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NEW RAMANUJAN-TYPE CONGRUENCES FOR 4-CORE PARTITIONS

BY

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Abstract. A partition is called a t-core if none of its hook lengths is divisible by t. Let $a_t(n)$ denote the number of t-cores of n. We obtain two infinite families of congruences modulo 5 for $a_4(n)$. For example, we prove that for $\ell \geq 1$ and $n \geq 0$,

$$a_4\left(5^{2\ell+1}n + \frac{21 \cdot 5^{2\ell} - 5}{8}\right) \equiv 0 \pmod 5.$$

We also establish three infinite families of congruences modulo 4.

1. Introduction. A partition is called a *t-core* if none of its hook lengths is divisible by t. Let $a_t(n)$ be the number of t-cores of n. From Garvan, Kim and Stanton [GKS], we know that the generating function for $a_t(n)$ is given by the following nice infinite product:

$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{(q^t;q^t)_\infty^t}{(q;q)_\infty}. \tag{1.1}$$

Throughout the paper, we will use the following standard q-series notation:

$$(a;q)_\infty := \lim_{n\to\infty} \prod_{k=1}^{n} (1 - aq^{k-1}).$$

The arithmetic properties of t-cores have been widely studied by many mathematicians (see, for example, [BO, C1, C2, G, HS1–HS4, KS, OS, RS, X]). In [HS1], Hirschhorn and Sellers established the congruences

$$a_4(9n+2) \equiv 0 \pmod 2, \tag{1.2}$$

$$a_4(9n+8) \equiv 0 \pmod 4. \tag{1.3}$$

They also found criteria for the parity of $a_4(n)$. Furthermore, in [HS2] they

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proved four unexpected relations for $a_4(n)$: for instance, for $\lambda \geq 1$, $n \geq 0$,

$$a_4(3^{2\lambda+1}n + (5 \cdot 3^{2\lambda} - 5)/8) = 3^{\lambda} a_4(3n). \tag{1.4}$$

In fact, in [HS1] and [HS2], Hirschhorn and Sellers established two new congruences modulo 8 and 16: for $n \geq 0$,

$$a_4(81n + 23) \equiv 0 \pmod 8, \tag{1.5}$$

$$a_4(81n + 77) \equiv 0 \pmod{16}. \tag{1.6}$$

In [HS2] they also conjectured similar multiplicative properties for $a_4(n)$ for other primes p, which were later confirmed by Ono and Sze [OS] by using the index formulae for class numbers. Recently, Baruah and Nath [BN] established infinite families of arithmetic identities for $a_4(n)$ in arithmetic progressions modulo powers of 5. For example, for $n \geq 0$ and $k \geq 1$,

$$a_4(125n + 65) = 5a_4(5n + 2), \tag{1.7}$$

$$a_4(125n + 90) = 5a_4(5n + 3), \tag{1.8}$$

$$a_4\left(5^{2k+2}n + \frac{5^{2k+1} - 5}{8}\right) = \frac{5^{k+1} - 1}{4} a_4(25n). \tag{1.9}$$

In this paper, we shall prove the following two theorems.

THEOREM 1.1. *For $\ell \geq 1$ and $n \geq 0$, we have*

$$a_4\left(5^{2\ell+1}n + \frac{21 \cdot 5^{2\ell} - 5}{8}\right) \equiv 0 \pmod 5, \tag{1.10}$$

$$a_4\left(5^{2\ell+1}n + \frac{29 \cdot 5^{2\ell} - 5}{8}\right) \equiv 0 \pmod 5. \tag{1.11}$$

REMARK. The case $\ell = 1$ of Theorem 1.1 also follows from (1.7) and (1.8).

THEOREM 1.2. *For $\ell, n \geq 0$, we have*

$$a_4\left(3^{4\ell+2}n + \frac{23 \cdot 3^{4\ell+1} - 5}{8}\right) \equiv 0 \pmod 4, \tag{1.12}$$

$$a_4\left(3^{4\ell+4}n + \frac{7 \cdot 3^{4\ell+3} - 5}{8}\right) \equiv 0 \pmod 4, \tag{1.13}$$

$$a_4\left(3^{4\ell+4}n + \frac{23 \cdot 3^{4\ell+3} - 5}{8}\right) \equiv 0 \pmod 4. \tag{1.14}$$

2. Preliminaries. Let $f(a, b)$ be Ramanujan's general theta function given by

$$f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1.$$

Jacobi's triple product identity can be stated in Ramanujan's notation as follows:

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

Thus,

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty},$$

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2},$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty.$$

It is easy to see that

$$\frac{\psi(q)^5}{\varphi(q)^2} = \frac{(q^4;q^4)_\infty^4}{(q;q)_\infty}.$$

Therefore, we can rewrite the generating function for $a_4(n)$ as follows:

$$\sum_{n=0}^{\infty} a_4(n)q^n = \frac{\psi(q)^5}{\varphi(q)^2}. \tag{2.1}$$

We now list the necessary preliminary results.

Lemma 2.1 ([B, p. 49, Corollary]).

$$\varphi(q) = \varphi(q^{25}) + 2qf(q^{15},q^{35}) + 2q^4 f(q^5,q^{45}), \tag{2.2}$$

$$\psi(q) = f(q^{10},q^{15}) + qf(q^5,q^{20}) + q^3\psi(q^{25}), \tag{2.3}$$

$$\psi(q) = f(q^3,q^6) + q\psi(q^9). \tag{2.4}$$

Lemma 2.2 ([AB, p. 26, Eq. 1.6.6]).

$$\varphi^2(q) - \varphi^2(q^5) = 4qf(q,q^9)f(q^3,q^7). \tag{2.5}$$

To end this section, we state a congruence relation which will be frequently applied throughout the paper without explicit mention.

Lemma 2.3. *For any prime p,*

$$(q;q)_\infty^p \equiv (q^p;q^p)_\infty \pmod{p}. \tag{2.6}$$

Proof. By the binomial theorem, we have

$$(1-q)^p \equiv 1 - q^p \pmod{p},$$

which yields the desired congruence relation. ∎

3. Infinite families of congruences modulo 5. The goal of this section is to prove Theorem 1.1. We first establish the following results.

THEOREM 3.1. *For $\ell \geq 1$,*

$$\sum_{n=0}^{\infty} a_4\left(5^{2\ell-1}n + \frac{5^{2\ell-1}-5}{8}\right)q^n \equiv \varphi(q)\psi(q)\varphi(q^5) \pmod{5}, \tag{3.1}$$

$$\sum_{n=0}^{\infty} a_4\left(5^{2\ell}n + \frac{5^{2\ell+1}-5}{8}\right)q^n \equiv \varphi(q)\varphi(q^5)\psi(q^5) \pmod{5}. \tag{3.2}$$

Proof. From (2.1), we see that

$$\sum_{n=0}^{\infty} a_4(n)q^n \equiv \frac{\psi(q^5)}{\varphi(q^5)}\varphi(q)^3 \pmod{5}.$$

By (2.2), we obtain

$$\sum_{n=0}^{\infty} a_4(n)q^n \equiv \frac{\psi(q^5)}{\varphi(q^5)} \cdot \left(\varphi(q^{25}) + 2qf(q^{15},q^{35}) + 2q^4f(q^5,q^{45})\right)^3 \pmod{5}.$$

Then

$$\sum_{n=0}^{\infty} a_4(5n)q^{5n} \equiv \frac{\psi(q^5)}{\varphi(q^5)} \cdot \left(\varphi^3(q^{25}) + 24q^5f(q^{15},q^{35})f(q^5,q^{45})\varphi(q^{25})\right) \pmod{5}.$$

This yields

$$\sum_{n=0}^{\infty} a_4(5n)q^n \equiv \frac{\psi(q)}{\varphi(q)} \cdot \varphi(q^5) \cdot \left(\varphi^2(q^5) + 4qf(q^3,q^7)f(q,q^9)\right) \pmod{5}.$$

By Lemma 2.2, we find that

$$\sum_{n=0}^{\infty} a_4(5n)q^n \equiv \varphi(q)\psi(q)\varphi(q^5) \pmod{5}. \tag{3.3}$$

By Lemma 2.1,

$$\sum_{n=0}^{\infty} a_4(5n)q^n \equiv \varphi(q^5)\left(\varphi(q^{25}) + 2qf(q^{15},q^{35}) + 2q^4f(q^5,q^{45})\right)$$
$$\times \left(f(q^{10},q^{15}) + qf(q^5,q^{20}) + q^3\psi(q^{25})\right) \pmod{5},$$

from which it follows that

$$\sum_{n=0}^{\infty} a_4(5(5n+3))q^{5n+3} \equiv q^3\varphi(q^5)\varphi(q^{25})\psi(q^{25}) \pmod{5}.$$

Thus,

$$\sum_{n=0}^{\infty} a_4(25n+15)q^n \equiv \varphi(q)\varphi(q^5)\psi(q^5) \pmod{5}. \tag{3.4}$$

Using (2.2) again, it is easy to derive that

(3.5)
$$\sum_{n=0}^{\infty} a_4(25(5n)+15)q^n = \sum_{n=0}^{\infty} a_4(125n+15)q^n \equiv \varphi(q)\psi(q)\varphi(q^5) \pmod 5.$$

From (3.3) and (3.5), we see that

$$a_4(5n) \equiv a_4(125n+15) \pmod 5. \tag{3.6}$$

Using (3.3), (3.4) and (3.6), by induction, we easily establish the desired results. ■

Proof of Theorem 1.1. From (3.2) and (2.2), we have

$$\sum_{n=0}^{\infty} a_4(5^{2\ell}(5n+2) + \frac{5^{2\ell+1}-5}{8})q^n \equiv 0 \pmod 5,$$
$$\sum_{n=0}^{\infty} a_4(5^{2\ell}(5n+3) + \frac{5^{2\ell+1}-5}{8})q^n \equiv 0 \pmod 5.$$

By equating the coefficients of q^n, the desired results immediately follow. ■

4. Infinite families of congruences modulo 4. In this section, we prove Theorem 1.2. We will need the following lemma.

LEMMA 4.1. *We have*

$$2f^3(q,q^2) \equiv 2\psi(q) \pmod 4.$$

Proof. It suffices to note that

$$f^3(q,q^2) \equiv f^3(-q,-q^2) \equiv (q;q)_\infty^3 \equiv \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} \pmod 2. \ ■$$

THEOREM 4.2. *For $\ell \geq 0$, we have*

$$\sum_{n=0}^{\infty} a_4\left(3^{4\ell}n + \frac{5\cdot 3^{4\ell}-5}{8}\right)q^n \equiv \psi^5(q) \pmod 4, \tag{4.1}$$

$$\sum_{n=0}^{\infty} a_4\left(3^{4\ell+1}n + \frac{7\cdot 3^{4\ell+1}-5}{8}\right)q^n \equiv q\psi^5(q^3) + 2\psi(q)\psi^2(q^3) \pmod 4, \tag{4.2}$$

$$\sum_{n=0}^{\infty} a_4\left(3^{4\ell+2}n + \frac{5\cdot 3^{4\ell+2}-5}{8}\right)q^n \equiv \psi^5(q) + 2\psi(q^3)\psi^2(q) \pmod 4, \tag{4.3}$$

$$\sum_{n=0}^{\infty} a_4\left(3^{4\ell+3}n + \frac{7\cdot 3^{4\ell+3}-5}{8}\right)q^n \equiv q\psi^5(q^3) \pmod 4. \tag{4.4}$$

Proof. Since $\varphi(q)^2 \equiv 1 \pmod 4$, from (2.1) we see that

$$\sum_{n=0}^{\infty} a_4(n)q^n \equiv \psi^5(q) \pmod 4. \tag{4.5}$$

Applying (2.4), we find that

$$\sum_{n=0}^{\infty} a_4(3n+2)q^{3n+2} \equiv q^5\psi^5(q^9) + 2q^2 f^3(q^3, q^6)\psi^2(q^9) \pmod 4.$$

Thus,

$$\sum_{n=0}^{\infty} a_4(3n+2)q^n \equiv q\psi^5(q^3) + 2f^3(q, q^2)\psi^2(q^3) \pmod 4.$$

Applying Lemma 4.1, we have

$$\sum_{n=0}^{\infty} a_4(3n+2)q^n \equiv q\psi^5(q^3) + 2\psi(q)\psi^2(q^3) \pmod 4. \tag{4.6}$$

By invoking (2.4) again, it follows that

$$\begin{aligned}\sum_{n=0}^{\infty} a_4(3(3n+1)+2)q^n \\ = \sum_{n=0}^{\infty} a_4(9n+5)q^n \equiv \psi^5(q) + 2\psi(q^3)\psi^2(q) \pmod 4.\end{aligned} \tag{4.7}$$

Using (4.5), we can rewrite the above equation as

$$\sum_{n=0}^{\infty} a_4(9n+5)q^n \equiv \sum_{n=0}^{\infty} a_4(n)q^n + 2\psi(q^3)\psi^2(q) \pmod 4.$$

Employing (2.4), we obtain

$$\sum_{n=0}^{\infty} a_4(27n+23)q^n \equiv \sum_{n=0}^{\infty} a_4(3n+2)q^n + 2\psi(q)\psi^2(q^3) \pmod 4.$$

Using (4.6), we see that

$$\sum_{n=0}^{\infty} a_4(27n+23)q^n \equiv q\psi^5(q^3) \pmod 4. \tag{4.8}$$

This yields

$$\sum_{n=0}^{\infty} a_4(27(3n+1)+23)q^n \equiv \psi^5(q) \pmod 4$$

and

$$a_4(81n+50) \equiv a_4(n) \pmod 4. \tag{4.9}$$

Using (4.5)–(4.8) and (4.9), by induction, it is easy to establish the desired results. ∎

Proof of Theorem 1.2. Employing (4.2) yields

$$a_4\left(3^{4\ell+1}(3n+2)+\frac{7\cdot 3^{4\ell+1}-5}{8}\right)\equiv 0 \pmod 4.$$

Furthermore, from (4.4), it can be seen that

$$a_4\left(3^{4\ell+3}\cdot 3n+\frac{7\cdot 3^{4\ell+3}-5}{8}\right)\equiv 0 \pmod 4,$$

$$a_4\left(3^{4\ell+3}\cdot (3n+2)+\frac{7\cdot 3^{4\ell+3}-5}{8}\right)\equiv 0 \pmod 4. \blacksquare$$

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