COLLOQUIUM MATHEMATICUM

VOL. 148 2017 NO. 2

HOMOGENEOUS ROTA–BAXTER OPERATORS ON THE 3-LIE ALGEBRA A_ω

BY

RUIPU BAI and YINGHUA ZHANG (Baoding)

Abstract. We study homogeneous Rota–Baxter operators with weight zero on an infinite-dimensional simple 3-Lie algebra A_ω over a field F ($\mathrm{ch}\, F = 0$), which is constructed from an associative commutative algebra A with a derivation Δ and an involution ω (Lemma 2.4). A homogeneous Rota–Baxter operator on A_ω is a linear map R of A_ω satisfying $R(L_m) = f(m)L_m$ for all generators of A_ω, where $f : \mathbb{Z} \to F$ is a function. We prove that R is a homogeneous Rota–Baxter operator on A_ω if and only if R is one of the five possibilities R_{0_1}, R_{0_2}, R_{0_3}, R_{0_4} and R_{0_5}, described in Theorems 3.2, 3.12, 3.15, 3.19 and 3.21. Using the operators R_{0_i}, we construct new 3-Lie algebras $(A, [\,,\,,\,]_i)$ for $1 \le i \le 5$, such that R_{0_i} is a homogeneous Rota–Baxter operator on the 3-Lie algebra $(A, [\,,\,,\,]_i)$.

1. Introduction. Rota–Baxter operators were originally defined on associative algebras by G. Baxter to solve an analytic problem in probability [Bt], and popularized by the work of Cartier and Rota [Ca, Ro1, Ro2]. They have been closely related to many fields in mathematics and mathematical physics. Rota–Baxter algebras have played an important role in the Hopf algebra approach to renormalization of perturbative quantum field theory of Connes and Kreimer [CK, EGK, EGM], as well as in the application of the renormalization method in solving divergence problems in number theory [GZ, MP].

Rota–Baxter operators on a Lie algebra are an operator form of the classical Yang–Baxter equations, and contribute to the study of integrable systems [Bai, BGN1, BGN2]. Semenov-Tian-Shansky's fundamental work [STS] shows that a Rota–Baxter operator of weight 0 on a Lie algebra is exactly the operator form of the classical Yang–Baxter equation (CYBE), which was regarded as a classical limit of the quantum Yang–Baxter equation [Bv]. The latter is an important topic in many fields such as symplectic geometry, integrable systems, quantum groups and quantum field theory [A, BBGN, EGK, G1, G2, GK1, GK2, LHB, GZ, Ro1, Ro2].

2010 *Mathematics Subject Classification*: Primary 17B05; Secondary 17D99.
Key words and phrases: 3-Lie algebra, homogeneous Rota–Baxter operator, Rota–Baxter 3-algebra.
Received 2 December 2015; revised 12 February 2016.
Published online 3 March 2017.

DOI: 10.4064/cm6829-2-2016

Rota–Baxter n-algebras and differential n-algebras were first introduced in [BGL]. They are the generalization of Rota–Baxter algebras to multiple algebraic systems. We know that n-Lie algebras [F] are a type of multiple algebraic systems, appearing in many fields of mathematics and mathematical physics [N, T, BL, HHM, HIM, HCK, G, P]. Especially, 3-Lie algebras and metric 3-Lie algebras are applied to the study of supersymmetry and gauge symmetry transformations of the world-volume theory of multiple coincident $M2$-branes; the Bagger–Lambert theory has a novel local gauge symmetry which is based on a metric 3-Lie algebra. The n-Jacobi identity in n-Lie algebras can be regarded as a generalized Plücker relation in the physics literature. The theory of n-Lie algebras has been widely studied [K, L, BSZ1, BSZ2, BBW, BHB, AI]. Recently, some researchers are interested in the structure of n-Lie algebras constructed from well-known algebras, for example, from Lie algebras, associative algebras, commutative associative algebras, cubic matrices, etc. [BBW, BW, BLZ, Dzhu, Poz1, Poz2].

The authors of [BGL] investigate Rota–Baxter operators on n-Lie algebras and study the structure of Rota–Baxter 3-Lie algebras, and they also give a method to construct Rota–Baxter 3-Lie algebras from other Rota–Baxter 3-Lie algebras, Rota–Baxter Lie algebras, Rota–Baxter pre-Lie algebras and Rota–Baxter commutative associative algebras and derivations.

In this paper we investigate a class of Rota–Baxter operators with weight zero on the simple 3-Lie algebra A_ω, which is constructed from a commutative associative algebra A with a derivation $\varDelta$ and an involution ω which satisfies $\varDelta\omega + \omega\varDelta = 0$ [BW]. This construction not only allows us to further study the structure of the simple Rota–Baxter 3-Lie algebra, but it is also a rich source of examples for Rota–Baxter 3-Lie algebras.

The paper is organized as follows. Section 2 describes the concepts of Rota–Baxter operators with weights for general n-ary algebras and some basic results used in the paper. Section 3 is devoted to homogeneous Rota–Baxter operators on A_ω with weight zero. In Section 4, new 3-Lie algebras are constructed from homogeneous Rota–Baxter operators on A_ω.

In this paper, we suppose that F is a field of characteristic zero, and $\mathbb{Z}$ is the set of integers.

2. Preliminaries. An *n-Lie algebra* [F] is a vector space A over a field F endowed with an n-ary multilinear skew-symmetric operation $[x_1, \dots, x_n]$ satisfying the *n-Jacobi identity*

$$[[x_1, \dots, x_n], y_2, \dots, y_n] = \sum_{i=1}^{n} [x_1, \dots, [x_i, y_2, \dots, y_n], \dots, x_n]. \tag{2.1}$$

In particular, a 3-*Lie algebra* is a vector space A endowed with a ternary multilinear skew-symmetric operation satisfying, for all $x, y, z, u, v \in A$,

$$\big[[x,y,z],u,v\big] = \big[[x,u,v],y,z\big] + \big[x,[y,u,v],z\big] + \big[x,y,[z,u,v]\big]. \tag{2.2}$$

DEFINITION 2.1. Let $\lambda \in F$ be fixed.

(1) A (*nonassociative*) n-*algebra* over a field F is a pair $(A, \langle\, , \ldots, \,\rangle)$ consisting of a vector space A over F and a multilinear multiplication

$$\langle\, , \ldots, \,\rangle : A^{\otimes n} \to A.$$

(2) A *derivation of weight* λ on an n-algebra $(A, \langle\, , \ldots, \,\rangle)$ is a linear map $d : A \to A$ such that for all $x_1, \ldots, x_n \in A$,

$$d(\langle x_1, \ldots, x_n\rangle) = \sum_{\emptyset \neq I \subseteq [n]} \lambda^{|I|-1} \langle \check{d}(x_1), \ldots, \check{d}(x_i), \ldots, \check{d}(x_n)\rangle, \tag{2.3}$$

where

$$\check{d}(x_i) := \check{d}_I(x_i) := \begin{cases} d(x_i), & i \in I, \\ x_i, & i \notin I. \end{cases}$$

An n-algebra equipped with such a derivation is called a *differential* n-*algebra of weight* λ. In particular, a *differential* 3-*algebra of weight* λ is a 3-algebra $(A, \langle\, , \, , \,\rangle)$ with a linear map $d : A \to A$ such that

$$\begin{aligned} d(\langle x_1, x_2, x_3\rangle) &= \langle d(x_1), x_2, x_3\rangle + \langle x_1, d(x_2), x_3\rangle + \langle x_1, x_2, d(x_3)\rangle \\ &\quad + \lambda\langle d(x_1), d(x_2), x_3\rangle + \lambda\langle d(x_1), x_2, d(x_3)\rangle + \lambda\langle x_1, d(x_2), d(x_3)\rangle \\ &\quad + \lambda^2\langle d(x_1), d(x_2), d(x_3)\rangle. \end{aligned} \tag{2.4}$$

(3) A *Rota–Baxter operator of weight* λ on an n-algebra $(A, \langle\, , \ldots, \,\rangle)$ is an F-linear map $R : A \to A$ such that for all $x_1, \ldots, x_n \in A$,

$$R\Big(\sum_{\emptyset \neq I \subseteq [n]} \lambda^{|I|-1} \langle \hat{R}(x_1), \ldots, \hat{R}(x_i), \ldots, \hat{R}(x_n)\rangle\Big) = \langle R(x_1), \ldots, R(x_n)\rangle, \tag{2.5}$$

where

$$\hat{R}(x_i) := \hat{R}_I(x_i) := \begin{cases} x_i, & i \in I, \\ R(x_i), & i \notin I. \end{cases}$$

An n-algebra equipped with such an operator is called a *Rota–Baxter* n-*algebra of weight* λ. In particular, a *Rota–Baxter* 3-*algebra* is a 3-algebra $(A, \langle\, , \, , \,\rangle)$ with a linear map $P : A \to A$ such that

$$\begin{aligned} \langle R(x_1), R(x_2), R(x_3)\rangle = R\big(&\langle R(x_1), R(x_2), x_3\rangle + \langle R(x_1), x_2, R(x_3)\rangle \\ &+ \langle x_1, R(x_2), R(x_3)\rangle + \lambda\langle R(x_1), x_2, x_3\rangle + \lambda\langle x_1, R(x_2), x_3\rangle \\ &+ \lambda\langle x_1, x_2, R(x_3)\rangle + \lambda^2\langle x_1, x_2, x_3\rangle\big). \end{aligned} \tag{2.6}$$

LEMMA 2.2 ([BGL]). *Let $(A, \langle\, , \ldots, \,\rangle)$ be an n-algebra over F. An invertible linear mapping $P : A \to A$ is a Rota–Baxter operator of weight λ on A if and only if P^{-1} is a derivation of weight λ on A.*

LEMMA 2.3. *Let $(A, \langle\, , \dots , \rangle, R)$ be a Rota–Baxter n-algebra over F with weight* 0. *Then for all $\lambda \in F \setminus \{0\}$, $(A, \langle\, , \dots , \rangle, \lambda R)$ is a Rota–Baxter n-algebra with weight* 0.

Proof. This follows directly from (2.5). ∎

LEMMA 2.4 ([BW]). *Let A be a vector space with a basis $\{L_n \mid n \in \mathbb{Z}\}$ over F. Then A is a simple* 3*-Lie algebra under the multiplication, for $l, m, n \in \mathbb{Z}$,*

$$[L_l, L_m, L_n] = \begin{vmatrix} (-1)^l & (-1)^m & (-1)^n \\ 1 & 1 & 1 \\ l & m & n \end{vmatrix} L_{l+m+n-1}. \tag{2.7}$$

In the following, the 3-Lie algebra A of Lemma 2.4 is denoted by A_ω, and the determinant $\begin{vmatrix} (-1)^l & (-1)^m & (-1)^n \\ 1 & 1 & 1 \\ l & m & n \end{vmatrix}$ is denoted by $D(l, m, n)$.

LEMMA 2.5. *$D(l, m, n) = 0$ if and only if either $(l-m)(l-n)(m-n) = 0$, or $l = 2k + 1, m = 2s + 1, n = 2t + 1$, or $l = 2k, m = 2s, n = 2t$, for some $k, s, t \in \mathbb{Z}$.*

Proof. This follows by direct computation. ∎

3. Homogeneous Rota–Baxter operators with weight 0 on A_ω. In this section we discuss Rota–Baxter operators with weight 0 on the 3-Lie algebra A_ω. By Definition 2.1, if $(A, [\,,\,,\,], R)$ is a Rota–Baxter 3-Lie algebra of weight $\lambda = 0$, then the F-linear map $R : A \to A$ satisfies, for all $x_1, x_2, x_3 \in A$,

$$\begin{aligned} &[R(x_1), R(x_2), R(x_3)] \\ &\quad = R\big([R(x_1), R(x_2), x_3] + [R(x_1), x_2, R(x_3)] + [x_1, R(x_2), R(x_3)]\big). \end{aligned} \tag{3.1}$$

A *homogeneous Rota–Baxter operator* R on the 3-Lie algebra A_ω is a Rota–Baxter operator such that there exists a $\mathbb{Z}$-linear function $f : \mathbb{Z} \to F$ satisfying

$$R(L_m) = f(m)L_m, \qquad \forall m \in \mathbb{Z}. \tag{3.2}$$

THEOREM 3.1. *Let $R : A_\omega \to A_\omega$ be a linear map defined as in* (3.2). *Then R is a homogeneous Rota–Baxter operator of weight* 0 *on A_ω if and only if, for all $l, m, n \in \mathbb{Z}$,*

$$\begin{aligned} &\big(f(l)f(n) + f(m)f(n) + f(l)f(m)\big)f(l + m + n - 1)D(l, m, n) \\ &\qquad = f(l)f(m)f(n)D(l, m, n). \end{aligned} \tag{3.3}$$

Proof. By (2.7), (3.1) and (3.2), we have

$$[R(L_l),R(L_m),R(L_n)] = f(l)f(m)f(n)D(l,m,n)L_{l+m+n-1}$$

and

$$\begin{aligned}&R\big([R(L_l),R(L_m),L_n]+[R(L_l),L_m,R(L_n)]+[L_l,R(L_m),R(L_n)]\big)\\&=\big(f(l)f(m)+f(l)f(n)+f(m)f(n)\big)f(l+m+n-1)D(l,m,n)L_{l+m+n-1}.\end{aligned}$$

Therefore, R is a homogeneous Rota–Baxter operator on A_ω if and only if (3.3) holds. ∎

3.1. Homogeneous Rota–Baxter operators with $f(0)+f(1)\neq 0$. In this section, we discuss the homogeneous Rota–Baxter operators with weight 0 defined by (3.2) with $f(0)+f(1)\neq 0$.

THEOREM 3.2. *Let $R : A_\omega \to A_\omega$ be a linear map defined as in* (3.2) *with $f(0)+f(1)\neq 0$. Then R is a homogeneous Rota–Baxter operator on A_ω if and only if*

$$f(m)=0 \quad \textit{for all } m\in\mathbb{Z}\setminus\{0,1\}.$$

Proof. If $f(m)=0$ for all $m\in\mathbb{Z}\setminus\{0,1\}$, then by a direct computation R is a homogeneous Rota–Baxter operator.

Conversely, if R is a homogeneous Rota–Baxter operator with $f(0)+f(1)\neq 0$, then (3.1) in the case $l=0$, $n=1$ becomes

$$f(0)f(m)f(1)=\{f(0)f(1)+f(m)f(1)+f(0)f(m)\}f(m),\ \forall m\in\mathbb{Z}\setminus\{0,1\}.$$

Since $f(0)+f(1)\neq 0$, we have $f(m)^2=0$ for all $m\in\mathbb{Z}\setminus\{0,1\}$. ∎

3.2. Homogeneous Rota–Baxter operators with $f(0)+f(1)=0$

LEMMA 3.3. *Let $R : A_\omega\to A_\omega$ be a linear map of A defined as in* (3.2) *with $f(0)+f(1)=0$. Then R is a homogeneous Rota–Baxter operator if and only if for all $l,m,n\in\mathbb{Z}$,*

$$\begin{aligned}f(2l+1)f(2m+1)f(2n)=\big(&f(2l+1)f(2m+1)+f(2l+1)f(2n)\\&+f(2m+1)f(2n)\big)f(2l+2m+2n+1)\quad \textit{if } m\neq l,\end{aligned}\tag{3.4}$$

$$\begin{aligned}f(2l+1)f(2m)f(2n)=\big(&f(2l+1)f(2m)+f(2l+1)f(2n)\\&+f(2m)f(2n)\big)f(2l+2m+2n)\quad \textit{if } m\neq n.\end{aligned}\tag{3.5}$$

Proof. For all $l,m,n\in\mathbb{Z}$ with $l\neq m$ and $m\neq n$, we have $D(2l+1, 2m+1, 2n)\neq 0$ and $D(2l+1,2m,2n)\neq 0$. Thanks to (2.7) and (3.2), we obtain (3.4) and (3.5). ∎

3.2.1. *Homogeneous Rota–Baxter operators with $f(0)=-f(1)\neq 0$.* Now we discuss the case $f(0)+f(1)=0$, but $f(0)\neq 0$. By Lemma 2.3, we can suppose $f(0)=1$, so $f(1)=-1$.

COROLLARY 3.4. *Let R be a homogeneous Rota–Baxter operator with $f(0) = -f(1) = 1$. Then, for all $l, m, n \in \mathbb{Z}$, we have*

(1) $f(2l+1)f(2m+1) = (f(2l+1) + f(2m+1) + f(2l+1)f(2m+1)) \cdot f(2l+2m+1)$ *if* $l \neq m$,
(2) $f(2l+1)f(2m) = (f(2l+1) + f(2m) + f(2l+1)f(2m))f(2l+2m)$ *if* $m \neq 0$,
(3) $f(2l+1)f(2n) = (f(2l+1) + f(2n) - f(2l+1)f(2n))f(2l+2n+1)$ *if* $l \neq 0$,
(4) $f(2m)f(2n) = (f(2m) + f(2n) - f(2m)f(2n))f(2m+2n)$ *if* $m \neq n$.

Proof. This follows from Lemma 3.3 and $f(0) = -f(-1) = 1$. ∎

THEOREM 3.5. *Let R be a homogeneous Rota–Baxter operator with $f(0) = -f(1) = 1$. Then*

$$f(1-m) + f(m) = 0 \quad \text{for all } m \in \mathbb{Z}. \tag{3.6}$$

Proof. According to Corollary 3.4, for all $n \in \mathbb{Z} \setminus \{0\}$, we have

$$\begin{aligned} f(2m+1)\big(f(2m+2n) - f(2m+2n+1)\big) \\ + f(2n)\big(f(2m+2n) - f(2m+2n+1)\big) \\ + f(2m+1)f(2n)\big(f(2m+2n) + f(2m+2n+1)\big) = 0. \end{aligned}$$

Then if $m = -n \neq 0$, we have $f(2m+1) + f(-2m) = 0$ and

$$f(2m+1) + f(1-(2m+1)) = 0.$$

Similarly, $f(1-2(-m)) + f(2(-m)) = 0$ for all $m \in \mathbb{Z}$. Hence (3.6) follows. ∎

COROLLARY 3.6. *If R is a homogeneous Rota–Baxter operator on A_ω satisfying $f(0) = -f(1) = 1$, and there exist $k, l, m, n \in \mathbb{Z}$ such that $(k-l)(m-n)klmn \neq 0$ and $f(2k) \neq 0$, $f(2l) \neq 0$, $f(2m+1) \neq 0$, $f(2n+1) \neq 0$, then*

(1) $f(2k+2l) \neq 0$, (2) $f(2k+2m) \neq 0$, (3) $f(2k+2m+1) \neq 0$,
(4) $f(2m+2n+1) \neq 0$, (5) $f(1-2k+2m) \neq 0$ *if* $k \neq -m$,
(6) $f(4k) \neq 0$, (7) $f(2m+2n+2k+1) \neq 0$,
(8) $f(2m+2k+2l) \neq 0$, (9) $f(2k-2m) \neq 0$ *if* $k \neq -m$,
(10) $f(1-2k-2m) \neq 0$, (11) $f(1-4k) \neq 0$.

Proof. (1)–(6) follow from Corollary 3.4 and $f(0) = -f(1) = 1$; (7)–(8) follow from Lemma 3.3; and (9)–(11) come from Theorem 3.5. ∎

LEMMA 3.7. *Let $R : A_\omega \to A_\omega$ be a linear map defined by (3.2). If $f(0) = -f(1) = 1$ and there exist finitely many distinct integers $m_i \neq 0, 1$ ($i = 1, \ldots, t$) such that $f(m_i) \neq 0$ for all i and $f(m) = 0$ for $m \in \mathbb{Z} \setminus \{m_1, \ldots, m_t, 0, 1\}$, then R is not a homogeneous Rota–Baxter operator on A_ω.*

Proof. If R were a homogeneous Rota–Baxter operator, then thanks to Theorem 3.5, $f(1-m_i) = -f(m_i) \neq 0$ for $1 \leq i \leq t$. We obtain $t \geq 2$. Without loss of generality, suppose m_1 is odd; then $m_2 = 1 - m_1$ is even and $f(m_2) \neq 0$. By Corollary 3.6(6), we have $f(2nm_2) \neq 0$ for all $n \in \mathbb{Z}$. This contradicts $t < \infty$. ∎

LEMMA 3.8. *Let $R : A_\omega \to A_\omega$ be a linear map defined by* (3.2)*. If $f(0) = -f(1) = 1$ and there exist finitely many distinct integers m_i $(i = 1, \ldots, t)$ such that $f(m_i) = 0$ for all i and $f(m) \neq 0$ for $m \in \mathbb{Z} \setminus \{m_1, \ldots, m_t\}$, then R is not a homogeneous Rota–Baxter operator on A_ω.*

Proof. Suppose R is a homogeneous Rota–Baxter operator. Thanks to Theorem 3.5, $f(1-m_i) = -f(m_i) = 0$ for $1 \leq i \leq t$, and for all $n \neq m_i$, $f(n) = -f(1-n) \neq 0$. It follows that there exist infinitely many $l \in \mathbb{Z}$ such that $f(2l+1) \neq 0$. Without loss of generality, suppose m_1 is even; then by Corollary 3.4, there exist infinitely many $l \in \mathbb{Z}$ such that $f(2l + 2m_1) = 0$. This contradicts $t < \infty$. ∎

THEOREM 3.9. *Let $R : A_\omega \to A_\omega$ be a linear map defined by* (3.2)*. If R is a homogeneous Rota–Baxter operator on A_ω with $f(0) = -f(1) = 1$ and there exists $m \in \mathbb{Z} \setminus \{0, 1\}$ such that $f(m) \neq 0$, then there exists $m_0 \in \mathbb{Z}_{>0}$ such that for any $m \in \mathbb{Z}$, $f(m) \neq 0$ if and only if*

$$m \in W_{m_0} = \{2m_0k \mid k \in \mathbb{Z}\} \cup \{1 - 2m_0k \mid k \in \mathbb{Z}\}.$$

We then say that W_{m_0} is the supporter of R.

Proof. From Theorem 3.5, there exists $W = \{2x_k \mid k \in \mathbb{Z}\} \cup \{1 - 2x_k \mid k \in \mathbb{Z}\} \subset \mathbb{Z}$ such that $f(m) \neq 0$ if and only if $m \in W$. Thanks to Lemmas 3.7 and 3.8, W is infinite. From Corollary 3.6, we can suppose that for all $k, s \in \mathbb{Z}$, $2x_k < 2x_s$ if and only if $k < s$, and $x_{-1} < 0 < x_1$.

By Corollary 3.6(2) and since $2x_2, -2x_1+1 \in W$, we have $2(x_2 - x_1) \in W$. Thanks to $0 < x_2 - x_1 < x_2$, we obtain $x_2 = 2x_1$.

Now suppose that $x_k - x_{k-1} = x_1$ for some $k > 0$, that is, $x_k = kx_1$. Since $2x_{k+1}, 2x_{k-1}, -2x_k+1 \in W$, from Corollary 3.6(8) we have $2(x_{k+1} - x_k + x_{k-1}) = 2(x_{k+1} - x_1) \in W$. Since $x_{k+1} - x_1 < x_{k+1}$ and $x_{k-1} = x_k - x_1 < x_{k+1} - x_1$, we obtain $x_{k-1} < x_{k+1} - x_1 < x_{k+1}$. Therefore, $x_{k+1} - x_1 = x_k$, that is, $x_{k+1} = x_k + x_1 = (k+1)x_1$.

By a completely similar discussion, we find that $x_k = -kx_{-1}$ for all $k < 0$.

Since $2x_{-1}, 2x_1 \in W$, from Corollary 3.6(1) we have $2(x_{-1} + x_1) \in W$. From $x_{-1} < 0 < x_1$ and $x_{-1} < x_{-1} + x_1 < x_1$, we obtain $x_{-1} + x_1 = 0$, that is, $x_{-1} = -x_1$. Denote $m_0 = x_1$. Then $2x_k = 2m_0k$ for all $2x_k \in W$. ∎

COROLLARY 3.10. *Let R be a homogeneous Rota–Baxter operator with $f(0) = -f(1) = 1$. If there exists $k \in \mathbb{Z}$ such that $f(2k) \neq 0$, then $f(-2k) \neq 0$ and $f(1+2k) \neq 0$.*

Proof. This follows from Theorems 3.9 and 3.5 directly. ∎

LEMMA 3.11. *Let R be a homogeneous Rota–Baxter operator with $f(0) = -f(1) = 1$, and let W_{m_0} be its support. Then $f(2m_0) \neq 1/2$, and for all $k, k_1, k_2, k_3 \in \mathbb{Z}$ with $k_2 \neq k_3$,*

$$\frac{1}{2f(2m_0k)} + \frac{1}{2f(-2m_0k)} = 1, \quad \frac{1}{2f(2m_0k)} - \frac{1}{2f(1+2m_0k)} = 1, \tag{3.7}$$

$$\frac{1}{f(2m_0k_2)} + \frac{1}{f(2m_0k_3)} = \frac{1}{f(2m_0k_1)} + \frac{1}{f(-2m_0k_1+2m_0k_2+2m_0k_3)}. \tag{3.8}$$

Proof. By Corollary 3.4(4), for all $k \in \mathbb{Z} \setminus \{0\}$,

$$f(2m_0k)f(-2m_0k) = f(2m_0k) + f(-2m_0k) - f(2m_0k)f(-2m_0k).$$

Hence (3.7) follows, and $f(2m_0) \neq 1/2$.

According to Lemma 3.3 and Theorem 3.5, for all distinct $m, n \in \mathbb{Z}$,

$$\{f(2l)f(2m) + f(2l)f(2n) - f(2m)f(2n)\}f(-2l+2m+2n) = f(2l)f(2m)f(2n).$$

For $l = m_0k_1$, $m = m_0k_2$, $n = m_0k_3$, we obtain (3.8). ∎

THEOREM 3.12. *Let $R : A_\omega \to A_\omega$ be a linear map defined as in (3.2) with $f(0) = -f(1) = 1$. Then R is a homogeneous Rota–Baxter operator on A_ω if and only if either $f(m) = 0$ for all $m \in \mathbb{Z} \setminus \{0,1\}$, or there exist $m_0 \in \mathbb{Z}_{>0}$ and $a \in F$, $a \neq (k-1)/k$ for $k \in \mathbb{Z} \setminus \{0\}$, such that W_{m_0} is the support of R and*

$$f(2m_0k) = -f(1-2m_0k) = \frac{1}{ka-(k-1)}, \quad \forall k \in \mathbb{Z}. \tag{3.9}$$

Furthermore, if $m_0 = 1$, then R is an invertible Rota–Baxter operator on A_ω, so R^{-1} is an invertible derivation of A_ω, and

$$R^{-1}(L_{2k}) = (ka-(k-1))L_{2k}, \quad R^{-1}(L_{1-2k}) = (-ka+(k-1))L_{1-2k}, \quad \forall k \in \mathbb{Z}.$$

Proof. If R is a homogeneous Rota–Baxter operator on A_ω and there exists $m \in \mathbb{Z} \setminus \{0,1\}$ such that $f(m) \neq 0$, then by Theorem 3.9, there exists $m_0 \in \mathbb{Z}_{>0}$, such that W_{m_0} is the supporter of R. Suppose $f(2m_0) = 1/a$. Then by Lemma 3.11, $a \neq 2$.

Now suppose $k \in \mathbb{Z}_{>0}$ satisfies $f(2m_0k) = \frac{1}{ka-(k-1)}$. By Lemma 3.11, $\frac{1}{f(2m_0(k+1))} + 1 = \frac{1}{f(2m_0k)} + \frac{1}{f(2m_0)} = ka-(k-1)+a$, that is, $f(2m_0(k+1)) = \frac{1}{(k+1)a-k}$, and $a \neq (k-1)/k$.

Since $\frac{1}{f(2m_0)}+\frac{1}{f(-2m_0)}=2$, we have $f(-2m_0)=\frac{1}{2-a}=\frac{1}{-a-(-1-1)}$. Now suppose that for some $k\in\mathbb{Z}_{<0}$ we have $f(2m_0k)=\frac{1}{ka-(k-1)}$. From

$$\frac{1}{f(2m_0(k-1))}+1=\frac{1}{f(2m_0k)}+\frac{1}{f(-2m_0)}=ka-(k-1)+2-a,$$

we obtain

$$f(2m_0(k-1))=\frac{1}{(k-1)a-(k-2)},\qquad a\neq\frac{k-2}{k-1}.$$

This yields (3.9).

Conversely, since for all $2l,2m,2n\notin W_{m_0}$ with $l\neq m$,

$$f(\pm 2l)=f(\pm 2m)=f(\pm 2n)=0,\quad f(1\pm 2l)=f(1\pm 2m)=f(1\pm 2n)=0,$$

identity (3.3) holds. So we only need to prove (3.3) in the following cases:

(1) If $2l,2m\notin W_{m_0}$, $2n\in W_{m_0}$, $l\neq m$, by Theorems 3.9 and 3.5, we have

$$\begin{aligned}&f(\pm 2l)=f(\pm 2m)=f(1\pm 2l)=f(1\pm 2m)=0,\\&f(\pm 2m\pm 2n\pm 2l)=f(\pm 2m\pm 2n\pm 2l+1)=0.\end{aligned}$$

Thus (3.3) holds.

(2) If $2l\notin W_{m_0}$, $2m,2n\in W_{m_0}$, and $m\neq n$, we have $f(\pm 2l)=f(1\pm 2l)=0$, $f(\pm 2l\pm 2m\pm 2n)=f(1\pm 2l\pm 2m\pm 2n)=0$. Thus (3.3) holds.

(3) If $2l,2m,2n\in W_{m_0}$, $l\neq m,n$ and $m\neq n$, suppose $2l=2m_0k_1$, $2m=2m_0k_2$, $2n=2m_0k_3$. From

$$f(1-2l)f(2m)f(2n)=\frac{-1}{k_1a-(k_1-1)}\,\frac{1}{k_2a-(k_2-1)}\,\frac{1}{k_3a-(k_3-1)},$$

$$\begin{aligned}&\big(f(1-2l)f(2m)+f(1-2l)f(2n)+f(2m)f(2n)\big)f(2(m+n-l))\\&=\left(\frac{-1}{k_1a-(k_1-1)}\,\frac{1}{k_2a-(k_2-1)}+\frac{-1}{k_1a-(k_1-1)}\,\frac{1}{k_3a-(k_3-1)}\right.\\&\qquad\left.+\frac{1}{k_2a-(k_2-1)}\,\frac{1}{k_3a-(k_3-1)}\right)\\&\qquad\times\frac{1}{(-k_1+k_2+k_3)a-(-k_1+k_2+k_3-1)}\\&=\frac{-1}{k_1a-(k_1-1)}\,\frac{1}{k_2a-(k_2-1)}\,\frac{1}{k_3a-(k_3-1)},\end{aligned}$$

$$f(1-2l)f(1-2m)f(2n)=\frac{-1}{k_1a-(k_1-1)}\,\frac{-1}{k_2a-(k_2-1)}\,\frac{1}{k_3a-(k_3-1)},$$

$$\begin{aligned}&\big(f(1-2l)f(1-2m)+f(1-2l)f(2n)+f(1-2m)f(2n)\big)f(1-2(l+m-n))\\&\quad=\bigg(\frac{-1}{k_1a-(k_1-1)}\,\frac{-1}{k_2a-(k_2-1)}+\frac{-1}{k_1a-(k_1-1)}\,\frac{1}{k_3a-(k_3-1)}\\&\qquad+\frac{-1}{k_2a-(k_2-1)}\,\frac{1}{k_3a-(k_3-1)}\bigg)\frac{-1}{(k_1+k_2-k_3)a-(k_1+k_2-k_3-1)}\\&\quad=\frac{1}{k_1a-(k_1-1)}\,\frac{1}{k_2a-(k_2-1)}\,\frac{1}{k_3a-(k_3-1)}\end{aligned}$$

identity (3.3) follows.

Altogether, we obtain the desired result. ■

Let $m_0=1$ and $a=3$. By Theorem 3.12, the linear map $R:A_\omega\to A_\omega$ defined by setting, for all $k\in\mathbb{Z}$,

$$R(L_{2k})=\frac{1}{3k-(k-1)}L_{2k}=\frac{1}{2k+1}L_{2k},\quad R(L_{1-2k})=-\frac{1}{2k+1}L_{1-2k}$$

is a homogeneous Rota–Baxter operator on A_ω, and it is invertible. Therefore, $D=R^{-1}:A_\omega\to A_\omega$ satisfying

$$D(L_{2k})=(2k+1)L_{2k},\quad D(L_{1-2k})=-(2k+1)L_{1-2k},\qquad k\in\mathbb{Z},$$

is an invertible derivation of A_ω.

Let $m_0=3$ and $a=\sqrt{2}$. Then the linear map $R:A_\omega\to A_\omega$ defined by

$$R(L_{6k})=\frac{1}{\sqrt{2}k-(k-1)}L_{6k},\quad R(L_{1-6k})=-\frac{1}{\sqrt{2}k-(k-1)}L_{1-6k},\quad k\in\mathbb{Z},$$

and setting the other $R(L_n)$ to zero, is a homogeneous Rota–Baxter operator on A_ω. But R is degenerate.

3.2.2. *Homogeneous Rota–Baxter operators with $f(0)=f(1)=0$.* In this section we discuss the case $f(0)=f(1)=0$.

LEMMA 3.13. *Let $R:A_\omega\to A_\omega$ be a homogeneous Rota–Baxter operator on A_ω with $f(0)=f(1)=0$. Then for all $l,m,n\in\mathbb{Z}$:*

(1) $f(2l+1)f(2m+1)f(2l+2m+1)=0$ *if* $l\neq m$.
(2) $f(2m+1)f(2n)f(2m+2n+1)=0$ *if* $m\neq 0$.
(3) $f(2l+1)f(2m)f(2l+2m)=0$ *if* $m\neq 0$.
(4) $f(2m)f(2n)f(2m+2n)=0$ *if* $m\neq n$.

Proof. The result follows from $D(2l+1,2m+1,0)\neq 0$, $D(1,2m+1,2n)\neq 0$, $D(2l+1,2m,0)\neq 0$, $D(1,2m,2n)\neq 0$, and Lemma 3.3. ■

COROLLARY 3.14. *Let $R:A_\omega\to A_\omega$ be a homogeneous Rota–Baxter operator with $f(0)=f(1)=0$, and suppose there exist $k,l,m,n\in\mathbb{Z}$ such that $(k-l)(m-n)klmn\neq 0$, $f(2k)\neq 0$, $f(2l)\neq 0$, $f(2m+1)\neq 0$, $f(2n+1)\neq 0$. Then*

(1) $f(2k+2l)=0$, (2) $f(2k+2m)=0$, (3) $f(2k+2m+1)=0$,

(4) $f(2m+2n+1)=0$, (5) $f(2m+2n+2k+1)\neq 0$,
(6) $f(2m+2k+2l)\neq 0$, (7) $f(2k-2m)=0$, $k\neq -m$, (8) $f(4k)=0$.

Proof. (1), (2), (3) and (4) follow from (4), (3), (2) and (1) of Lemma 3.13, respectively. (5) and (6) follow from (3.4) and (3.5), respectively. (7) and (8) follow from (4) and (3) of Lemma 3.13, respectively. ■

THEOREM 3.15. *Let $R: A_\omega \to A_\omega$ be a homogeneous Rota–Baxter operator with $f(0)=f(1)=0$, and suppose there exist $m_1,\dots,m_s\in\mathbb{Z}$ such that $f(m_i)\neq 0$ for all i and $f(m)=0$ for all $m\neq m_i$. Then either*

(1) *$s=1$, and then we can suppose $f(m_1)=1$ and $f(m)=0$ for all $m\in\mathbb{Z}\setminus\{m_1\}$, or*
(2) *$s=2$ and $m_1+m_2=1$, so we can suppose that $f(m_1)=1$, $f(1-m_1)=b\neq 0$, and $f(m)=0$ for all $m\in\mathbb{Z}\setminus\{m_1,1-m_1\}$.*

Proof. First, if there exists $m_1\in\mathbb{Z}$ such that $f(m_1)\neq 0$ and $f(m)=0$ for all $m\in\mathbb{Z}\setminus\{m_1\}$, then by Lemma 3.3 and a direct computation, R is a homogeneous Rota–Baxter operator. Thanks to Lemma 2.3, we can suppose $f(m_1)=1$.

Second, if there exist distinct $m_1,m_2\in\mathbb{Z}$ satisfying $m_1+m_2=1$ such that $f(m_1),f(m_2)\neq 0$ and $f(m)=0$ for all $m\in\mathbb{Z}\setminus\{m_1,m_2\}$, then $D(m_1,m_2,m)\neq 0$ for all $m\in\mathbb{Z}$. By a direct computation, $f(l)$, $f(m)$ and $f(n)$ satisfy (3.4) and (3.5) for all $l,m,n\in\mathbb{Z}$. Therefore, R is a homogeneous Rota–Baxter operator. By Lemma 2.3, we can suppose $f(m_1)=1$, $f(m_2)=f(1-m_1)=b\neq 0$.

Third, if R is a homogeneous Rota–Baxter operator such that there exist distinct $m_1,m_2\in\mathbb{Z}$ with $f(m_i)\neq 0$ and $f(m)=0$ for all $m\in\mathbb{Z}\setminus\{m_1,m_2\}$, then there exists $m\in\mathbb{Z}$ such that $D(m_1,m_2,m)\neq 0$. Thanks to Lemma 3.3,

$$f(m_1+m_2+m-1)=0.$$

Then $m_1+m_2+m-1\neq m_1$ and $m_1+m_2+m-1\neq m_2$, that is, $m\neq 1-m_1$ and $m\neq 1-m_2$. It follows that $1-m_1=m_2$.

Lastly, suppose R is a homogeneous Rota–Baxter operator satisfying $f(m_i)\neq 0$ for $1\le i\le s$, $s\ge 3$, and $f(m)=0$ for all $m\neq m_i$; then for every $1\le i\le s$, we have $f(1-m_i)\neq 0$.

In fact, suppose $f(1-m_1)=0$. From $D(m_1,m_2,1-m_1)\neq 0$, (3.4) and (3.5), we have the contradiction $f(m_1+m_2+(1-m_1)-1)=f(m_2)=0$. Therefore, $f(1-m_1)\neq 0$. From $s\ge 3$ and a similar discussion, we find that $f(1-m_i)\neq 0$ for $1\le i\le s$.

Therefore, s is even and $s\ge 4$, and we can suppose $m_1<\cdots<m_s$. Then there exists $m\in\mathbb{Z}\setminus\{0,1\}$ such that $f(m)=0$ and $D(m_1,m_2,m)\neq 0$. Thanks to (3.4) and (3.5), $f(m_1+m_2+m-1)=0$. Thus $m_1+m_2+m-1\neq m_s$, that is, $m\neq m_s-m_1-m_2+1$. By the above discussion and $s\ge 4$,

there exists $i \geq 3$ such that $m_s - m_1 - m_2 + 1 = 1 - m_i$. We obtain the contradiction $m_1 + m_2 = m_i + m_s$.

Altogether, we obtain the desired result. ■

LEMMA 3.16. *Let $R : A_\omega \to A_\omega$ be a homogeneous Rota–Baxter operator with $f(0) = f(1) = 0$, and suppose there exist infinitely many $m \in \mathbb{Z}$ such that $f(m) \neq 0$. Then there exist infinitely many $n \in \mathbb{Z}$ such that $f(n) = 0$, and for all $m \in \mathbb{Z}$, if $f(m) \neq 0$, then $f(1 - m) \neq 0$ and*

$$f(m) + f(1 - m) = 0.$$

Proof. If there exists $m \in \mathbb{Z}$ such that $f(m) \neq 0$, but $f(1-m) = 0$, then for all $n \in \mathbb{Z} \setminus \{m, 1 - m\}$, by (3.4), (3.5) and $D(m, n, 1 - m) \neq 0$, we have the contradiction $f(m+n+1-m-1) = f(n) = 0$. Therefore, if $f(m) \neq 0$, then $f(1-m) \neq 0$. Thanks to Corollary 3.14(8), there exist infinitely many $n \in \mathbb{Z}$ such that $f(n) = 0$.

Now for distinct $2m, 2n \in \mathbb{Z}$ with $f(2m) \neq 0$, $f(2n) \neq 0$, by (3.5),

$$\begin{aligned} &f(1 - 2m)f(2m)f(2n) \\ &= \big(f(1-2m)f(2m)+f(1-2m)f(2n)+f(2m)f(2n)\big)f(1-2m+2n+2m-1). \end{aligned}$$

It follows that $f(1 - 2m) + f(2m) = 0$ for all $m \in \mathbb{Z}$. ■

THEOREM 3.17. *Let $R : A_\omega \to A_\omega$ be a homogeneous Rota–Baxter operator with $f(0) = f(1) = 0$, and suppose there exist infinitely many $m \in \mathbb{Z}$ such that $f(m) \neq 0$. Then there exist positive integers m_0 and s_0 satisfying $1 \leq s_0 < m_0$ such that $f(m) \neq 0$ if and only if*

$$m \in W_{m_0,s_0} := \{2m_0k + 2s_0 \mid k \in \mathbb{Z}\} \cup \{1 - 2m_0k - 2s_0 \mid k \in \mathbb{Z}\},$$

so $W_{m_0s_0}$ is the supporter of R.

Proof. By Lemma 3.16, we can suppose that

$$W := \{2x_k \mid k \in \mathbb{Z}\} \cup \{1 - 2x_k \mid k \in \mathbb{Z}\}$$

is the set of integers such that $f(m) \neq 0$ if and only if $m \in W$. By Lemma 3.16 and Corollary 3.4, we can suppose that for $2x_k, 2x_s \in W$, $2x_k < 2x_s$ if and only if $k < s$, and $2x_{-1} < 0$, $2x_0 > 0$.

Denote $x_1-x_0 = m_0$ and $x_2-x_1 = m_1$. Then $m_0, m_1 > 0$. From $2x_0 \in W$, $-2x_1 + 1 \in W$, $2x_2 \in W$ and Corollary 3.14(6), we have

$$2(x_2 - x_1 + x_0) = 2(m_0 + x_0 - x_0 + x_0 - m_1) = 2(x_2 - m_0) \in W.$$

Since $x_0 = x_1 - m_0 < x_2 - m_0 < x_2$, we have $x_2 - m_0 = x_1$, that is, $m_1 = m_0$.

Now suppose $x_k - x_{k-1} = m_0$ for some $k > 0$. Denote $x_{k+1} - x_k = m_k$. According to Corollary 3.14(6), we have

$$2(x_{k+1} - x_k + x_{k-1}) = 2(m_k + x_k - x_k + x_k - m_0) = 2(x_{k+1} - m_0) \in W.$$

Thanks to $x_{k-1} = x_k - m_0 < x_{k+1} - m_0 < x_{k+1}$, we obtain $x_{k+1} - m_0 = x_k$, that is, $m_k = m_0$. Therefore, $2x_k = 2km_0 + 2x_0$ for $k \in \mathbb{Z}_{>0}$.

By a similar discussion we have $2x_k = 2km_0 + 2x_0$ for all $k \in \mathbb{Z}_{<0}$.

Therefore, $W = \{2km_0 + 2x_0 \mid\ k \in \mathbb{Z}, x_0 > 0\}$, where $m_0 > 0$.

By Lemma 3.16 and Corollary 3.14(1), $2x_1 + 2x_{-1} = 2x_0 + 2x_0 \notin W$, that is, m_0 is not a factor of x_0. So there exist integers s_0 and q such that $1 \le s_0 < m_0$ and $x_0 = qm_0 + s_0$.

Therefore, $2x_k = 2(k+q)m_0 + 2s_0$ for all $k \in \mathbb{Z}$, and the result follows. ∎

By Lemma 2.3, if W_{m_0,s_0} is the supporter of R, *we can suppose that* $f(2s_0) = 1$.

Lemma 3.18. *Let $R : A_\omega \to A_\omega$ be a homogeneous Rota–Baxter operator with supporter W_{m_0,s_0}, and $f(0) = f(1) = 0$. Then for all $k_i \in \mathbb{Z}$, and $k_i \neq k_j$, $1 \le i \neq j \le 3$, we have*

$$\begin{aligned}(3.10)\qquad &\frac{1}{f(2m_0k_1 + 2s_0)} + \frac{1}{f(2m_0k_2 + 2s_0)} + \frac{1}{f(2m_0k_3 + 2s_0)} \\ &= \frac{1}{f(2m_0(k_1 + k_2 - k_3) + 2s_0)} + \frac{1}{f(2m_0(k_1 - k_2 + k_3) + 2s_0)} \\ &\quad + \frac{1}{f(2m_0(-k_1 + k_2 + k_3) + 2s_0)}.\end{aligned}$$

Therefore,

$$\frac{1}{f(2m_0k + 2s_0)} + \frac{1}{f(-2m_0k + 2s_0)} = 2 \tag{3.11}$$

and $f(2m_0k + 2s_0) \neq 1/2$ for all $k \in \mathbb{Z}$.

Proof. By Lemmas 3.3 and 3.16, for all $k_1, k_2, k_3 \in \mathbb{Z}$ with $k_1 \neq k_2$ we have

$$\begin{aligned}&f(2m_0k_1 + 2s_0)f(2m_0k_2 + 2s_0)f(2m_0k_3 + 2s_0) \\ &\quad = \big(-f(2m_0k_1 + 2s_0)f(2m_0k_2 + 2s_0) + f(2m_0k_1 + 2s_0)f(2m_0k_3 + 2s_0) \\ &\qquad + f(2m_0k_2 + 2s_0)f(2m_0k_3 + 2s_0)\big)f(2m_0(k_1 + k_2 - k_3) + 2s_0) \neq 0.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{1}{f(2k_1m_0 + 2s_0)} + \frac{1}{f(2k_2m_0 + 2s_0)} - \frac{1}{f(2k_3m_0 + 2s_0)} \\ &\qquad\qquad = \frac{1}{f(2(k_1 + k_2 - k_3)m_0 + 2s_0)}.\end{aligned}$$

For $k_1 = -k_2$ and $k_3 = 0$, we obtain (3.11).

Similarly, for $k_1 \neq k_3$,

$$\frac{1}{f(2k_1m_0+2s_0)}+\frac{1}{f(2k_3m_0+2s_0)}-\frac{1}{f(2k_2m_0+2s_0)} = \frac{1}{f(2(k_1+k_3-k_2)m_0+2s_0)},$$

and for $k_2 \neq k_3$,

$$\frac{1}{f(2k_2m_0+2s_0)}+\frac{1}{f(2k_3m_0+2s_0)}-\frac{1}{f(2k_1m_0+2s_0)} = \frac{1}{f(2(k_2+k_3-k_1)m_0+2s_0)}.$$

This yields (3.10). ∎

THEOREM 3.19. *Let $R : A_\omega \to A_\omega$ be a linear map defined as in* (3.2) *with the property that there exist infinitely many $m \in \mathbb{Z}$ such that $f(m) = f(0) = f(1) = 0$. Then R is a homogeneous Rota–Baxter operator on A_ω if and only if there exist $m_0, s_0 \in \mathbb{Z}_{>0}$ and $a \in F$ such that W_{m_0,s_0} is the supporter of R, and*

$$f(2m_0k+2s_0) = -f(1-2m_0k-2s_0) = \frac{1}{ka-(k-1)}, \quad \forall k \in \mathbb{Z}, \tag{3.12}$$

where $1 \le s_0 < m_0$ and $a \neq (k-1)/k$ for all $k \in \mathbb{Z}$ and $k \neq 0$.

Proof. The proof is similar to that of Theorem 3.12. ∎

Let $m_0 = 7$, $a = 2$ and $s_0 = 2$. By Theorem 3.19, the linear map $R : A_\omega \to A_\omega$ defined by

$$R(L_{14k+4}) = \frac{1}{k+1}L_{14k+4}, \qquad R(L_{-14k-3}) = -\frac{1}{k+1}L_{-14k-3},$$

for all $k \in \mathbb{Z}$, and with all other $R(L_n)$ zero, is a homogeneous Rota–Baxter operator of weight 0 with supporter

$$W_{7,2} = \{14k+4 \mid k \in \mathbb{Z}\} \cup \{-14k-3 \mid k \in \mathbb{Z}\}.$$

If $m_0 = 4$, $s_0 = 3$ and $a = 3/5$, then the linear map $R : A_\omega \to A_\omega$ defined by, for all $k \in \mathbb{Z}$,

$$R(L_{8k+6}) = \frac{5}{5-2k}L_{8k+6}, \qquad R(L_{-8k-5}) = -\frac{5}{5-2k}L_{-8k-5},$$

and with all other $R(L_n)$ zero, is a homogeneous Rota–Baxter operator of weight 0 with supporter

$$W_{4,3} = \{8k+6 \mid k \in \mathbb{Z}\} \cup \{-8k-5 \mid k \in \mathbb{Z}\}.$$

3.3. 3-Lie algebras constructed from A_ω and homogeneous Rota–Baxter operators. Constructing 3-Lie algebras from known algebras is always of interest. In this section, we construct some 3-Lie algebras from the 3-Lie algebra A_ω and homogeneous Rota–Baxter operators.

Let $(A, [\, , \, , \,])$ be a 3-Lie algebra and R be a Rota–Baxter operator with weight λ. Using the notation of (2.5), we define a ternary operation $[\, , \, , \,]_R$ on A by setting, for all $x, y, z \in A$,

$$[x_1, x_2, x_3]_R = \sum_{\emptyset \neq I \subseteq [3]} \lambda^{|I|-1} [\widehat{R}_I(R(x_1)), \widehat{R}_I(R(x_2)), \widehat{R}_I(R(x_3))]. \tag{3.13}$$

Thus, in the case $\lambda = 0$ we have, for all $x, y, z \in A$,

$$[x_1, x_2, x_3]_R = [R(x), R(y), z] + [R(x), y, R(z)] + [x, R(y), R(z)]. \tag{3.14}$$

THEOREM 3.20 ([BGL]). *Let $(A, [\, , \, , \,])$ be a 3-Lie algebra and R be a Rota–Baxter operator on A of weight λ. Then $(A, [\, , \, , \,]_R)$ is a 3-Lie algebra under the multiplication defined in* (3.13)*, and R is a Rota–Baxter operator on it.*

So, if R is a homogeneous Rota–Baxter operator on the 3-Lie algebra A_ω of weight 0, then $(A, [\, , \, , \,]_R)$ is a 3-Lie algebra under the multiplication defined in (3.14), where $A = A_\omega$ as vector spaces, and R is also a homogeneous Rota–Baxter operator on $(A, [\, , \, , \,]_R)$.

THEOREM 3.21. *Let $R : A_\omega \to A_\omega$ be a linear map defined in* (3.2)*. Then R is a homogeneous Rota–Baxter operator of weight* 0 *on the* 3*-Lie algebra A_ω if and only if R is one of the following:*

$R_{0_1}(L_0) = L_0,\ R_{0_1}(L_1) = bL_1,\ R_{0_1}(L_m) = 0$ *for all* $m \in \mathbb{Z} \setminus \{0, 1\}$;

$$R_{0_2}(L_m) = \begin{cases} L_0, & m = 0, \\ -L_1, & m = 1, \\ \frac{1}{ka-(k-1)} L_{2m_0k}, & m = 2m_0k \in W_{m_0}, \\ -\frac{1}{ka-(k-1)} L_{1-2m_0k}, & m = 1 - 2m_0k \in W_{m_0}, \\ 0, & \textit{otherwise}; \end{cases}$$

$$R_{0_3}(L_m) = \begin{cases} \frac{1}{ka-(k-1)} L_{2m_0k+2s_0}, & m = 2m_0k + 2s_0 \in W_{m_0',s_0}, \\ -\frac{1}{ka-(k-1)} L_{1-2m_0k-2s_0}, & m = 1 - 2m_0k - 2s_0 \in W_{m_0',s_0}, \\ 0, & m \notin W_{m_0',s_0}; \end{cases}$$

$$R_{0_4}(L_m) = \begin{cases} L_{m_1}, & m = m_1, \\ 0, & m \neq m_1; \end{cases}$$

$$R_{0_5}(L_m)=\begin{cases} L_{m_1}, & m=m_1,\\ bL_{1-m_1}, & m=1-m_1,\\ 0, & m\neq m_1, 1-m_1.\end{cases}$$

Here $m_1, m_0, m_0', s_0 \in \mathbb{Z}$, $m_1 \neq 0,1$; $m_0 > 0$; $1 \le s_0 < m_0'$; $a, b \in F$, $a \neq (k-1)/k$, $b \neq 0$, $W_{m_0} = \{2m_0k \mid k \in \mathbb{Z}\} \cup \{1-2m_0k \mid k \in \mathbb{Z}\}$, $W_{m_0', s_0} = \{2m_0'k + 2s_0 \mid k \in \mathbb{Z}\} \cup \{1-2m_0'k-2s_0 \mid k \in \mathbb{Z}\}$.

Proof. This follows from Theorems 3.2, 3.12, 3.15 and 3.19. ∎

For convenience, denote $\lambda_k = ka-(k-1)$ for all $k \in \mathbb{Z}$, $a \neq (k-1)/k$, and denote the multiplication $[\,,\,,\,]_{R_{0_i}}$ defined as in (3.14) by $[\,,\,,\,]_i$, $1 \le i \le 5$. Then we obtain 3-Lie algebras $(A, [\,,\,,\,]_i)$ with the homogeneous Rota–Baxter operators R_{0_i} for $1 \le i \le 5$, where $A = A_\omega$ as vector spaces. Below, we omit the zero products of basis vectors in the multiplication of the 3-Lie algebras $[A, [\,,\,,\,]_i])$ for $1 \le i \le 5$.

(1) $([A, [\,,\,,\,]_1)$ with the multiplication, for all $m \in \mathbb{Z}$, $m \neq 0,1$,

$$[L_0, L_1, L_m]_1 = b(2m-1+(-1)^m)L_m, \qquad b \in F,\, b \neq 0.$$

(2) $([A, [\,,\,,\,]_2)$ with the multiplication

$$[L_0, L_1, L_{2m}]_2 = -4mL_{2m},$$
$$[L_0, L_1, L_{2m+1}]_2 = -4mL_{2m+1},$$
$$[L_0, L_{1-2m_0k_1}, L_{2m}]_2 = \tfrac{-4m}{\lambda_{k_1}} L_{2m-2m_0k_1},$$
$$[L_0, L_{2m_0k_1}, L_{2m+1}]_2 = -\tfrac{4m_0k_1}{\lambda_{k_1}} L_{2m+2m_0k_1},$$
$$[L_1, L_{2m_0k_1}, L_{2m}]_2 = -\tfrac{4m_0k_1-4m}{\lambda_{k_1}} L_{2m+2m_0k_1},$$
$$[L_1, L_{2m_0k_1}, L_{2m+1}]_2 = \tfrac{4m}{\lambda_{k_1}} L_{2m+2m_0k_1+1},$$
$$[L_1, L_{1-2m_0k_1}, L_{2m}]_2 = -\tfrac{4m_0k_1}{\lambda_{k_1}} L_{2m-2m_0k_1+1},$$
$$[L_0, L_{2m_0k_1}, L_{1-2m_0k_2}]_2 = \tfrac{-4m_0k_1(\lambda_{k_2}-\lambda_{k_1}-1)}{\lambda_{k_1}\lambda_{k_2}} L_{2m_0(k_1-k_2)},$$
$$[L_0, L_{1-2m_0k_1}, L_{1-2m_0k_2}]_2 = \tfrac{4m_0(k_1-k_2)(-\lambda_{k_2}-\lambda_{k_1}+1)}{\lambda_{k_1}\lambda_{k_2}} L_{-2m_0(k_1+k_2)+1},$$
$$[L_0, L_{1-2m_0k_1}, L_{2m+1}]_2 = -\tfrac{4m+4m_0k_1}{\lambda_{k_1}} L_{2m-2m_0k_1+1},$$
$$[L_1, L_{2m_0k_1}, L_{1-2m_0k_2}]_2 = \tfrac{4m_0k_2(\lambda_{k_1}-\lambda_{k_2}-1)}{\lambda_{k_1}\lambda_{k_2}} L_{2m_0(k_1-k_2)+1},$$
$$[L_1, L_{2m_0k_1}, L_{2m_0k_2}]_2 = \tfrac{4m_0(k_1-k_2)(-\lambda_{k_2}-\lambda_{k_1}+1)}{\lambda_{k_1}\lambda_{k_2}} L_{2m_0(k_1+k_2)},$$
$$[L_{2m_0k_1}, L_{1-2m_0k_2}, L_{2m}]_2 = -\tfrac{4m-4m_0k_1}{\lambda_{k_1}\lambda_{k_2}} L_{2m+2m_0(k_1-k_2)},$$

$$[L_{2m_0k_1}, L_{1-2m_0k_2}, L_{2m+1}]_2 = -\tfrac{4m+4m_0k_2}{\lambda_{k_1}\lambda_{k_2}} L_{2m+2m_0(k_1-k_2)+1},$$
$$[L_{2m_0k_1}, L_{2m_0k_2}, L_{1-2m_0k_3}]_2 = \tfrac{4m_0(k_1-k_2)(\lambda_{k_3}-\lambda_{k_2}-\lambda_{k_1})}{\lambda_{k_1}\lambda_{k_2}\lambda_{k_3}} L_{2m_0(k_1+k_2-k_3)},$$
$$[L_{2m_0k_1}, L_{1-2m_0k_2}, L_{1-2m_0k_3}]_2 = \tfrac{4m_0(k_2-k_3)(\lambda_{k_1}-\lambda_{k_2}-\lambda_{k_3})}{\lambda_{k_1}\lambda_{k_2}\lambda_{k_3}} L_{2m_0(k_1-k_2-k_3)+1},$$
$$[L_{2m_0k_1}, L_{2m_0k_2}, L_{2m+1}]_2 = \frac{4m_0(k_1-k_2)}{\lambda_{k_1}\lambda_{k_2}} L_{2m+2m_0(k_1+k_2)},$$
$$[L_{1-2m_0k_1}, L_{1-2m_0k_2}, L_{2m}]_2 = \frac{4m_0(k_1-k_2)}{\lambda_{k_1}\lambda_{k_2}} L_{2m-2m_0(k_1+k_2)+1},$$

for all $2m+1, 2m \in \mathbb{Z}$ and $2m, 2m+1 \notin W_{m_0}$, where $m_0 \in \mathbb{Z}$, $m_0 > 0$.

(2) $([A, [\,,\,,\,]_3)$ with the multiplication, for all $2m+1, 2m \in \mathbb{Z}$ and $2m, 2m+1 \notin W_{m_0,s_0}$,

$$[L_{2m_0k_1+2s_0}, L_{2m_0k_2+2s_0}, L_{1-2m_0k_3-2s_0}]_3 = \tfrac{4m_0(k_1-k_2)(\lambda_{k_3}-\lambda_{k_2}-\lambda_{k_1})}{\lambda_{k_1}\lambda_{k_2}\lambda_{k_3}} L_{2m_0(k_1+k_2-k_3)+2s_0},$$
$$[L_{2m_0k_1+2s_0}, L_{1-2m_0k_2-2s_0}, L_{1-2m_0k_3-2s_0}]_3 = \tfrac{4m_0(k_2-k_3)(\lambda_{k_1}-\lambda_{k_2}-\lambda_{k_3})}{\lambda_{k_1}\lambda_{k_2}\lambda_{k_3}} L_{2m_0(k_1-k_2-k_3)-2s_0+1},$$
$$[L_{2m_0k_1+2s_0}, L_{1-2m_0k_2-2s_0}, L_{2m+1}]_3 = -\tfrac{4(m+m_0k_2+s_0)}{\lambda_{k_1}\lambda_{k_2}} L_{2m+2m_0(k_1-k_2)+1},$$
$$[L_{2m_0k_1+2s_0}, L_{1-2m_0k_2-2s_0}, L_{2m}]_3 = \tfrac{4(m-m_0k_1-s_0)}{\lambda_{k_1}\lambda_{k_2}} L_{2m+2m_0(k_1-k_2)},$$
$$[L_{2m_0k_1+2s_0}, L_{2m_0k_2+2s_0}, L_{2m+1}]_3 = \tfrac{4m_0(k_1-k_2)}{\lambda_{k_1}\lambda_{k_2}} L_{2m+2m_0(k_1+k_2)+4s_0},$$
$$[L_{1-2m_0k_1-2s_0}, L_{1-2m_0k_2-2s_0}, L_{2m}]_3 = \tfrac{4m_0(k_1-k_2)}{\lambda_{k_1}\lambda_{k_2}} L_{2m-2m_0(k_1+k_2)-4s_0+1},$$

where $m_0, s_0 \in \mathbb{Z}$, $1 \le s_0 < m_0$.

(4) $([A, [\,,\,,\,]_4)$ is an abelian 3-Lie algebra.

(6) $([A, [\,,\,,\,]_5)$ with the multiplication, for all $m \in \mathbb{Z}$, $m \ne m_1$,

$$[L_{m_1}, L_{1-m_1}, L_m]_5 = bD(m_1, 1-m_1, m)L_m,$$

where $m_1 \in \mathbb{Z}$, $m_1 \ne 0, 1$, $b \in F$, $b \ne 0$.

Acknowledgements. The first author was supported in part by the Natural Science Foundation (11371245) and the Natural Science Foundation of Hebei Province (A2014201006).

REFERENCES

[A] M. Aguiar, *Pre-Poisson algebras*, Lett. Math. Phys. 54 (2000), 263–277.

[AI] J. A. de Azcárraga and J. M. Izquierdo, *n-ary algebras: a review with applications*, J. Phys. A 43 (2010), 293001; arXiv:1005.1028 [math-ph] (2010).

[BL] J. Bagger and N. Lambert, *Gauge symmetry and supersymmetry of multiple $M2$-branes*, Phys. Pev. D 77 (2008), 065008.

[Bai] C. Bai, *A unified algebraic approach to classical Yang–Baxter equation*, J. Phys. A 40 (2007), 11073–11082.

[BBGN] C. Bai, O. Bellier, L. Guo and X. Ni, *Spliting of operations, Manin products, and Rota–Baxter operators*, Int. Math. Res. Notices 2013, 485–524.

[BGN1] C. Bai, L. Guo and X. Ni, *Generalizations of the classical Yang–Baxter equation and $\mathcal{O}$-operators*, J. Math. Phys. 52 (2011), 063515, 17 pp.

[BGN2] C. Bai, L. Guo and X. Ni, *Nonabelian generalized Lax pairs, the classical Yang–Baxter equation and PostLie algebras*, Comm. Math. Phys. 297 (2010), 553–596.

[BBW] R. Bai, C. Bai and J. Wang, *Realizations of 3-Lie algebras*, J. Math. Phys. 51 (2010), 063505.

[BGL] R. Bai, L. Guo and J. Li, *Rota–Baxter 3-Lie algebras*, J. Math. Phys. 54 (2013), 063504.

[BHB] R. Bai, W. Han and C. Bai, *The generating index of an n-Lie algebra*, J. Phys. A 44 (2011), 185201, 14 pp.

[BLZ] R. Bai, H. Liu and M. Zhang, *3-Lie algebras realized by cubic matrices*, Chin. Ann. Math. 35B (2014), 261–270.

[BSZ1] R. Bai, C. Shen and Y. Zhang, *3-Lie algebras with an ideal N*, Linear Algebra Appl. 431 (2009), 673–700.

[BSZ2] R. Bai, G. Song and Y. Zhang, *On classification of n-Lie algebras*, Front. Math. China 6 (2011), 581–606.

[BW] R. Bai and Y. Wu, *Constructions of 3-Lie algebras*, Linear Multilinear Algebra 63 (2015), 2171–2186.

[Bt] G. Baxter, *An analytic problem whose solution follows from a simple algebraic identity*, Pacific J. Math. 10 (1960), 731–742.

[Bv] A. A. Belavin, *Dynamical symmetry of integrable quantum systems*, Nuclear Phys. B 180 (1981), 189–200.

[Ca] P. Cartier, *On the structure of free Baxter algebras,* Adv. Math. 9 (1972), 253–265.

[CK] A. Connes and D. Kreimer, *Hopf algebras, renormalisation and noncommutative geometry*, Comm. Math. Phys. 199 (1988), 203–242.

[Dzhu] A. S. Dzhumadil'daev, *Identities and derivations for Jacobi algebras*, arXiv: math/0202040v1 (2002).

[EGK] K. Ebrahimi-Fard, L. Guo and D. Kreimer, *Spitzer's identity and the algebraic Birkhoff decomposition in pQFT*, J. Phys. A 37 (2004), 11037–11052.

[EGM] K. Ebrahimi-Fard, L. Guo and D. Manchon, *Birkhoff type decompositions and the Baker–Campbell–Hausdorff recursion*, Comm. Math. Phys. 267 (2006), 821–845.

[F] V. T. Filippov, *n-Lie algebras*, Sibirsk. Mat. Zh. 26 (1985), 126–140 (in Russian).

[G1] L. Guo, *What is a Rota–Baxter algebra?*, Notices Amer. Math. Soc. 56 (2009), 1436–1437.

[G2] L. Guo, *Introduction to Rota–Baxter Algebra*, Int. Press and Higher Education Press, 2012.

[GK1] L. Guo and W. Keigher, *Baxter algebras and shuffle products*, Adv. Math. 150 (2000), 117–149.

[GK2] L. Guo and W. Keigher, *On differential Rota–Baxter algebras*, J. Pure Appl. Algebra 212 (2008), 522–540.

[GZ] L. Guo and B. Zhang, *Renormalization of multiple zeta values*, J. Algebra 319 (2008), 3770–3809.

[G] A. Gustavsson, *Algebraic structures on parallel M2-branes*, Nucl. Phys. B 811 (2009), 66–76.

[HCK] P. Ho, M. Chebotar and W. Ke, *On skew-symmetric maps on Lie algebras*, Proc. Roy. Soc. Edinburgh Sect. A 113 (2003), 1273–1281.

[HHM] P. Ho, R. Hou and Y. Matsuo, *Lie 3-algebra and multiple M2-branes*, J. High Energy Phys. 2008, no. 6, 020, 30 pp.

[HIM] P. Ho, Y. Imamura and Y. Matsuo, *M2 to D2 revisited*, J. High Energy Phys. 2008, no. 7, 003, 17 pp.

[K] S. Kasymov, *On a theory of n-Lie algebras*, Algebra i Logika 26 (1987), 277–297 (in Russian).

[LHB] X. Li, D.Hou and C. Bai, *Rota–Baxter operators on pre-Lie algebras*, J. Nonlinear Math. Phys. 14 (2007), 269–289.

[L] W. Ling, *On the structure of n-Lie algebras*, Dissertation, University-GHS-Siegen, Siegen, 1993.

[MP] D. Manchon and S. Paycha, *Nested sums of symbols and renormalised multiple zeta values*, Int. Math. Res. Papers 24 (2010), 4628–4697.

[N] Y. Nambu, *Generalized Hamiltonian dynamics*, Phys. Pev. D 7 (1973), 2405–2412.

[P] G. Papadopoulos, *M2-branes, 3-Lie algebras and Plucker relations*, J. High Energy Phys. 2008, no. 5, 054, 9 pp.

[Poz1] A. P. Pozhidaev, *Simple quotient algebras and subalgebras of Jacobian algebras*, Siberian Math. J. 39 (1998), 512–517.

[Poz2] A. P. Pozhidaev, *Monomial n-Lie algebras*, Algebra i Logika 37 (1998), 542–567 (in Russian).

[Ro1] G.-C. Rota, *Baxter algebras and combinatorial identities I, II*, Bull. Amer. Math. Soc. 75 (1969), 325–329, 330–334.

[Ro2] G.-C. Rota, *Baxter operators, an introduction*, in: Gian-Carlo Rota on Combinatorics, Introductory Papers and Commentaries, J. P. S. Kung (ed.), Birkhäuser, Boston, 1995, 504–512.

[STS] M. A. Semenov-Tian-Shansky, *What is a classical r-matrix?*, Funct. Anal. Appl. 17 (1983), 259–272.

[T] L. Takhtajan, *On foundation of the generalized Nambu mechanics*, Comm. Math. Phys. 160 (1994), 295–315.

Ruipu Bai, Yinghua Zhang
College of Mathematics and Information Science
Hebei University
Baoding 071002, China
E-mail: bairuipu@126.com
zhangyinghua1234@163.com