_3|,\dots,|a_n|\}\\ &\le 2^{\sum_{k=2}^{n}1/q_k}\max\{|a_1|,|a_2|,\dots,|a_n|\}.\end{aligned}$$

Therefore, by letting $n \to \infty$, for any $a \in X^*$ we obtain

$$\|a\|_\infty \le \|a\|_{X^*} \le \rho^{-1}\|a\|_\infty,$$

i.e. the $\|\cdot\|_{X^*}$-norm is equivalent to the $\|\cdot\|_\infty$-norm, and $X^* = (\ell_\infty, \|\cdot\|_{X^*})$ is a Banach space.

We claim that X^* is the dual of X. Indeed, for any $x \in X$ and any $a \in X^*$ the Hölder inequality yields, for any $N \in \mathbb{N}$,

$$\begin{aligned}\sum_{n=1}^{N} |a_n|\,|x_n| &\le \nu_2(a)\vartheta_2(x) + \sum_{n=3}^{N} |a_n|\,|x_n| \le \cdots \\ &\le \nu_{N-1}(a)\vartheta_{N-1}(x) + |a_N|\,|x_N| \le \nu_N(a)\vartheta_N(x),\end{aligned}$$

and therefore

$$|a(x)| = \lim_{N\to\infty}\Big|\sum_{n=1}^{N} a_n x_n\Big| \le \lim_{N\to\infty}\sum_{n=1}^{N} |a_n|\,|x_n| \le \|a\|_{X^*}\|x\|_X.$$

Similarly, using induction we obtain, for any functional $a(x) = \sum_{n=1}^{\infty} a_j x_j$ on X,

$$\sup_{x\in S_X} a(x) = \|a\|_{X^*},$$

which completes the proof of the claim.

Consider the spaces $X^n = (\ell_1^n, \vartheta_n(\cdot))$ and $X^{*n} = (\ell_\infty^n, \nu_n(\cdot))$, the initial segments of X and X^* respectively. We use induction to show that for any $n \ge 1$ the space X^{*n} is strictly convex. Indeed, $X^{*1} = (\mathbb{R}, |\cdot|)$ is strictly convex, and for any $n > 1$,

$$X^{*n} = X^{*n-1} \oplus_{q_n} \mathbb{R}$$

is strictly convex as a q_n-sum of strictly convex spaces with $1 < q_n < \infty$ (see, e.g., [1]). Therefore X^n is smooth as a predual of a strictly convex space X^{*n} (e.g. [1]).

Lastly, we will need the following simple lemma to prove smoothness of X.

Lemma 5. *Let $x = \sum_{n=1}^{\infty} x_n e_n \in X$ and $F_x = \sum_{n=1}^{\infty} a_n e_n^*$ be a norming functional for x. Then for any $m \in \mathbb{N}$,*

$$F_x^m = \frac{\sum_{n=1}^{m} a_n e_n^*}{\nu_m(a)}$$

is a norming functional for $x^m = \sum_{n=1}^{m} x_n e_n \in X^m$.

Proof. Assume that $F_x^m(x^m) < \|x^m\|_{X^m} = \vartheta_m(x)$, i.e. F_x^m is not a norming functional for x^m. Then

$$\sum_{n=1}^{m} a_n x_n < \nu_m(a)\vartheta_m(x)$$

and for any $N > m$ by Hölder's inequality

$$\begin{aligned} F_x(x) &= \sum_{n=1}^{\infty} a_n x_n \le \sum_{n=1}^{\infty} |a_n|\,|x_n| \\ &< \nu_m(a)\vartheta_m(x) + \sum_{n=m+1}^{\infty} |a_n|\,|x_n| \\ &\le \nu_N(a)\vartheta_N(x) + \sum_{n=N+1}^{\infty} |a_n|\,|x_n|. \end{aligned}$$

Letting as $N \to \infty$ we get $F_x(x) < \|a\|_{X^*}\|x\|_X = \|x\|_X$, which contradicts $F_x(x) = \|x\|$. ∎

LEMMA 6 [1]. *The space $X = (\ell_1, \|\cdot\|_X)$ is smooth.*

Proof. Assume that there is an $x \in X$ with two distinct norming functionals: $F_x = \sum_{n=1}^{\infty} a_n e_n^*$ and $G_x = \sum_{n=1}^{\infty} b_n e_n^*$. Then Lemma 5 and the smoothness of initial segments imply, for any $N \in \mathbb{N}$,

$$\frac{\sum_{n=1}^{N} a_n e_n^*}{\nu_N(a)} = F_x^N = G_x^N = \frac{\sum_{n=1}^{N} b_n e_n^*}{\nu_N(b)}.$$

Pick $m \in \mathbb{N}$ such that $a_m \neq b_m$. Then, letting $N \to \infty$ and taking into account that $\|a\|_{X^*} = \|b\|_{X^*} = 1$, we get

$$a_m = \lim_{N\to\infty} \frac{a_m}{\nu_N(a)} = \lim_{N\to\infty} F_x^N(e_m) = \lim_{N\to\infty} G_x^N(e_m) = \lim_{N\to\infty} \frac{b_m}{\nu_N(b)} = b_m,$$

which contradicts $a_m \neq b_m$, and thus X is smooth. ∎

Finally, we need to find the norming functionals on X. Take any $x = \sum_{n=1}^{\infty} x_n e_n \in X$ and consider a sequence $\{\mathcal{F}_x^n\}_{n=0}^{\infty}$ of functionals on X defined as follows: for any $y = \sum_{n=1}^{\infty} y_n e_n \in X$,

$$\begin{aligned} \mathcal{F}_x^0(y) &= 0, \\ \mathcal{F}_x^n(y) &= \frac{\vartheta_{n-1}^{p_n-1}(x)\mathcal{F}_x^{n-1}(y) + \operatorname{sgn} x_n\,|x_n|^{p_n-1} y_n}{\vartheta_n^{p_n-1}(x)} \\ &= \vartheta_n^{1-p_{n+1}}(x) \sum_{k=1}^{n} \Big(\operatorname{sgn} x_k\,|x_k|^{p_k-1} y_k \prod_{j=k}^{n} \vartheta_j^{p_{j+1}-p_j}(x) \Big) \quad \text{for } n \ge 1. \end{aligned}$$

LEMMA 7. *Let $x = \sum_{n=1}^{m} x_n e_n \in X$. Then $\mathcal{F}_x^m$ is a norming functional for x.*

Proof. We will use induction on m. For $m = 1$,

$$\mathcal{F}_x^1(y) = \operatorname{sgn} x_1\, y_1,$$

[1] This proof is due to S. J. Dilworth.

and $\mathcal{F}_x^1(x) = \vartheta_1(x) = \|x\|_X$, $|\mathcal{F}_x^1(y)| = \vartheta_1(y) = \|y\|_X$. For $m > 1$,

$$\mathcal{F}_x^m(y) = \frac{\vartheta_{m-1}^{p_m-1}(x)\mathcal{F}_x^{m-1}(y) + \operatorname{sgn} x_m\, |x_m|^{p_m-1} y_m}{\vartheta_m^{p_m-1}(x)}.$$

Then

$$\mathcal{F}_x^m(x) = \frac{\vartheta_{m-1}^{p_m}(x) + |x_m|^{p_m}}{\vartheta_m^{p_m-1}(x)} = \vartheta_m(x) = \|x\|_X,$$

and the induction hypothesis and Hölder's inequality show that

$$\begin{aligned}|\mathcal{F}_x^m(y)| &\le \frac{\vartheta_{m-1}^{p_m-1}(x)|\mathcal{F}_x^{m-1}(y)| + |x_m|^{p_m-1}|y_m|}{\vartheta_m^{p_m-1}(x)} \\ &\le \frac{\vartheta_{m-1}^{p_m-1}(x)\vartheta_{m-1}(y) + |x_m|^{p_m-1}|y_m|}{\vartheta_m^{p_m-1}(x)} \\ &\le (\vartheta_{m-1}^{p_m}(y) + |y_m|^{p_m})^{1/p_m} = \vartheta_n(y) = \|y\|_X. \ \blacksquare\end{aligned}$$

Thus, we have established the norming functionals $\mathcal{F}_n$ in the initial segments X^n. In particular, for any $x, y \in X$,

$$\mathcal{F}_x^1(y) = \operatorname{sgn} x_1\, y_1,$$

$$\mathcal{F}_x^2(y) = \frac{\operatorname{sgn} x_1\, |x_1|^{p_2-1} y_1 + \operatorname{sgn} x_2\, |x_2|^{p_2-1} y_2}{\vartheta_2^{p_2-1}(x)},$$

$$\mathcal{F}_x^3(y) = \frac{(\operatorname{sgn} x_1\, |x_1|^{p_2-1} y_1 + \operatorname{sgn} x_2\, |x_2|^{p_2-1} y_2)\vartheta_2^{p_3-p_2}(x) + \operatorname{sgn} x_3\, |x_3|^{p_3-1} y_3}{\vartheta_3^{p_3-1}(x)}. \ \blacksquare$$

We now choose a dictionary $\mathcal{D}$ in X and an element $f \in X$ such that the WCGA for f diverges. Without loss of generality assume $t_n = 1$ for each $n \ge 1$, i.e. an element of the dictionary that maximizes $F_{f_{n-1}}$ is chosen at each step. Let

$$\begin{aligned} g_0 &= e_1 + e_2 + e_3, \\ g_k &= e_k + e_{k+1} \quad \text{for each } k \ge 1, \end{aligned}$$

and set $\mathcal{D} = \{\pm g_n/\|g_n\|_X\}_{n=0}^{\infty}$. Note that for any $k \ge 1$,

$$\|g_k\|_X = 2^{1/p_{k+1}} \le 2^{1/p_2} < (1 + 2^{p_3/p_2})^{1/p_3} = \|g_0\|_X. \tag{4.3}$$

Let $f = e_1 \in X$. Then $f = g_0 - g_2 \in A_0(\mathcal{D})$. We will show that the WCGA diverges even for such a simple element. We claim that for any $m \ge 1$,

$$\phi_m = \pm g_m/\|g_m\|_X, \tag{4.4}$$

where $\pm$ neans plus or minus. We will prove this claim by induction on m.

Consider the case $m = 1$. Lemma 7 yields $F_f = \mathcal{F}_f^1$, thus

$$|\mathcal{F}_f^1(g_0)| = 1, \quad |\mathcal{F}_f^1(g_1)| = 1, \quad |\mathcal{F}_f^1(g_k)| = 0 \quad \text{for any } k > 1.$$

Then estimate (4.3) guarantees that $\phi_1 = \pm g_1/\|g_1\|_X$.

Let now $m > 1$. By the induction hypothesis the elements

$$\pm g_1/\|g_1\|_X, \dots, \pm g_{m-1}/\|g_{m-1}\|_X$$

were chosen at previous steps. Then $f_{m-1} = \sum_{n=1}^{m} c_n e_n$ for some coefficients $\{c_n\}_{n=1}^{m}$, and therefore $F_{f_{m-1}} = \mathcal{F}^m_{f_{m-1}}$ by Lemma 7. Note that $f_{m-1} \in X^m$, which is a uniformly smooth space since it is smooth and finite-dimensional. Hence, applying Lemma 1 we find that $F_{f_{m-1}}(g_k) = 0$ for any $k = 1, \dots, m-1$, i.e.

$$\begin{aligned}
\mathcal{F}^m_{f_{m-1}}(g_1) &= \frac{\operatorname{sgn} c_1\, |c_1|^{p_2-1} + \operatorname{sgn} c_2\, |c_2|^{p_2-1}}{\vartheta_2^{p_2-1}(f_{m-1}) \dots \vartheta_m^{p_m-1}(f_{m-1})} = 0,\\
\mathcal{F}^m_{f_{m-1}}(g_2) &= \frac{\operatorname{sgn} c_2\, |c_2|^{p_2-1}\vartheta_2^{p_3-p_2}(f_{m-1}) + \operatorname{sgn} c_3\, |c_3|^{p_3-1}}{\vartheta_3^{p_3-1}(f_{m-1}) \dots \vartheta_m^{p_m-1}(f_{m-1})} = 0,\\
&\dots\\
\mathcal{F}^m_{f_{m-1}}(g_{m-1}) &= \frac{\operatorname{sgn} c_{m-1}\, |c_{m-1}|^{p_{m-1}-1}\vartheta_{m-1}^{p_m-p_{m-1}}(f_{m-1}) + \operatorname{sgn} c_m\, |c_m|^{p_m-1}}{\vartheta_m^{p_m-1}(f_{m-1})}\\
&= 0.
\end{aligned}$$

From these equalities we derive

$$\begin{aligned}
|c_2|^{p_2-1} &= |c_1|^{p_2-1},\\
|c_3|^{p_3-1} &= |c_2|^{p_2-1}\vartheta_2^{p_3-p_2}(f_{m-1}),\\
&\dots\\
|c_m|^{p_m-1} &= |c_{m-1}|^{p_{m-1}-1}\vartheta_{m-1}^{p_m-p_{m-1}}(f_{m-1}),
\end{aligned}$$

which implies that for any $k = 3, \dots, m$,

$$|c_k|^{p_k-1} = |c_1|^{p_2-1} \prod_{n=2}^{k-1} \vartheta_n^{p_{n+1}-p_n}(f_{m-1}). \tag{4.5}$$

Therefore

$$\begin{aligned}
|\mathcal{F}^m_{f_{m-1}}(g_0)| &= |\mathcal{F}^m_{f_{m-1}}(g_0 - g_1)|\\
&= \vartheta_m^{1-p_{m+1}}(f_{m-1})\Big(|c_3|^{p_3-1} \prod_{j=3}^{m} \vartheta_j^{p_{j+1}-p_j}(f_{m-1})\Big),\\
|\mathcal{F}^m_{f_{m-1}}(g_m)| &= \frac{|c_m|^{p_m-1}}{\vartheta_m^{p_m-1}(f_{m-1})},\\
|\mathcal{F}^m_{f_{m-1}}(g_k)| &= 0 \quad \text{for any } k \in \mathbb{N} \setminus \{m\}.
\end{aligned}$$

Thus, by (4.5),

$$|\mathcal{F}^m_{f_{m-1}}(g_0)| = \vartheta_m^{1-p_{m+1}}(f_{m-1})\Big(|c_1|^{p_2-1} \prod_{j=2}^{m} \vartheta_j^{p_{j+1}-p_j}(f_{m-1})\Big) = |\mathcal{F}^m_{f_{m-1}}(g_m)|,$$

and estimate (4.3) guarantees that $\phi_m = \pm g_m/\|g_m\|_X$, which completes the proof of (4.4).

Hence, the element $\pm g_0/\|g_0\|_X$ will not be chosen and

$$\Phi_n = \operatorname{span}\{g_1, \dots, g_n\}$$

for any $n \geq 1$. Then the equivalence (4.2) of norms implies that

$$\|f_n\|_X = \inf_{G\in\Phi_n} \|f - G\|_X \geq \rho \inf_{G\in\Phi_n} \|f - G\|_1 = \rho \nrightarrow 0 \quad \text{as } n \to \infty,$$

i.e. the WCGA of f diverges.

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