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Lineability in sequence and function spaces

by

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Abstract. We prove the existence of large algebraic structures—including large vector subspaces or infinitely generated free algebras—inside, among others, the family of Lebesgue measurable functions that are surjective in a strong sense, the family of nonconstant differentiable real functions vanishing on dense sets, and the family of discontinuous separately continuous real functions. Lineability in special spaces of sequences is also investigated.

1. Introduction and notation. Lebesgue ([27], 1904) was probably the first to show an example of a real function on the reals with the rather surprising property that it takes on each real value in any nonempty open set (see also [22, 23]). The functions satisfying this property are called *everywhere surjective* (functions with even more stringent properties can be found in [18, 26]). Of course, such functions are nowhere continuous but, as we will see later, it is possible to construct a *Lebesgue measurable* everywhere surjective function. Entering a very different realm, in 1906 Pompeiu [29] was able to construct a nonconstant differentiable function on the reals whose derivative *vanishes on a dense set.*

Passing to several variables, the first problem one meets related to the "minimal regularity" of functions at a elementary level is that of whether separate continuity implies continuity, the answer being in the negative.

In this paper, we will consider the families consisting of each of these kinds of functions, as well as two special families of sequences, and analyze the existence of large algebraic structures inside each of these families. Nowadays

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the topic of lineability has had a major resonance in many different areas on mathematics, from real and complex analysis [9], to set theory [21], operator theory [25], and even (more recently) in probability theory [17]. Our main goal here is to continue this research.

Let us now fix some notation. As usual, we denote by $\mathbb{N}, \mathbb{Q}$ and $\mathbb{R}$ the set of positive integers, the set of rational numbers and the set of all real numbers, respectively. The symbol $\mathcal{C}(I)$ will stand for the vector space of all real continuous functions defined on an interval $I \subset \mathbb{R}$. In the special case $I = \mathbb{R}$, the space $\mathcal{C}(\mathbb{R})$ will be endowed with the topology of uniform convergence on compacta. It is well known that $\mathcal{C}(\mathbb{R})$ under this topology is an F-space, that is, a complete metrizable topological vector space.

We denote by $\mathcal{MES}$ the family of Lebesgue measurable everywhere surjective functions $\mathbb{R} \to \mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be a *Pompeiu function* (see Figure 1) if it is differentiable and f' vanishes on a dense set in $\mathbb{R}$. The symbols $\mathcal{P}$ and $\mathcal{DP}$ stand for the vector spaces of Pompeiu functions and of the derivatives of Pompeiu functions, respectively. Other notation will be rather standard and, when needed, definitions will be provided.

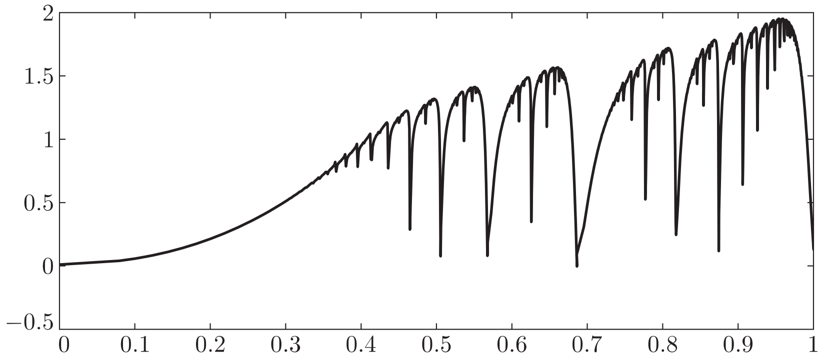

Fig. 1. Rough sketch of the graph of Pompeiu's original example

The organization of this paper is as follows. In Section 2, a number of concepts concerning the linear or algebraic structure of sets inside a vector space or a linear algebra, together with some examples related to everywhere surjectivity and special derivatives, will be recalled. Sections 3, 4, and 5 will focus on diverse lineability properties of the families $\mathcal{MES}$, $\mathcal{P}$, $\mathcal{DP}$, and certain subsets of discontinuous functions, thus completing or extending a number of known results about several strange classes of real functions. Concerning sequence spaces, Section 6 will deal with subsets of convergent and divergent series for which classical tests of convergence fail; finally, in Section 7 convergence in measure versus convergence almost everywhere will be analyzed in the space of sequences of measurable Lebesgue functions on the unit interval.

2. Lineability notions. A number of terms have been coined in order to describe the algebraic size of a given set—see [3, 5, 10, 12, 15, 16, 24, 31] (see also the survey paper [13] and the book [1] for an account of lineability properties of specific subsets of vector spaces). Namely, if X is a vector space, α is a cardinal number and $A \subset X$, then A is said to be:

- *lineable* if there is an infinite-dimensional vector space M such that $M \setminus \{0\} \subset A$,
- *α-lineable* if there exists a vector space M with $\dim(M) = \alpha$ and $M \setminus \{0\} \subset A$ (hence lineability means $\aleph_0$-lineability, where $\aleph_0 = \operatorname{card}(\mathbb{N})$, the cardinality of $\mathbb{N}$), and
- *maximal lineable* in X if A is $\dim(X)$-lineable.

If, in addition, X is a topological vector space, then A is said to be:

- *dense-lineable* in X whenever there is a dense vector subspace M of X satisfying $M \setminus \{0\} \subset A$ (hence dense-lineability implies lineability as soon as $\dim(X) = \infty$), and
- *maximal dense-lineable* in X whenever there is a dense vector subspace M of X satisfying $M \setminus \{0\} \subset A$ and $\dim(M) = \dim(X)$.

And, according to [4, 8], if X is a topological vector space contained in some (linear) algebra, then A is called:

- *algebrable* if there is an algebra M such that $M \setminus \{0\} \subset A$ and M is infinitely generated, that is, the cardinality of any system of generators of M is infinite,
- *densely algebrable* in X if, in addition, M can be taken dense in X,
- *α-algebrable* if there is an α-generated algebra M with $M \setminus \{0\} \subset A$,
- *strongly α-algebrable* if there exists an α-generated *free* algebra M with $M \setminus \{0\} \subset A$ (for $\alpha = \aleph_0$, we simply say *strongly algebrable*),
- *densely strongly α-algebrable* if, in addition, the free algebra M can be taken dense in X.

Note that if X is contained in a commutative algebra then a set $B \subset X$ is a generating set of some free algebra contained in A if and only if for any $N \in \mathbb{N}$, any nonzero polynomial P in N variables without constant term and any distinct $f_1, \ldots, f_N \in B$, we have $P(f_1, \ldots, f_N) \neq 0$ and $P(f_1, \ldots, f_N) \in A$. Observe that strong α-algebrability $\Rightarrow$ α-algebrability $\Rightarrow$ α-lineability, and none of these implications can be reversed (see [13, p. 74]).

In [3] the authors proved that the set of *everywhere surjective* functions $\mathbb{R} \to \mathbb{R}$ is $2^{\mathfrak{c}}$-lineable, which is the best possible result in terms of dimension ($\mathfrak{c}$ being the cardinality of the continuum). In other words, the last set is maximal lineable in the space of all real functions. Other results establishing the degree of lineability of more stringent classes of functions can be found in [13] and the references therein.

Turning to the setting of more regular functions, in [20] the following results are proved: the set of *differentiable* functions on $\mathbb{R}$ whose derivatives are discontinuous almost everywhere is $\mathfrak{c}$-lineable; given a nonvoid compact interval $I \subset \mathbb{R}$, the family of differentiable functions whose derivatives are discontinuous almost everywhere on I is dense-lineable in the space $\mathcal{C}(I)$, endowed with the supremum norm; and the class of differentiable functions on $\mathbb{R}$ that are monotone on no interval is $\mathfrak{c}$-lineable.

Finally, recall that every *bounded variation* function f on an interval $I \subset \mathbb{R}$ (that is, with $\sup\{\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| : \{t_1 < \cdots < t_n\} \subset I,\, n \in \mathbb{N}\} < \infty$) is differentiable almost everywhere. A continuous bounded variation function $f : I \to \mathbb{R}$ is called *strongly singular* whenever $f'(x) = 0$ for almost every $x \in I$ and, in addition, f is nonconstant on any subinterval of I. Balcerzak et al. [6] showed that the set of strongly singular functions on $[0,1]$ is densely strongly $\mathfrak{c}$-algebrable in $\mathcal{C}([0,1])$.

A number of related results will be shown in the next two sections.

3. Measurable functions. Our aim in this section is to study the lineability of the family of Lebesgue measurable functions $\mathbb{R} \to \mathbb{R}$ that are everywhere surjective, denoted $\mathcal{MES}$. This result is quite surprising, since (as we can see in [19, 20]) the class of everywhere surjective functions contains a $2^{\mathfrak{c}}$-lineable set of nonmeasurable ones (called *Jones functions*).

THEOREM 3.1. *The set $\mathcal{MES}$ is $\mathfrak{c}$-lineable.*

Proof. Firstly, we consider the everywhere surjective function furnished in [20, Example 2.2]. For the sake of convenience, we reproduce its construction. Let $(I_n)_{n\in\mathbb{N}}$ be the collection of all open intervals with rational endpoints. The interval I_1 contains a Cantor type set, call it C_1. Now, $I_2 \setminus C_1$ also contains a Cantor type set, call it C_2. Next, $I_3 \setminus (C_1 \cup C_2)$ contains a Cantor type set, C_3. Inductively, we construct a family of pairwise disjoint Cantor type sets, $(C_n)_{n\in\mathbb{N}}$, such that $I_n \setminus \bigcup_{k=1}^{n-1} C_k \supset C_n$ for every $n \in \mathbb{N}$. Now, for every $n \in \mathbb{N}$, take any bijection $\phi_n : C_n \to \mathbb{R}$, and define $f : \mathbb{R} \to \mathbb{R}$ as

$$f(x) = \begin{cases} \phi_n(x) & \text{if } x \in C_n \text{ for some } n, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is clearly everywhere surjective. Indeed, let I be any interval in $\mathbb{R}$. There exists $k \in \mathbb{N}$ such that $I_k \subset I$. Thus $f(I) \supset f(I_k) \supset f(C_k) = \phi_k(C_k) = \mathbb{R}$.

But the novelty of the last function is that f is, in addition, zero almost everywhere, and in particular, it is (Lebesgue) *measurable*. That is, $f \in \mathcal{MES}$.

Now, taking advantage of the approach of [3, Proposition 4.2], we are going to construct a vector space that will be useful later on. Let

$$\Lambda := \operatorname{span}\{\varphi_\alpha : \alpha > 0\}, \quad \text{where} \quad \varphi_\alpha(x) := e^{\alpha x} - e^{-\alpha x}.$$

Then M is a $\mathfrak{c}$-dimensional vector space because the φ_α ($\alpha > 0$) are linearly independent. Indeed, assume that there are scalars $c_1, \dots, c_p$ (not all 0) as well as positive reals $\alpha_1, \dots, \alpha_p$ such that $c_1\varphi_{\alpha_1}(x) + \dots + c_p\varphi_{\alpha_p}(x) = 0$ for all $x \in \mathbb{R}$. Without loss of generality, we may assume that $p \geq 2$, $c_p \neq 0$ and $\alpha_1 < \dots < \alpha_p$. Then $\lim_{x\to\infty}(c_1\varphi_{\alpha_1}(x) + \dots + c_p\varphi_{\alpha_p}(x)) = \infty$ or $-\infty$, which is clearly a contradiction. Therefore $c_1 = \dots = c_p = 0$ and we are done. Note that each nonzero member $g = \sum_{i=1}^p c_i\varphi_{\alpha_i}$ (with the c_i's and the α_i's as before) of Λ is (continuous and) surjective because $\lim_{x\to\infty} g(x) = \infty$ and $\lim_{x\to-\infty} g(x) = -\infty$ if $c_p > 0$ (with the values of the limits interchanged if $c_p < 0$).

Next, we define the vector space

$$M := \{g \circ f : g \in \Lambda\}.$$

Observe that since f is measurable and all functions g in Λ are continuous, the members of M are measurable. Fix any $h \in M \setminus \{0\}$. Then, again, there are finitely many scalars $c_1, \dots, c_p$ with $c_p \neq 0$ and positive reals $\alpha_1 < \dots < \alpha_p$ such that $g = c_1\varphi_{\alpha_1} + \dots + c_p\varphi_{\alpha_p}$ and $h = g \circ f$. Now, fix a nondegenerate interval $J \subset \mathbb{R}$. Then $h(J) = g(f(J)) = g(\mathbb{R}) = \mathbb{R}$, which shows that h is everywhere surjective. Hence $M \setminus \{0\} \subset \mathcal{MES}$.

Finally, by using the linear independence of the functions φ_α and the fact that f is surjective, it is easy to see that the functions $\varphi_\alpha \circ f$ ($\alpha > 0$) are linearly independent, which shows that M has dimension $\mathfrak{c}$, as required. ∎

In [33, Example 2.34] a sequence of measurable everywhere surjective functions tending pointwise to zero is exhibited. With Theorem 3.1 in hand, we now get a plethora of such sequences, and even in a much easier way than in [33].

Corollary 3.2. *The family of sequences $\{f_n\}_{n\geq 1}$ of Lebesgue measurable functions $\mathbb{R} \to \mathbb{R}$ such that f_n converges pointwise to zero and $f_n(I) = \mathbb{R}$ for any n and each nondegenerate interval I, is $\mathfrak{c}$-lineable.*

Proof. Consider the family $\widetilde{M}$ consisting of all sequences $\{h_n\}_{n\geq 1}$ given by $h_n(x) = h(x)/n$ where the functions h run over the vector space M constructed in the last theorem. It is easy to see that $\widetilde{M}$ is a $\mathfrak{c}$-dimensional vector subspace of $(\mathbb{R}^{\mathbb{R}})^{\mathbb{N}}$, each h_n is measurable, $h_n(x) \to 0$ ($n \to \infty$) for every $x \in \mathbb{R}$, and each h_n is everywhere surjective if h is not the zero function. ∎

Remark 3.3. It would be interesting to know whether $\mathcal{MES}$ is—like the set of everywhere surjective functions—maximal lineable in $\mathbb{R}^{\mathbb{R}}$ (that is, $2^{\mathfrak{c}}$-lineable).

4. Special differentiable functions. In this section, we analyze the lineability of the set of Pompeiu functions that are not constant on any interval. Of course, this set is not a vector space.

Firstly, the following version of the well-known Stone–Weierstrass density theorem (see e.g. [30]) for the space $\mathcal{C}(\mathbb{R})$ will be relevant to the proof of our main result. Its proof is a simple application of the original Stone–Weierstrass theorem for $\mathcal{C}(S)$ (the Banach space of continuous functions $S \to \mathbb{R}$, endowed with the uniform distance, where S is a compact topological space) together with the fact that convergence in $\mathcal{C}(\mathbb{R})$ means convergence on each compact subset of $\mathbb{R}$. So we omit the proof.

LEMMA 4.1. *Suppose that $\mathcal{A}$ is a subalgebra of $\mathcal{C}(\mathbb{R})$ satisfying the following properties:*

(a) *Given $x_0 \in \mathbb{R}$ there is $F \in \mathcal{A}$ with $F(x_0) \neq 0$.*
(b) *Given distinct $x_0, x_1 \in \mathbb{R}$, there exists $F \in \mathcal{A}$ such that $F(x_0) \neq F(x_1)$.*

Then $\mathcal{A}$ is dense in $\mathcal{C}(\mathbb{R})$.

In [6, Proposition 7], Balcerzak, Bartoszewicz and Filipczak established a nice algebrability result by using *exponential-like functions* $\varphi : \mathbb{R} \to \mathbb{R}$, of the form

$$\varphi(x) = \sum_{j=1}^{m} a_j e^{b_j x}$$

for some $m \in \mathbb{N}$, $a_1, \ldots, a_m \in \mathbb{R} \setminus \{0\}$ and distinct $b_1, \ldots, b_m \in \mathbb{R} \setminus \{0\}$. We denote by $\mathcal{E}$ the class of exponential-like functions. The following lemma (see [11] or [1, Chapter 7]) is a slight variant of the above mentioned result of [6].

LEMMA 4.2. *Let Ω be a nonempty set and $\mathcal{F}$ be a family of functions $\Omega \to \mathbb{R}$. Assume that there exists a function $f \in \mathcal{F}$ such that $f(\Omega)$ is uncountable and $\varphi \circ f \in \mathcal{F}$ for every $\varphi \in \mathcal{E}$. Then $\mathcal{F}$ is strongly $\mathfrak{c}$-algebrable. More precisely, if $H \subset (0, \infty)$ is a set with $\operatorname{card}(H) = \mathfrak{c}$ and linearly independent over the field $\mathbb{Q}$, then*

$$\{\exp \circ (rf) : r \in H\}$$

is a free system of generators of an algebra contained in $\mathcal{F} \cup \{0\}$.

Lemma 4.3 below is an adaptation of a result that is implicitly contained in [7, Section 6]. We sketch the proof for the sake of completeness.

LEMMA 4.3. *Let $\mathcal{F}$ be a family of functions in $\mathcal{C}(\mathbb{R})$. Assume that there exists a strictly monotone function $f \in \mathcal{F}$ such that $\varphi \circ f \in \mathcal{F}$ for every exponential-like function φ. Then $\mathcal{F}$ is densely strongly $\mathfrak{c}$-algebrable in $\mathcal{C}(\mathbb{R})$.*

Proof. If $\Omega = \mathbb{R}$ then $f(\Omega)$ is a nondegenerate interval, so it is an uncountable set. Thus, it is sufficient to show that the algebra $\mathcal{A}$ generated by the system $\{\exp \circ (rf) : r \in H\}$ given in Lemma 4.2 is dense. For this, we invoke Lemma 4.1. Take any $\alpha \in H \subset (0, \infty)$. Given $x_0 \in \mathbb{R}$, the function $F(x) := e^{\alpha f(x)}$ belongs to $\mathcal{A}$ and satisfies $F(x_0) \neq 0$. Moreover, for any distinct $x_0, x_1 \in \mathbb{R}$, the same function F fulfills $F(x_0) \neq F(x_1)$, because both f and $x \mapsto e^{\alpha x}$ are one-to-one. In conclusion, $\mathcal{A}$ is dense in $\mathcal{C}(\mathbb{R})$. ∎

Now we state and prove the main result of this section.

THEOREM 4.4. *The set of functions in $\mathcal{P}$ that are nonconstant on any nondegenerate interval of $\mathbb{R}$ is densely strongly $\mathfrak{c}$-algebrable in $\mathcal{C}(\mathbb{R})$.*

Proof. From [33, Example 3.11] (see also [32, Example 13.3]) we know that there exists a differentiable *strictly increasing* real-valued function $f : (a, b) \to (0, 1)$ (with $f((a, b)) = (0, 1)$) whose derivative vanishes on a dense set and yet does not vanish everywhere. By composition with the function $x \mapsto \frac{b-a}{\pi} \arctan x + \frac{a+b}{2}$, we get a strictly monotone function $f : \mathbb{R} \to \mathbb{R}$ such that

$$D := \{x \in \mathbb{R} : f'(x) = 0\}$$

is dense in $\mathbb{R}$ but $D \neq \mathbb{R}$. Observe that, in particular, f is a Pompeiu function that is nonconstant on any interval.

According to Lemma 4.3, our only task is to prove that, given a prescribed function $\varphi \in \mathcal{E}$, the function $\varphi \circ f$ belongs to $\mathcal{F}$, where

$$\mathcal{F} := \{f \in \mathcal{P} : f \text{ is nonconstant on any interval of } \mathbb{R}\}.$$

By the chain rule, $\varphi \circ f$ is a differentiable function and $(\varphi \circ f)'(x) = \varphi'(f(x))f'(x)$ ($x \in \mathbb{R}$). Hence $(\varphi \circ f)'$ vanishes at least on D, so this derivative vanishes on a dense set. It remains to prove that $\varphi \circ f$ is nonconstant on any open interval of $\mathbb{R}$.

In order to see this, fix such an interval J. Clearly, φ' also belongs to $\mathcal{E}$. Now, φ' is a nonzero entire function, so the set

$$S := \{x \in \mathbb{R} : \varphi'(x) = 0\}$$

is discrete in $\mathbb{R}$. In particular, it is closed in $\mathbb{R}$ and countable, so $\mathbb{R} \setminus S$ is open and dense in $\mathbb{R}$. Of course, $S \cap (0, 1)$ is discrete in $(0, 1)$. Since $f : \mathbb{R} \to (0, 1)$ is a homeomorphism, the set $f^{-1}(S)$ is discrete in $\mathbb{R}$. Hence $J \setminus f^{-1}(S)$ is a nonempty open set of J. On the other hand, since D is dense in $\mathbb{R}$, it follows that the set D^0 of interior points of D is empty. Indeed, if there existed an interval $(c, d) \subset D$, then $f' = 0$ on (c, d), so f would be constant on (c, d), which is not possible because f is strictly increasing. Therefore $\mathbb{R} \setminus D$ is dense in $\mathbb{R}$, from which one derives that $J \setminus D$ is dense in J. Thus $(J \setminus f^{-1}(S)) \cap (J \setminus D) \neq \emptyset$. Pick any point x_0 in the last set. This means that $x_0 \in J$, $f(x_0) \notin S$ (so $\varphi'(f(x_0)) \neq 0$) and $x_0 \notin D$ (so $f'(x_0) \neq 0$). Thus

$$(\varphi \circ f)'(x_0) = \varphi'(f(x_0))f'(x_0) \neq 0,$$

which implies that $\varphi \circ f$ is nonconstant on J, as required. ∎

REMARKS 4.5. 1. In view of the last theorem one might believe that the phrase "$f' = 0$ on a dense set" (see the definition of $\mathcal{P}$) could be replaced by the stronger one "$f' = 0$ almost everywhere". But this is not possible because every differentiable function is an N-function, that is, it sends sets of null measure into sets of null measure (see [32, Theorem 21.9]), and every continuous N-function on an interval whose derivative vanishes almost everywhere must be a constant (see [32, Theorem 21.10]).

2. If a real function f is a derivative then f^2 may not be a derivative (see [32, p. 86]). This leads us to conjecture that the set $\mathcal{DP}$ of Pompeiu derivatives (and of course, any subset of it) is not algebrable.

3. Nevertheless, from Theorem 3.6 (and also from Theorem 4.1) of [20] it follows that the family $\mathcal{BDP}$ of bounded Pompeiu derivatives is $\mathfrak{c}$-lineable. A quicker way to see this is by invoking the fact that $\mathcal{BDP}$ is a vector space that becomes a Banach space under the supremum norm [14, pp. 33–34]. Since it is not finite-dimensional, a simple application of Baire's category theorem yields $\dim(\mathcal{BDP}) = \mathfrak{c}$. Now, on the one hand, trivially, $\mathcal{BDP}$ is dense-lineable in itself. On the other hand, the set of derivatives that are positive on a dense set and negative on another is a dense G_δ set in $\mathcal{BDP}$ [14, p. 34]. Therefore, as already indicated in [20], it would be interesting to see whether this set is also dense-lineable.

5. Discontinuous functions.

Let $n \geq 2$ and consider the function $f : \mathbb{R}^n \to \mathbb{R}$ given by

$$f(x_1, \ldots, x_n) = \begin{cases} \dfrac{x_1 \cdots x_n}{x_1^{2n} + \cdots + x_n^{2n}} & \text{if } x_1^2 + \cdots + x_n^2 \neq 0, \\ 0 & \text{if } x_1 = \cdots = x_n = 0. \end{cases} \tag{5.1}$$

Observe that f is discontinuous at the origin since arbitrarily near to $0 \in \mathbb{R}^n$ there exist points of the form $x_1 = \cdots = x_n = t$ at which f has the value $1/(nt^n)$. On the other hand, for fixed $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathbb{R}^{n-1}$, the function $\psi : \mathbb{R} \to \mathbb{R}$ given by $\psi : x_i \mapsto f(x_1, \ldots, x_n)$ is everywhere continuous. Indeed, this is trivial if all x_j's ($j \neq i$) are not 0, while $\psi \equiv 0$ if some x_j is zero. Of course, f is continuous at any point of $\mathbb{R}^n \setminus \{0\}$.

Given $x_0 \in \mathbb{R}^n$, we denote by $\mathcal{SC}(\mathbb{R}^n, x_0)$ the vector space of all *separately continuous* functions $\mathbb{R}^n \to \mathbb{R}$ that are *continuous on* $\mathbb{R}^n \setminus \{x_0\}$. Since $\mathrm{card}(\mathcal{C}(\mathbb{R}^n \setminus \{x_0\})) = \mathfrak{c}$, it is easy to see that the cardinality (so the dimension) of $\mathcal{SC}(\mathbb{R}^n, x_0)$ equals $\mathfrak{c}$. Theorem 5.1 below will show the algebrability of the family

$$\mathcal{DSC}(\mathbb{R}^n, x_0) := \{f \in \mathcal{SC}(\mathbb{R}^n, x_0) : f \text{ is discontinuous at } x_0\}$$

in a maximal sense.

THEOREM 5.1. *Let $n \in \mathbb{N}$ with $n \geq 2$, and let $x_0 \in \mathbb{R}^n$. Then the set $\mathcal{DSC}(\mathbb{R}^n, x_0)$ is strongly $\mathfrak{c}$-algebrable.*

Proof. We can suppose without loss of generality that $x_0 = 0 = (0, \dots, 0)$. Let $f \in \mathcal{DSC}(\mathbb{R}^n, 0)$ be given by (5.1). For each $c > 0$, we set

$$\varphi_c(x) := e^{|x|^c} - e^{-|x|^c}.$$

It is easy to see that these functions generate a free algebra. Indeed, if $P(t_1, \dots, t_p)$ is a nonzero polynomial in p variables with $P(0, \dots, 0) = 0$ and $c_1, \dots, c_p$ are distinct positive real numbers, let $M := \{j \in \{1, \dots, p\}$: the variable t_j appears explicitly in the expression of $P\}$, and $c_0 := \max\{c_j : j \in M\}$. Then $P(\varphi_{c_1}, \dots, \varphi_{c_p})$ has the form $De^{m|x|^{c_0} + g(x)} + h(x)$, where $D \in \mathbb{R} \setminus \{0\}$, $m \in \mathbb{N}$, g is a finite sum $\sum_k m_k |x|^{\alpha_k}$ with m_k integers and $\alpha_k < c_0$, and h is a finite linear combination of functions of the form $e^{q(x)}$ where, in turn, each $q(x)$ is a finite sum $\sum_k n_k |x|^{\gamma_k}$, with each γ_k satisfying either $\gamma_k < c_0$, or $\gamma_k = c_0$ and $n_k < 0$. Then

$$\lim_{x \to \infty} |P(\varphi_{c_1}(x), \dots, \varphi_{c_p}(x))| = \infty, \tag{5.2}$$

in particular $P(\varphi_{c_1}, \dots, \varphi_{c_p})$ is not identically zero. This shows that the algebra Λ generated by the φ_c's is free.

Now, define

$$\mathcal{A} = \{\varphi \circ f : \varphi \in \Lambda\}.$$

Plainly, $\mathcal{A}$ is an algebra of functions $\mathbb{R}^n \to \mathbb{R}$ continuous on $\mathbb{R}^n \setminus \{0\}$. In addition, this algebra is freely generated by the functions $\varphi_c \circ f$ $(c > 0)$. To see this, assume that

$$\Phi = P(\varphi_{c_1} \circ f, \dots, \varphi_{c_p} \circ f) \in \mathcal{A},$$

where $P, c_1, \dots, c_p$ are as above. Suppose that $\Phi = 0$. Evidently, the function f is onto (note that, for example, $f(x, x, \dots, x) = 1/(nx^n)$, $f(-x, x, x, \dots, x) = -1/(nx^n)$ and $f(0, \dots, 0) = 0$). Therefore $P(\varphi_{c_1}(x), \dots, \varphi_{c_p}(x)) = 0$ for all $x \in \mathbb{R}$, so $P \equiv 0$, which is absurd because $P(\varphi_{c_1}, \dots, \varphi_{c_p})$ becomes large as $x \to \infty$.

Hence our only task is to prove that every function $\Phi \in \mathcal{A} \setminus \{0\}$ as in the last paragraph belongs to $\mathcal{DSC}(\mathbb{R}^n, 0)$. Firstly, the continuity of each φ_c implies that $\Phi \in \mathcal{SC}(\mathbb{R}^n, 0)$. Finally, Φ is discontinuous at the origin: indeed, for all $x \neq 0$,

$$|\Phi(x, \dots, x)| = \left| P\left(\varphi_{c_1}\left(\frac{1}{nx^n}\right), \dots, \varphi_{c_p}\left(\frac{1}{nx^n}\right)\right)\right| \to \infty$$

as $x \to 0$, due to (5.2). ∎

6. Series for which the ratio test or the root test fail. Every real sequence (a_n) generates a real series $\sum_n a_n$. In order to make the notation of this section consistent, we adopt the convention $a/0 := \infty$ for every $a \in (0, \infty)$, and $0/0 := 0$. And a series $\sum_n a_n$ will be called divergent whenever it does not converge. As is commonly known, given a series $\sum_n a_n$, a refinement of the classical *ratio test* states that

(i) if $\limsup_{n\to\infty} |a_{n+1}|/|a_n| < 1$ then $\sum_n a_n$ converges, and
(ii) if $\liminf_{n\to\infty} |a_{n+1}|/|a_n| > 1$ then $\sum_n a_n$ diverges.

However, we can have convergent (positive) series with $\limsup_{n\to\infty} a_{n+1}/a_n > 1$ and $\liminf_{n\to\infty} a_{n+1}/a_n < 1$ simultaneously. For instance, consider the series

$$\sum_n 2^{-n+(-1)^n} = \frac{1}{2^2} + \frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^3} + \cdots = 1.$$

Setting $a_n = 2^{-n+(-1)^n}$, we have

$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = 2 \quad \text{and} \quad \liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{8}.$$

Now, the series $\sum_n b_n = \sum_n 2^{n+(-1)^n}$ diverges with the same corresponding limsup and liminf.

Analogously, a refinement of the classical *root test* asserts that

(i) if $\limsup_{n\to\infty} |a_n|^{1/n} < 1$ then $\sum_n a_n$ converges, and
(ii) if $\limsup_{n\to\infty} |a_n|^{1/n} > 1$ then $\sum_n a_n$ diverges.

But none of these conditions is necessary because, for instance, the positive series $\sum_n 1/n^2$ converges, the series $\sum_n n$ diverges but $\limsup_{n\to\infty} a_n^{1/n} = 1$ for both of them.

Our goal in this section is to show that the set of convergent series for which the ratio test or the root test fails—that is, the refinements of both tests provide no information whatsoever—is lineable in a rather strong sense (see Theorem 6.2 below). The same result will be shown to hold for divergent series.

In order to put these properties into an appropriate context, we are going to consider the space $\omega := \mathbb{R}^{\mathbb{N}}$ of all real sequences and its subset ℓ_1, the space of all absolutely summable real sequences. Recall that ω becomes a Fréchet space under the product topology, while ℓ_1 becomes a Banach space (so a Fréchet space as well) if it is endowed with the 1-norm $\|(a_n)\|_1 := \sum_{n\geq 1} |x_n|$. Moreover, the set $c_{00} := \{(a_n) \in \omega : \exists n_0 = n_0((a_n)) \in \mathbb{N}$ such that $a_n = 0$ $\forall n > n_0\}$ is a dense vector subspace of both ω and ℓ_1. A standard application of Baire's category theorem together with the separability of these spaces shows that their dimension equals $\mathfrak{c}$.

We need an auxiliary, general result about lineability. Let X be a vector space and A, B be two subsets of X. According to [2], we say that A *is stronger than* B whenever $A + B \subset A$. The following assertion—of which many variants have been proved—can be found in [1, 13] and the references therein.

LEMMA 6.1. *Assume that X is a metrizable topological vector space. Let $A \subset X$ be maximal lineable. Suppose that there exists a dense-lineable subset $B \subset X$ such that A is stronger than B and $A \cap B = \emptyset$. Then A is maximal dense-lineable in X.*

THEOREM 6.2. *The following four sets are maximal* ($\mathfrak{c}$-)*dense-lineable in ℓ_1, ℓ_1, ω and ω, respectively:*

(a) *The set of sequences in ℓ_1 that generate series for which the ratio test fails.*
(b) *The set of sequences in ℓ_1 that generate series for which the root test fails.*
(c) *The set of sequences in ω that generate divergent series for which the ratio test fails.*
(d) *The set of sequences in ω that generate divergent series for which the root test fails.*

Proof. We shall only show the first item, even in a very strong form. Namely, our aim is to prove that the set

$$\mathcal{A} := \left\{ (a_n) \in \ell_1 : \limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \infty \quad \text{and} \quad \liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 0 \right\}$$

is maximal dense-lineable. The remaining items can be handled in a similar manner and are left to the reader: as a hint, suffice it to say that, instead of the collection of sequences $\{(ns)^{-n+(-1)^n}\}$ used for (a), one may use $\{n^{-s}\}$, $\{(ns)^{n+(-1)^n}\}$ and $\{n^s\}$, respectively, to prove (b), (c) and (d).

Let us prove (a). Consider, for every real $s > 1$, the positive sequence $a_{n,s} = (ns)^{-n+(-1)^n}$ for $n \in \mathbb{N}$. Since $a_{n,s} \le n^{-2}$ for all $n \ge 3$, the comparison test yields $(a_{n,s}) \in \ell_1$. Next, take

$$E = \operatorname{span}\{(a_{n,s})_{n\ge 1} : s > 1\},$$

which is a vector subspace of ℓ_1. It can be easily seen that $\dim(E) = \mathfrak{c}$. Indeed, suppose that a linear combination of the type

$$x = (x_n) \quad \text{with} \quad x_n = \sum_{j=1}^{k} \alpha_j a_{n,s_j} \quad (n \ge 1) \tag{6.1}$$

is identically 0. Then, supposing without loss of generality that $k \ge 2$ and $s_1 > \cdots > s_k$, and dividing the previous expression by $(ns_k)^{-n+(-1)^n}$ we obtain

$$0=\alpha_1\left(\frac{s_1}{s_k}\right)^{-n+(-1)^n}+\cdots+\alpha_{k-1}\left(\frac{s_{k-1}}{s_k}\right)^{-n+(-1)^n}+\alpha_k.$$

Letting $n\to\infty$, we obtain $\alpha_k=0$. Inductively we can show that all α_j's are 0, proving that the set of sequences $\{a_{n,s}, s>1\}$ is linearly independent, thus $\dim(E)=\mathfrak{c}$.

Next, let us show that, given any sequence $x=(x_n)_{n\geq 1}\in E\setminus\{0\}$ as in (6.1) (with $\alpha_k\neq 0$ and $s_1>\cdots>s_k$), the ratio test does not provide any information on the convergence of $\sum_{n\geq 1}x_n$. Dividing numerators and denominators by $\alpha_k(ns_k)^{-n+(-1)^n}$, we get

$$\begin{aligned}\left|\frac{x_{n+1}}{x_n}\right|&=\left|\frac{\alpha_1((n+1)s_1)^{-n-1+(-1)^{n+1}}+\cdots+\alpha_k((n+1)s_k)^{-n-1+(-1)^{n+1}}}{\alpha_1(ns_1)^{-n+(-1)^n}+\cdots+\alpha_k(ns_k)^{-n+(-1)^n}}\right|\\&=\left|\frac{\beta_{1,n}+\cdots+\beta_{k,n}}{\gamma_n+1}\right|,\end{aligned}$$

where $\gamma_n\to 0$ $(n\to\infty)$ and

$$\beta_{j,n}=\begin{cases}\dfrac{\alpha_j}{\alpha_k}\left(\dfrac{n}{n+1}\right)^n\dfrac{s_j^{-2}s_k^{-1}}{n(n+1)}\left(\dfrac{s_k}{s_j}\right)^n & \text{if } n \text{ is even},\\ \dfrac{\alpha_j}{\alpha_k}\left(\dfrac{n}{n+1}\right)^n ns_k\left(\dfrac{s_k}{s_j}\right)^n & \text{if } n \text{ is odd},\end{cases}$$

for $j=1,\ldots,k$. Note that

$$\lim_{\substack{n\to\infty\\ n \text{ even}}}\beta_{j,n}=0 \quad \text{for } j\in\{1,\ldots,k\},$$

$$\lim_{\substack{n\to\infty\\ n \text{ odd}}}\beta_{j,n}=0 \quad \text{for } j\in\{1,\ldots,k-1\},\qquad \lim_{\substack{n\to\infty\\ n \text{ odd}}}|\beta_{k,n}|=\infty.$$

Then

$$\liminf_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right|=0 \quad\text{and}\quad \limsup_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right|=\infty.$$

Consequently, x belongs to $\mathcal{A}$, as we wished. This shows that $\mathcal{A}$ is maximal lineable in ℓ_1.

Finally, an application of Lemma 6.1 with $X=\ell_1$, $A=\mathcal{A}$ and $B=c_{00}$ proves the maximal dense-lineability of $\mathcal{A}$. ∎

Concerning parts (c) and (d) of the last theorem, one might believe that they hold because the root and ratio tests are rather non-sharp. To be more precise, given a divergent series $\sum_n c_n$ with positive terms (notice that we may have $c_n\to 0$, for instance with $c_n=1/n$), one might believe that there are not many sequences (x_n) essentially lower than (c_n) such that $\sum_n x_n$ still diverges. The following theorem will show that this is far from being true. In order to formulate it properly, a piece of notation is again needed. For a given sequence $(c_n) \subset (0, \infty)$, we denote by $c_0((c_n))$ the vector space of all sequences $(x_n) \in \omega$ satisfying $\lim_{n\to\infty} x_n/c_n = 0$. It is a standard exercise to prove that, when endowed with the norm

$$\|(x_n)\| = \sup_{n\geq 1} |x_n/c_n|,$$

the set $c_0((c_n))$ becomes a separable Banach space such that c_{00} is a dense subspace of it.

THEOREM 6.3. *Assume that (c_n) is a sequence of positive real numbers such that the series $\sum_n c_n$ diverges. Then the family of sequences $(x_n) \in c_0((c_n))$ such that the series $\sum_n x_n$ diverges is maximal ($\mathfrak{c}$-)dense-lineable in $c_0((c_n))$.*

Proof. By Baire's theorem, $\dim(c_0((c_n))) = \mathfrak{c}$. We denote

$$A := \Big\{(x_n) \in c_0((c_n)) : \text{the series} \sum_n x_n \text{ diverges}\Big\}.$$

Obviously, $A + c_{00} \subset A$ and $A \cap c_{00} = \emptyset$. Let us apply Lemma 6.1 with

$$X = c_0((c_n)) \quad \text{and} \quad B = c_{00}.$$

Then it is enough to show that A is maximal lineable, that is, $\mathfrak{c}$-lineable.

To this end, we use the divergence of $\sum_n c_n$ and the fact that $c_n > 0$ $(n \geq 1)$. Letting $n_0 := 1$, we can obtain inductively a sequence $\{n_1 < \cdots < n_k < \cdots\} \subset \mathbb{N}$ satisfying

$$c_{n_{k-1}+1} + \cdots + c_{n_k} > k \quad (k = 1, 2, \dots).$$

Now, define the collection of sequences $\{(d_{n,t})_n : 0 < t < 1\}$ by

$$d_{j,t} := c_j/k^t \quad (j = n_{k-1} + 1, \dots, n_k; k \in \mathbb{N}).$$

Since $k^t \to \infty$ as $k \to \infty$, each sequence $(d_{n,t})_n$ belongs to $c_0((c_n))$. We set

$$M := \text{span}\{(d_{n,t})_n : 0 < t < 1\}.$$

This vector space is $\mathfrak{c}$-dimensional. Indeed, if this were not the case, then there would exist $s \in \mathbb{N}$, $\lambda_1, \dots, \lambda_s \in \mathbb{R}$ with $\lambda_1 \neq 0$ and $0 < t_1 < \cdots < t_s < 1$ such that the sequence

$$\Phi = (x_n)_n = (\lambda_1 d_{n,t_1} + \cdots + \lambda_s d_{n,t_2})_n \tag{6.2}$$

is identically zero. From the triangle inequality, for each $k \in \mathbb{N}$,

$$\begin{aligned}\Big|\sum_{j=n_{k-1}+1}^{n_k} x_j\Big| &= \Big|\sum_{j=n_{k-1}+1}^{n_k} \sum_{\nu=1}^{s} \lambda_\nu d_{j,t_\nu}\Big| \\ &\geq \Big||\lambda_1| \sum_{j=n_{k-1}+1}^{n_k} d_{j,t_1} - \sum_{\nu=2}^{s} |\lambda_\nu| \sum_{j=n_{k-1}+1}^{n_k} d_{j,t_\nu}\Big|\end{aligned}$$

$$= \Big(\sum_{j=n_{k-1}+1}^{n_k} c_j \Big) \bigg| \frac{|\lambda_1|}{k^{t_1}} - \sum_{\nu=2}^{s} \frac{|\lambda_\nu|}{k^{t_\nu}} \bigg|$$

$$> \bigg| |\lambda_1| k^{1-t_1} - \sum_{\nu=2}^{s} |\lambda_\nu| k^{1-t_\nu} \bigg| \to \infty \quad \text{as } k \to \infty,$$

which is absurd. Hence, the sequences $(d_{n,t})$ $(0 < t < 1)$ are linearly independent and $\dim(M) = \mathfrak{c}$.

Finally, we prove that each $\Phi = (x_n) \in M \setminus \{0\}$ belongs to A. Note that Φ has the shape given in (6.2) with $\lambda_1 \neq 0$ and $0 < t_1 < \cdots < t_s < 1$. But the fact $|\sum_{j=n_{k-1}+1}^{n_k} x_j| \to \infty$ $(k \to \infty)$ as shown above entails that the Cauchy convergence criterion for series does not hold for $\sum_n x_n$. Consequently, this series diverges, as required. ∎

Remark 6.4. The property given in Theorem 6.3 is topologically generic too, that is, the set A above is *residual* in $c_0((c_n))$. Indeed, we find that $A \supset \bigcap_{M \in \mathbb{N}} \bigcap_{N \in \mathbb{N}} \bigcup_{m > n > N} A_{M,N}$, where

$$A_{M,N} := \big\{ (x_n) \in c_0((c_n)) : |x_n + x_{n+1} + \cdots + x_m| > M \big\},$$

and each set $B_{M,N} := \bigcup_{m > n > N} A_{M,N}$ is open and dense in $c_0((c_n))$. To prove this, fix $M, N, m, n \in \mathbb{N}$ with $m > n > N$ and observe that $A_{M,N} = c_0((c_n)) \cap \Psi^{-1}((-\infty, -M) \cup (M, \infty))$, where $\Psi : \omega \to \mathbb{R}$ is given by $\Psi((x_j)) = x_m + \cdots + x_n$. The continuity of the projections $(x_j) \in \omega \mapsto x_k \in \mathbb{R}$ $(k \in \mathbb{N})$ entails the continuity of Ψ, so $\Psi^{-1}((-\infty, -M) \cup (M, \infty))$ is open in ω. The inclusion $c_0((c_n)) \hookrightarrow \omega$ being continuous, we deduce that $A_{M,N}$ is open in $c_0((c_n))$. Therefore $B_{M,N}$ is also open in $c_0((c_n))$. As for the density of $B_{M,N}$, note that, due to the density of c_{00} in $c_0((c_n))$, it is enough to show that, given $\Phi = (y_j) = (y_1, \ldots, y_s, 0, 0, \ldots) \in c_{00}$ and $\varepsilon > 0$, there exist $m > n > N$ and $(x_j) \in c_0((c_n))$ with $|x_m + \cdots + x_n| > M$ and $\|(x_j) - \Phi\| < \varepsilon$. Since the positive series $\sum_j c_j$ diverges, there exist $m, n \in \mathbb{N}$ such that $m > n > \max\{s, N\}$ and $c_n + \cdots + c_m > 2M/\varepsilon$. Define $(x_j) \in c_{00} \subset c_0((c_n))$ by

$$x_j = \begin{cases} y_j & \text{if } j \in \{1, \ldots, s\}, \\ \varepsilon c_j / 2 & \text{if } j \in \{m, \ldots, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

By construction, $|x_m + \cdots + x_n| > M$. Finally, $\|(x_j) - \Phi\| = \sup_{j \geq 1} |x_j - y_j| / c_j = \sup_{n \leq j \leq m} |x_j| / c_j = \varepsilon/2 < \varepsilon$, as required.

7. Convergence in measure versus convergence almost everywhere. Let m be the Lebesgue measure on $\mathbb{R}$. In this section we will restrict ourselves to the interval $[0,1]$, which of course has finite measure $m([0,1]) = 1$. Denote by L_0 the vector space of all Lebesgue measurable functions $[0,1] \to \mathbb{R}$, where two functions are identified whenever they are equal almost everywhere (a.e.) in $[0,1]$. Two natural kinds of convergence of functions of L_0 are a.e.-convergence and convergence in measure. Recall that a sequence (f_n) of measurable functions is said to *converge in measure* to a measurable function $f : [0,1] \to \mathbb{R}$ provided that

$$\lim_{n\to\infty} m(\{x \in [0,1] : |f_n(x) - f(x)| > \alpha\}) = 0 \quad \text{for all } \alpha > 0.$$

Convergence in measure is specially pleasant because it can be described by a natural metric on L_0 (see e.g. [28]). Namely, the distance

$$\rho(f,g) = \int_{[0,1]} \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}\,dx \quad (f,g \in L_0)$$

has the property that

$$f_n \xrightarrow[n\to\infty]{\rho} f \quad \text{if and only if} \quad f_n \xrightarrow[n\to\infty]{} f \text{ in measure}$$

(the finiteness of the measure of $[0,1]$ is crucial). Under the topology generated by ρ, the space L_0 becomes a complete metrizable topological vector space for which the set $\mathcal{S}$ of simple (i.e. of finite image) measurable functions forms a dense vector subspace. Actually, L_0 is separable because the set S_0 $(\subset S)$ of finite linear combinations with rational coefficients of functions of the form $\chi_{[p,q]}$ $(0 \le p < q \le 1$ rational numbers) is countable and dense in L_0. Here χ_A denotes the indicator function of the set A.

Convergence in measure of a sequence (f_n) to f implies a.e.-convergence to f of some subsequence (f_{n_k}) (see [28, Theorem 21.9]). But, generally, this convergence cannot be obtained for the whole sequence (f_n). For instance, the so-called "typewriter sequence" given by

$$T_n = \chi_{[j2^{-k},(j+1)2^{-k}]}$$

(where, for each n, the nonnegative integers j and k are uniquely determined by $n = 2^k + j$ and $0 \le j < 2^k$) satisfies $T_n \to f \equiv 0$ in measure but, for every $x_0 \in [0,1]$, the sequence $\{T_n(x_0)\}_{n\ge1}$ does not converge.

In order to discuss the lineability of this phenomenon, we need, once more, to put the problem in an adequate framework. Let $L_0^{\mathbb{N}}$ be the space of all sequences of measurable functions $[0,1] \to \mathbb{R}$, endowed with the product topology. Since L_0 is metrizable and separable, the space $L_0^{\mathbb{N}}$ is also a complete metrizable separable topological vector space. Again, by Baire's theorem, this implies $\dim(L_0^{\mathbb{N}}) = \mathfrak{c}$. Moreover, the set

$$\text{(7.1)} \quad \big\{\Phi = (f_n) \in L_0^{\mathbb{N}} : \exists N = N(\Phi) \in \mathbb{N} \text{ such that } f_n = 0 \text{ for all } n \ge N\big\}$$

is dense in the product space. Now, we are ready to state our next theorem, with which we finish this paper.

Theorem 7.1. *The family of Lebesgue classes of sequences* $(f_n) \in L_0^{\mathbb{N}}$ *such that* $f_n \to 0$ *in measure but* (f_n) *does not converge almost everywhere in* $[0,1]$ *is maximal* $(\mathfrak{c}$-$)$*dense-lineable in* $L_0^{\mathbb{N}}$.

Proof. Let (T_n) be the typewriter sequence defined above, and let A be the family described in the statement of the theorem, so that $(T_n) \in A$. Extend each T_n to the whole $\mathbb{R}$ by defining $T_n(x) = 0$ for all $x \notin [0,1]$. It is readily seen that, for each $t \in (0,1/2)$, the translated-dilated sequence $T_{n,t}(x) := T_n(2(x-t))$ $(n \geq 1)$ also tends to 0 in measure. Consider the vector space

$$M := \operatorname{span}\{(T_{n,t}) : t > 0\}.$$

The sequences $(T_{n,t})$ $(0 < t < 1/2)$ are linearly independent. Indeed, if this were not the case, there would be $0 < t_1 < \cdots < t_s < 1/2$ as well as real numbers $c_1, \ldots, c_s$ with $c_s \neq 0$ such that $c_1 T_{n,t_1} + \cdots + c_s T_{n,t_s} = 0$ for all $n \in \mathbb{N}$. In particular, $c_1 T_{1,t_1}(x) + \cdots + c_s T_{1,t_s}(x) = 0$ for almost all $x \in \mathbb{R}$. But $T_1 = \chi_{[0,1]}$, so $T_{1,t} = \chi_{[t,t+1/2]}$ for all $t > 0$. Therefore

$$c_1 \chi_{[t_1,t_1+1/2]}(x) + \cdots + c_s \chi_{[t_s,t_s+1/2]}(x) = 0 \quad \text{for almost all } x \in [0,1].$$

But, for every $x \in (\max\{t_{s-1} + 1/2, t_s\}, t_s + 1/2]$, the left-hand side of the last expression equals $0 + \cdots + 0 + c_s \cdot 1 = c_s \neq 0$, which is absurd. Thus $\dim(M) = \mathfrak{c}$. Moreover, since L_0 is a topological vector space carrying the topology of convergence in measure, every member of $(F_n) := (c_1 T_{n,t_1} + \cdots + c_s T_{n,t_s}) \in M$ is a sequence tending to 0 in measure.

Next, fix any $(F_n) \in M$ as above, with $0 < t_1 < \cdots < t_s$ and $c_s \neq 0$. For all $x \in (\max\{t_s, t_{s-1} + 1/2\}, t_s + 1/2]$, we have

$$F_n(x) = \sum_{j=1}^{s} c_j T_{n,t_j}(x) = \sum_{j=1}^{s} c_j T_n(2(x - t_j)) = c_s T_n(2(x - t_s)).$$

Since $c_s \neq 0$ and $(T_n(y))$ converges for no $y \in (\max\{0, 2(t_{s-1} - t_s) + 1\}, 1]$ $(\subset [0,1])$, we derive that, for each $x \in (\max\{t_s, t_{s-1} + 1/2\}, t_s + 1/2]$, the sequence $(F_n(x))$ does not converge. This shows that $M \setminus \{0\} \subset A$. Thus, A is $\mathfrak{c}$-lineable. Finally, an application of Lemma 6.1 with

$$X = L_0^{\mathbb{N}} \quad \text{and} \quad B = \text{the set given by (7.1)}$$

completes the proof. ∎

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