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Completeness of symmetric Δ-normed spaces of τ-measurable operators

by

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Abstract. Let $\mathcal{M}$ be an arbitrary semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ and let E be a complete symmetric Δ-normed function space. We show that the corresponding symmetric space $E(\mathcal{M},\tau)$ of τ-measurable operators in $S(\mathcal{M},\tau)$ is a complete symmetrically Δ-normed ideal.

1. Introduction. Let H be a complex separable Hilbert space and let $\mathcal{M}$ be an arbitrary semifinite von Neumann algebra acting on H equipped with a faithful normal semifinite trace τ. Let $S(\mathcal{M},\tau)$ be the $*$-algebra of all τ-measurable operators affiliated with $\mathcal{M}$ (see Section 2 for the precise definition). In the special case when $\mathcal{M} = B(H)$ is the type I von Neumann algebra of all bounded operators on H, the algebra $S(\mathcal{M},\tau)$ coincides with the algebra $B(H)$ itself.

The main object of this paper is to study the "Calkin correspondence"

$$E \leftrightarrow E(\mathcal{M},\tau),$$

where E is an absolutely solid rearrangement invariant function space and $E(\mathcal{M},\tau)$ is the corresponding bimodule of τ-measurable operators in $S(\mathcal{M},\tau)$. This correspondence was introduced by J. Calkin [Cal] in the special case when E is a symmetric sequence space on $\mathbb{N}$ and $\mathcal{M} = B(H)$. When the algebra $\mathcal{M}$ is a factor of type I, II_1, or II_∞ and E lives, respectively, on $\mathbb{N}, (0,1)$, or $(0,\infty)$, the Calkin correspondence is one-to-one (see e.g. [LSZ, Chapter 2]).

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When E is equipped with a symmetric norm (or quasi-norm) $\|\cdot\|_E$, the Calkin correspondence can be extended to

$$(E, \|\cdot\|_E) \leftrightarrow (E(\mathcal{M},\tau), \|\cdot\|_{E(\mathcal{M},\tau)}), \tag{1.1}$$

where $\|\cdot\|_{E(\mathcal{M},\tau)}$ is a symmetric norm (or quasi-norm) on $E(\mathcal{M},\tau)$, explicitly determined by $\|\cdot\|_E$ (see details in Section 3 below). In this case, it is shown in [KSu, Su] that the correspondence (1.1) preserves completeness.

The first main objective of this paper is to extend these results to the case when $\|\cdot\|_E$ is a symmetric Δ-norm. We recall that every topological vector space with a countable base of neighbourhoods of zero and so satisfying the first axiom of countability can be equipped with a Δ-norm (see e.g. [K, Chapter 3]). In general, the topology of a Δ-normed space does not need to be either locally bounded or locally convex, and, in fact, the class of (symmetric) Δ-normed spaces is strictly larger than the class of all (quasi-)normed (symmetric) spaces. The question whether the noncommutative Δ-normed space $(E(\mathcal{M},\tau), \|\cdot\|_{E(\mathcal{M},\tau)})$ is complete, provided that its commutative counterpart, a symmetric Δ-normed function space $(E, \|\cdot\|_E)$ is complete, has been tackled before (see e.g. [B, Ci]) for some special classes of Δ-normed spaces. Our first main result, Theorem 3.8, shows that the correspondence (1.1) preserves completeness also in the more general framework of Δ-normed spaces. This result complements Nelson's well-known result [Nel] (underlying the modern theory of noncommutative integration) that the Δ-space $S(\mathcal{M},\tau)$ is complete with respect to the Δ-norm defined via the measure topology. It should be noted though that the completeness of the Δ-normed space $S(\mathcal{M},\tau)$ together with the techniques developed in [KSu, LSZ, Su] are crucially used in our approach.

In the special case, when $\mathcal{M}$ is a type I factor, A. Pietsch [Pi1], [Pi2] suggested supplementing Calkin correspondence with another correspondence (which we shall refer to as "Pietsch correspondence"). More precisely, he introduced another class of sequence spaces, called shift-monotone, and suggested a one-to-one correspondence between shift-monotone sequence spaces, symmetric sequence spaces and two-sided ideals of compact operators. In [LPSZ] it was established that this correspondence preserves completeness when these spaces are equipped with the corresponding quasi-norms. Our second main result, Theorem 4.5, extends this result to the case of Δ-normed spaces.

As an example of an important symmetric Δ-normed operator space we consider the space $\exp(L_1)(\mathcal{M},\tau)$ of all operators A affiliated with a type II_1 factor $\mathcal{M}$, equipped with a faithful normal finite trace τ possessing the property that $\log_+(|A|) \in L_1(\mathcal{M},\tau)$, where $\log_+(\lambda) = \max\{\log(\lambda), 0\}$, $\lambda > 0$. U. Haagerup and H. Schultz [HS] showed that both the Brown measure and the Fuglede–Kadison determinant can be extended to the algebra

$\exp(L_1)(\mathcal{M},\tau)$. In [DSZ] it was proved that this algebra equipped with a topology becomes a complete topological $*$-algebra. Our main result, Theorem 3.8, gives an alternative proof of this result.

2. Preliminaries. In this section, we recall main notions of the theory of noncommutative integration and define Δ-normed symmetric operator spaces.

In what follows, H is a Hilbert space, $B(H)$ is the $*$-algebra of all bounded linear operators on H, and $\mathbf{1}$ is the identity operator on H. Let $\mathcal{M}$ be a von Neumann algebra on H. We denote by $P(\mathcal{M})$ the lattice of all projections in $\mathcal{M}$. For details on von Neumann algebra theory, the reader is referred to e.g. [Dix], [KR1], [KR2] or [Tak1]. General facts concerning measurable operators may be found in [Nel], [Se] (see also [Tak2, Chapter IX] and the forthcoming book [DPS]). For the convenience of the reader, some of the basic definitions are recalled.

Recall that two projections $e, f \in P(\mathcal{M})$ are called *equivalent* (notation: $e \sim f$) if there exists a partial isometry $u \in \mathcal{M}$ such that $u^*u = e$ and $uu^* = f$. A projection $0 \neq p \in P(\mathcal{M})$ is called *finite* if the conditions $q \le p$ and $q \sim p$ imply that $q = p$.

A linear operator $X : \mathfrak{D}(X) \to H$, whose domain $\mathfrak{D}(X)$ is a linear subspace of H, is said to be *affiliated* with $\mathcal{M}$ if $YX \subseteq XY$ for all $Y \in \mathcal{M}'$, where $\mathcal{M}'$ is the commutant of $\mathcal{M}$. A linear operator $X : \mathfrak{D}(X) \to H$ is termed *measurable* with respect to $\mathcal{M}$ if X is closed, densely defined, affiliated with $\mathcal{M}$ and there exists a sequence $(P_n)_{n=1}^{\infty}$ in the lattice $P(\mathcal{M})$ of all projections of $\mathcal{M}$, such that $P_n \uparrow \mathbf{1}$, $P_n(H) \subseteq \mathfrak{D}(X)$ and $\mathbf{1} - P_n$ is a finite projection (with respect to $\mathcal{M}$) for all n. The collection of all measurable operators with respect to $\mathcal{M}$ is denoted by $S(\mathcal{M})$, which is a unital $*$-algebra with respect to strong sums and products (denoted simply by $X + Y$ and XY for all $X, Y \in S(\mathcal{M})$).

Let X be a self-adjoint operator affiliated with $\mathcal{M}$. We denote its spectral measure by $\{E^X\}$. It is known that if X is a closed operator affiliated with $\mathcal{M}$ with polar decomposition $X = U|X|$, then $U \in \mathcal{M}$ and $E \in \mathcal{M}$ for all projections $E \in \{E^{|X|}\}$. Moreover, $X \in S(\mathcal{M})$ if and only if X is closed, densely defined, affiliated with $\mathcal{M}$ and $E^{|X|}(\lambda, \infty)$ is a finite projection for some $\lambda > 0$. It follows immediately that when $\mathcal{M}$ is a von Neumann algebra of type *III* or a type *I* factor, we have $S(\mathcal{M}) = \mathcal{M}$. For type *II* von Neumann algebras, this is no longer true. From now on, let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ.

An operator $X \in S(\mathcal{M})$ is called *τ-measurable* if there exists a sequence $(P_n)_{n=1}^{\infty}$ in $P(\mathcal{M})$ such that $P_n \uparrow \mathbf{1}$, $P_n(H) \subseteq \mathfrak{D}(X)$ and $\tau(\mathbf{1} - P_n) < \infty$ for all n. The collection of all τ-measurable operators is a unital $*$-subalgebra of $S(\mathcal{M})$ denoted by $S(\mathcal{M}, \tau)$. It is well known that a linear operator X

belongs to $S(\mathcal{M},\tau)$ if and only if $X \in S(\mathcal{M})$ and there exists $\lambda > 0$ such that $\tau(E^{|X|}(\lambda,\infty)) < \infty$. Alternatively, an unbounded operator X affiliated with $\mathcal{M}$ is τ-measurable (see [FK]) if and only if

$$\tau(E^{|X|}(1/n,\infty)) = o(1), \quad n \to \infty.$$

For convenience of the reader we also recall the definition of the measure topology t_τ on the algebra $S(\mathcal{M},\tau)$. For every $\varepsilon,\delta > 0$, we define

$$V(\varepsilon,\delta) = \{X \in S(\mathcal{M},\tau) : \ \exists P \in P(\mathcal{M}) \text{ with } \|X(\mathbf{1}-P)\| \le \varepsilon, \ \tau(P) \le \delta\}.$$

The topology generated by the sets $V(\varepsilon,\delta)$, $\varepsilon,\delta > 0$, is called the *measure topology* t_τ on $S(\mathcal{M},\tau)$ [DPS, FK, Nel]. It is well known that the algebra $S(\mathcal{M},\tau)$ equipped with the measure topology is a complete metrizable topological algebra [Nel] (see also [MC]). We note that a sequence $(x_n)_{n=1}^\infty \subset S(\mathcal{M},\tau)$ converges to zero with respect to measure topology t_τ if and only if $\tau(E^{|x_n|}(\varepsilon,\infty)) \to 0$ as $n \to \infty$ for all $\varepsilon > 0$ [DPS].

Let L_0 be the space of Lebesgue measurable functions either on $(0,1)$ or $(0,\infty)$ or $\mathbb{N}$, finite almost everywhere (with identification m-a.e.). Here m is Lebesgue measure or else the counting measure on $\mathbb{N}$. Define S as the subalgebra of L_0 which consists of all functions x such that $m(\{|x| > s\})$ is a finite for some s. If m is finite measure, then $L_0 = S$.

The notation $\mu(x)$ stands for the nonincreasing right-continuous rearrangement of $x \in S$ given by

$$\mu(t;x) = \inf\big\{s \ge 0 : m(\{|x| \ge s\}) \le t\big\}.$$

When x is a sequence we denote by $\mu(x)$ the usual decreasing rearrangement of the sequence $|x|$.

Definition 2.1. Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ and let $X \in S(\mathcal{M},\tau)$. The *generalized singular value function* $\mu(X) : t \mapsto \mu(t,X)$ of the operator X is defined by setting

$$\mu(s,X) = \inf\{\|XP\|_\infty : P = P^* \in \mathcal{M} \text{ is a projection}, \ \tau(\mathbf{1}-P) \le s\}.$$

There exists an equivalent definition which involves the distribution function of X. For every self-adjoint operator $X \in S(\mathcal{M},\tau)$, setting

$$d_X(t) = \tau(E^X(t,\infty)), \quad t > 0,$$

we have (see e.g. [FK])

$$\mu(t,X) = \inf\{s \ge 0 : d_{|X|}(s) \le t\}.$$

Consider the algebra $\mathcal{M} = L^\infty(0,\infty)$ of all Lebesgue measurable essentially bounded functions on $(0,\infty)$. The algebra $\mathcal{M}$ can be seen as an abelian von Neumann algebra acting via multiplication on the Hilbert space $\mathcal{H} = L^2(0,\infty)$, with the trace given by integration with respect to Lebesgue measure m. It is easy to see that the set of all τ-measurable operators affiliated with $\mathcal{M}$ can be identified with the space S. It should also be pointed out that the generalized singular value function $\mu(x)$ is precisely the decreasing rearrangement $\mu(x)$ of the function x, defined above.

If $\mathcal{M} = B(H)$ (respectively, l_∞) and τ is the standard trace Tr (respectively, the counting measure on $\mathbb{N}$), then it is not difficult to see that $S(\mathcal{M}) = S(\mathcal{M}, \tau) = \mathcal{M}$. In this case, for $X \in S(\mathcal{M}, \tau)$ we have

$$\mu(n, X) = \mu(t, X), \quad t \in [n, n+1), \quad n \geq 0.$$

The sequence $(\mu(n, X))_{n\geq 0}$ is just the sequence of singular values of the operator X.

For the convenience of the reader, we recall the definition of Δ-norm. Let Ω be a linear space over the field $\mathbb{C}$. A function $\|\cdot\|$ from Ω to $\mathbb{R}$ is a *Δ-norm* if for all $x, y \in \Omega$ the following properties hold:

(1) $\|x\| \geq 0$; $\|x\| = 0 \Leftrightarrow x = 0$;
(2) $\|\alpha x\| \leq \|x\|$ for all $|\alpha| \leq 1$;
(3) $\lim_{\alpha\to 0} \|\alpha x\| = 0$;
(4) $\|x+y\| \leq C_\Omega(\|x\| + \|y\|)$ for a constant $C_\Omega \geq 1$ independent of x, y.

The couple $(\Omega, \|\cdot\|)$ is called a *Δ-normed space*. We note that the definition of a Δ-norm given above is the same as that given in [KPR]. It is well-known that every Δ-normed space $(\Omega, \|\cdot\|)$ is metrizable [KPR] and conversely every metrizable space can be equipped with a Δ-norm (see e.g. [K], [KPR]). Note that properties (2) and (4) of a Δ-norm imply that for any $\alpha \in \mathbb{C}$, there exists a constant M such that $\|\alpha x\| \leq M\|x\|$ for all $x \in \Omega$, in particular, if $(x_n)_{n=1}^\infty \subset \Omega$ with $\|x_n\| \to 0$, then $\|\alpha x_n\| \to 0$.

Let E be a space of real-valued Lebesgue measurable functions either on $(0,1)$ or $(0,\infty)$ (with identification m-a.e.) or on $\mathbb{N}$, equipped with a Δ-norm $\|\cdot\|_E$. The space E is said to be *absolutely solid* if $x \in E$ and $|y| \leq |x|$, $y \in L_0$ implies that $y \in E$ and $\|y\|_E \leq \|x\|_E$. An absolutely solid space $E \subseteq S$ is said to be *symmetric* if for every $x \in E$ and every y the assumption $\mu(y) = \mu(x)$ implies that $y \in E$ and $\|y\|_E = \|x\|_E$ (see e.g. [KPS]).

We now come to the definition of the main object of this paper.

Definition 2.2. Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ. Let $\mathcal{E}$ be a linear subset in $S(\mathcal{M}, \tau)$ equipped with a Δ-norm $\|\cdot\|_{\mathcal{E}}$. We say that $\mathcal{E}$ is a *symmetric operator space* (on $\mathcal{M}$, or in $S(\mathcal{M}, \tau)$) if for all $X \in \mathcal{E}$ and $Y \in S(\mathcal{M}, \tau)$ the assumption $\mu(Y) \leq \mu(X)$ implies that $Y \in \mathcal{E}$ and $\|Y\|_{\mathcal{E}} \leq \|X\|_{\mathcal{E}}$.

The fact that every (normed) symmetric operator space $\mathcal{E}$ is (an absolutely solid) $\mathcal{M}$-bimodule of $S(\mathcal{M}, \tau)$ is well known (see e.g. [KSu, SC] and references therein).

It is clear that in the special case when $\mathcal{M} = L_\infty(0,1)$, $\mathcal{M} = L_\infty(0,\infty)$, or $\mathcal{M} = l_\infty$, the definition of a symmetric Δ-normed operator space coincides with the definition of a symmetric function (or sequence) space. When $\mathcal{M} = B(H)$ and τ is the standard trace Tr, we shall call a symmetric Δ-normed operator space introduced in Definition 2.2 a *symmetric Δ-normed operator ideal* (for symmetrically normed ideals we refer to [GK1, GK2, Si]).

LEMMA 2.3. *Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ and let $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ be a symmetric Δ-normed space in $S(\mathcal{M},\tau)$ with constant $C_{\mathcal{E}}$.*

(i) *If p and q are projections from the algebra $\mathcal{M}$, $p \in \mathcal{E}$ and $\tau(q) \le \tau(p)$, then $q \in \mathcal{E}$ and $\|q\|_{\mathcal{E}} \le \|p\|_{\mathcal{E}}$.*
(ii) *If $p, q \in P(\mathcal{M})$, $p \in \mathcal{E}$ and $p \sim q$, then $q \in \mathcal{E}$ and $\|q\|_{\mathcal{E}} = \|p\|_{\mathcal{E}}$.*
(iii) *If p, q are projections from $\mathcal{E}$, then the projection $p \vee q$ also belongs to $\mathcal{E}$ and $\|p \vee q\|_{\mathcal{E}} \le C_{\mathcal{E}}(\|p\|_{\mathcal{E}} + \|q\|_{\mathcal{E}})$.*

Proof. (i) Since p and q are projections, we have

$$\mu(q) = \chi_{[0,\tau(q))} \le \chi_{[0,\tau(p))} = \mu(p),$$

which implies that $q \in \mathcal{E}$ and $\|q\|_{\mathcal{E}} \le \|p\|_{\mathcal{E}}$ for all n.

(ii) By definition, there exists a partial isometry $v \in \mathcal{M}$ such that $p = v^*v$ and $q = vv^*$. This implies that $p = v^*qv$ and $q = vpv^*$. Hence, $q \in \mathcal{E}$. In addition,

$$\mu(q) = \mu(vpv^*) \le \|v\|_\infty \|v^*\|_\infty \mu(p) = \mu(p),$$

and therefore

$$\|q\|_{\mathcal{E}} \le \|p\|_{\mathcal{E}}.$$

Similarly, $p = v^*qv$ implies that $\|p\|_{\mathcal{E}} \le \|q\|_{\mathcal{E}}$, and so $\|q\|_{\mathcal{E}} = \|p\|_{\mathcal{E}}$.

(iii) Since $p - p\wedge q \le p$, it is clear that $p - p\wedge q \in \mathcal{E}$ and $\|p - p\wedge q\|_{\mathcal{E}} \le \|p\|_{\mathcal{E}}$. Using the fact that

$$p \vee q - q \sim p - p \wedge q,$$

it follows from (ii) that $p \vee q - q \in \mathcal{E}$ and $\|p \vee q - q\|_{\mathcal{E}} = \|p - p \wedge q\|_{\mathcal{E}}$. Since $p \vee q = (p \vee q - q) + q$, it is now clear that $p \vee q \in \mathcal{E}$ and

$$\|p \vee q\|_{\mathcal{E}} \le C_{\mathcal{E}}(\|p \vee q - q\|_{\mathcal{E}} + \|q\|_{\mathcal{E}}) \le C_{\mathcal{E}}(\|p\|_{\mathcal{E}} + \|q\|_{\mathcal{E}}). \ \blacksquare$$

We now extend [DDP, Lemma 4.4] to symmetric Δ-normed operator spaces.

LEMMA 2.4. *Let $\mathcal{M}$ be a semifinite von Neumann algebra, and let $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ be a symmetric Δ-normed space on $\mathcal{M}$ with constant $C_{\mathcal{E}}$. Then the embedding of $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ into $S(\mathcal{M},\tau)$ equipped with the measure topology t_τ is continuous.*

Proof. Let $(X_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{E}$ such that $\|X_n\|_{\mathcal{E}} \to 0$. We claim that $X_n \xrightarrow{t_\tau} 0$ in $S(\mathcal{M}, \tau)$. It is sufficient to show that $\tau(E^{|X_n|}(\varepsilon, \infty)) \to 0$ for all $\varepsilon > 0$. Fix $\varepsilon > 0$ and set $p_n = E^{|X_n|}(\varepsilon, \infty)$. It follows from functional calculus that

$$0 \le E^{|X_n|}(\varepsilon, \infty) \le \frac{1}{\varepsilon}|X_n|E^{|x_n|}(\varepsilon, \infty),$$

and hence for every $n \in \mathbb{N}$ the projection p_n belongs to $\mathcal{E}$ and

$$\|p_n\|_{\mathcal{E}} \le \left\|\frac{1}{\varepsilon}|X_n|E^{|x_n|}(\varepsilon, \infty)\right\|_{\mathcal{E}} \le \text{const } \|X_n\|_{\mathcal{E}} \to 0$$

as $n \to \infty$.

Suppose that $\tau(p_n) \nrightarrow 0$ as $n \to \infty$. Passing to a subsequence, it may be assumed that $\tau(p_n) \ge \delta$ for all n and some $\delta > 0$. Let

$$\gamma = \inf\{\tau(e) : 0 \ne e \in P(\mathcal{M})\}.$$

Assuming first that $\gamma = 0$, there exists $e \in P(\mathcal{M})$ such that $0 < \tau(e) \le \delta \le \tau(p_n)$. Lemma 2.3(i) implies that $e \in \mathcal{E}$ and $0 < \|e\|_{\mathcal{E}} \le \|p_n\|_{\mathcal{E}}$ for all n. This clearly contradicts the fact that $\|p_n\|_{\mathcal{E}} \to 0$ as $n \to \infty$.

Assume now that $\gamma > 0$, in which case each projection p_n dominates a minimal projection $e_n \in P(\mathcal{M})$ satisfying $\tau(e_n) \ge \gamma$. Observe that a fixed minimal projection can only be dominated by finitely many p_n's, as $\|p_n\|_{\mathcal{E}} \to 0$. Therefore, by passing to a subsequence if necessary, it may be assumed that the minimal projections $\{e_n\}_{n=1}^{\infty}$ are mutually distinct. Defining $\alpha = \inf_n \tau(e_n)$, choose $n_0 \in \mathbb{N}$ such that $\tau(e_{n_0}) < 2\alpha$. Since the minimal projections e_n and e_{n+1} are distinct, it follows that $e_n \wedge e_{n+1} = 0$, and hence

$$\tau(e_n \vee e_{n+1}) = \tau(e_n) + \tau(e_{n+1}) \ge 2\alpha, \qquad n \in \mathbb{N}.$$

Consequently,

$$\tau(e_{n_0}) \le 2\alpha \le \tau(e_n \vee e_{n+1}),$$

and so, by Lemma 2.3(i), we have $\|e_{n_0}\|_{\mathcal{E}} \le \|e_n \vee e_{n+1}\|_{\mathcal{E}}$. By Lemma 2.3(iii), this implies that

$$0 < \|e_{n_0}\|_{\mathcal{E}} \le \|e_n \vee e_{n+1}\|_{\mathcal{E}} \le C_{\mathcal{E}}(\|e_n\|_{\mathcal{E}} + \|e_{n+1}\|_{\mathcal{E}}), \qquad n \in \mathbb{N}.$$

Since $\|e_n\|_{\mathcal{E}} \le \|p_n\|_{\mathcal{E}} \to 0$ as $n \to \infty$, this is clearly a contradiction, which completes the proof. ■

3. Calkin correspondence. In this section we introduce the Calkin correspondence between symmetric Δ-normed operator spaces and symmetric Δ-normed function spaces. In Theorem 3.8 below, which is the first main result of this paper, we prove that in this setting the Calkin correspondence preserves completeness.

Let E be a symmetric function space on the positive semi-axis equipped with a Δ-norm $\|\cdot\|_E$, and let $\mathcal{M}$ be an arbitrary semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ. Define

$$E(\mathcal{M},\tau) := \{X \in S(\mathcal{M},\tau) : \mu(X) \in E\}, \qquad \|X\|_{E(\mathcal{M},\tau)} := \|\mu(X)\|_E. \tag{3.1}$$

Our first aim is to prove that $(E(\mathcal{M},\tau), \|\cdot\|_{E(\mathcal{M},\tau)})$ is a symmetric Δ-normed space. Introduce the dilation operator σ_s on $S(0,\infty)$, $s > 0$, by setting

$$(\sigma_s(x))(t) = x(t/s), \qquad t > 0.$$

It is well known that $\mu(X+Y) \le \sigma_2(\mu(X)+\mu(Y))$, $X, Y \in S(\mathcal{M},\tau)$ [LSZ]. We also note that

$$\|\sigma_{2^k} x\|_E \le (2C_E)^k \|x\|_E \tag{3.2}$$

for all $x \in E$ and $k \in \mathbb{N}$ (see e.g. [KPS]).

PROPOSITION 3.1. *Let E be a symmetric Δ-normed function space on $(0,\infty)$ with constant C_E. Then $(E(\mathcal{M},\tau), \|\cdot\|_{E(\mathcal{M},\tau)})$ defined by (3.1) is a symmetric Δ-normed operator space with constant $2C_E^2$.*

Proof. It immediately follows from the definition of $(E(\mathcal{M},\tau), \|\cdot\|_{E(\mathcal{M},\tau)})$ that this space is symmetric and properties (1)–(3) of Δ-norm are satisfied. Thus, we only need to show that $\|\cdot\|_{E(\mathcal{M},\tau)}$ satisfies the quasi-triangular inequality.

Let $X, Y \in E(\mathcal{M},\tau)$. We have

$$\begin{aligned} \|X+Y\|_{E(\mathcal{M},\tau)} &= \|\mu(X+Y)\|_E \le \|\sigma_2(\mu(X)+\mu(Y))\|_E \\ &\overset{(3.2)}{\le} 2C_E\|\mu(X)+\mu(Y)\|_E \\ &\le 2C_E^2(\|X\|_{E(\mathcal{M},\tau)} + \|Y\|_{E(\mathcal{M},\tau)}). \ \blacksquare \end{aligned}$$

Prior to proceeding to the proof of completeness of $(E(\mathcal{M},\tau), \|\cdot\|_{E(\mathcal{M},\tau)})$, we need some more preliminaries.

DEFINITION 3.2 ([KSu, Su]). For all positive $x, y \in l_\infty$, we set

$$[[y,x]] = \inf\Big\{N : y \le \sum_{j=1}^{N} x_j,\ \mu(x_j) \le \mu(x),\ x_j \ge 0\Big\}.$$

We employ the same notation for functions $x, y \in S(0,\infty)$ (or $x, y \in L_0(0,1)$). $[[y,x]]$ is taken to be ∞ if no representation as above exists.

DEFINITION 3.3 ([KSu]). Let $\mathcal{M}$ be a semifinite von Neumann algebra and let $X, Y \in S(\mathcal{M},\tau)$. We say that Y is *uniformly majorized* by X (written

$Y \lhd X$) if there exists $\lambda \in \mathbb{N}$ such that

$$\int\limits_{\lambda a}^{b} \mu(s, Y)\, ds \le \int\limits_{a}^{b} \mu(s, X)\, ds$$

for all a, b such that $0 \le \lambda a < b$.

PROPOSITION 3.4 ([Su, Proposition 13]). *If $x = \mu(x) \in l_\infty$ and $y = \mu(y) \in l_\infty$ are such that $y \lhd x$ with $\lambda = 2$, then*

$$[[y, 2x]] \le 212.$$

The same assertion holds for functions $x, y \in S(0,\infty)$ (or $x, y \in L_0(0,1)$).

COROLLARY 3.5 (cf. [Su, Corollary 14]). *Let E be a symmetric Δ-normed function space on $(0,\infty)$ with constant C_E. If $x = \mu(x) \in E$ and $y = \mu(y) \in l_\infty$ are such that $y \lhd x$ with $\lambda = 2$, then $y \in E$ and $\|y\|_E \le 424 C_E^9 \|x\|_E$. A similar assertion also holds for functions $x, y \in S(0,\infty)$ (or $x, y \in L_0(0,1)$).*

Proof. By Proposition 3.4, we can represent y as $\sum_{k=1}^{212} y_k$ with $\mu(y_k) \le 2\mu(x)$, $1 \le k \le 212$. So, $y \in E$ and applying the quasi-triangular inequality repeatedly we obtain

$$\begin{aligned}\|y\|_E = \Big\|\sum_{k=1}^{212} y_k\Big\|_E &\le C_E\Big(\Big\|\sum_{k=1}^{106} y_k\Big\|_E + \Big\|\sum_{k=107}^{212} y_k\Big\|_E\Big)\\ &\le C_E^8 \sum_{k=1}^{212} \|y_k\|_E \le 212 C_E^8 \|2x\|_E \le 424 C_E^9 \|x\|_E. \ \blacksquare\end{aligned}$$

LEMMA 3.6 ([Su, Lemma 16]). *Let $\mathcal{M}$ be a semifinite von Neumann algebra and let $X_k \in S(\mathcal{M},\tau)$ for all $k \in \mathbb{N}$. If the series $\sum_{k=1}^{\infty} X_k$ converges in measure in $S(\mathcal{M},\tau)$, and $\sum_{k=1}^{\infty} \sigma_{2^k}\mu(X_k)$ converges in measure in S, then*

$$\sum_{k=1}^{\infty} X_k \lhd 2 \sum_{k=1}^{\infty} \sigma_{2^k}\mu(X_k),$$

with constant $\lambda = 2$.

We also need the following straightforward result.

LEMMA 3.7. *Let $(\Omega, \|\cdot\|)$ be a complete Δ-normed space with constant C_Ω. If $x_k \in \Omega$, $k \in \mathbb{N}$, then*

$$\Big\|\sum_{k=1}^{\infty} x_k\Big\| \le \sum_{k=1}^{\infty} C_\Omega^k \|x_k\|. \tag{3.3}$$

Here, the finiteness of the right hand side implies the convergence of the series on the left hand side.

Proof. If $m > n$, then

$$\Big\|\sum_{k=n+1}^{m} x_k\Big\| \le C_\Omega\Big(\|x_{n+1}\| + \Big\|\sum_{k=n+2}^{m} x_k\Big\|\Big)$$
$$\le C_\Omega\|x_{n+1}\| + C_\Omega^2\Big(\|x_{n+2}\| + \|\sum_{k=n+3}^{m} x_k\|\Big) \le \cdots .$$

Hence,

$$\Big\|\sum_{k=n+1}^{m} x_k\Big\| \le \sum_{k=n+1}^{m} C_\Omega^{k-n}\|x_k\| \le \sum_{k=n+1}^{m} C_\Omega^{k}\|x_k\|.$$

Thus, $(\sum_{k=1}^{n} x_k)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\Omega, \|\cdot\|)$. Since $(\Omega, \|\cdot\|)$ is complete, the convergence follows. The inequality (3.3) is now obvious. ■

Now, we are ready to prove the first main result of this paper.

THEOREM 3.8. *Let E be a symmetric function space on the positive semi-axis equipped with a complete Δ-norm $\|\cdot\|_E$ and let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ. Then the space $(E(\mathcal{M},\tau), \|\cdot\|_{E(\mathcal{M},\tau)})$ defined in* (3.1) *is also complete.*

Proof. Let (X_n) be a Cauchy sequence in $(E(\mathcal{M},\tau), \|\cdot\|_{E(\mathcal{M},\tau)})$. By Lemma 2.4, there exists an operator $X \in S(\mathcal{M},\tau)$ such that $X_n \to X$ in measure. We shall prove that $X \in E(\mathcal{M},\tau)$ and $X_n \to X$ in $E(\mathcal{M},\tau)$.

For every $k > 0$, there exists m_k such that $\|X_m - X_{m_k}\|_{E(\mathcal{M},\tau)} \le (2C_E)^{-2k}$ for $m \ge m_k$. Set $Y_k = X_{m_{k+1}} - X_{m_k}$. Clearly, $\|Y_k\|_{E(\mathcal{M},\tau)} \le (2C_E)^{-2k}$ for all $k \in \mathbb{N}$. In particular, the sequence of partial sums of the series $\sum_{k=1}^{\infty} Y_k$ is a Cauchy sequence in $(E(\mathcal{M},\tau), \|\cdot\|_{E(\mathcal{M},\tau)})$. Hence, it is a Cauchy sequence in measure in $S(\mathcal{M},\tau)$, and therefore the series $\sum_{k=1}^{\infty} Y_k$ converges in $S(\mathcal{M},\tau)$ in measure to the operator $X - X_{m_1}$.

Let us show that the sum $\sum_{k=n}^{\infty} \sigma_{2^{k-n+1}}\mu(Y_k)$ is well defined. Clearly,

$$\sum_{k=n}^{\infty} C_E^k\|\sigma_{2^{k-n+1}}\mu(Y_k)\|_E \le \sum_{k=n}^{\infty} C_E^k \cdot (2C_E)^{k-n+1}\|Y_k\|_{E(\mathcal{M},\tau)}$$
$$\le \sum_{k=n}^{\infty} C_E^k \cdot (2C_E)^{k-n+1} \cdot (2C_E)^{-2k} = 2^{1-n}.$$

Hence, by Lemma 3.7 the series $\sum_{k=n}^{\infty} \sigma_{2^{k-n+1}}\mu(Y_k)$ converges in $(E, \|\cdot\|_E)$, and moreover

$$\Big\|\sum_{k=n}^{\infty} \sigma_{2^{k-n+1}}\mu(Y_k)\Big\|_E \le 2^{1-n}.$$

Since $\sum_{k=n}^{\infty} \sigma_{2^{k-n+1}}\mu(Y_k)$ converges in $(E, \|\cdot\|_E)$, Lemma 2.4 implies that $\sum_{k=n}^{\infty} \sigma_{2^{k-n+1}}\mu(Y_k)$ converges in measure in S.

It follows from Lemma 3.6 that

$$\sum_{k=n}^{\infty} Y_k = \sum_{k=1}^{\infty} Y_{k+n-1} \lhd 2\sum_{k=1}^{\infty} \sigma_{2^k}\mu(Y_{k+n-1}) = 2\sum_{k=n}^{\infty} \sigma_{2^{k-n+1}}\mu(Y_k), \quad n \geq 1.$$

Hence, by Corollary 3.5, $\sum_{k=n}^{\infty} Y_k \in E(\mathcal{M},\tau)$ (in particular, $X - X_{m_n} \in E(\mathcal{M},\tau)$ and hence $X \in E(\mathcal{M},\tau)$) and

$$\Big\|\sum_{k=n}^{\infty} Y_k\Big\|_{E(\mathcal{M},\tau)} \leq 424 C_E^9 \cdot 2^{1-n}.$$

Since $\sum_{k=n}^{\infty} Y_k = X - X_{m_n}$, it follows that the subsequence $(X_{m_n})_{n=1}^{\infty}$ converges to X in $E(\mathcal{M},\tau)$. Since $E(\mathcal{M},\tau)$ is metrizable (as a Δ-normed space), the Cauchy sequence $(X_n)_{n\in\mathbb{N}}$ is itself convergent. This completes the proof of Theorem 3.8. ∎

Depending on the type of the von Neumann algebra $\mathcal{M}$, Theorem 3.8 admits a converse:

THEOREM 3.9 (cf. [LSZ, Theorem 2.5.3]). *Let $\mathcal{M}$ be a semifinite von Neumann algebra and let $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ be a complete symmetric Δ-normed operator space on $\mathcal{M}$.*

(i) *If the algebra $\mathcal{M}$ is atomic, then the pair $(E, \|\cdot\|_E)$ defined by setting*

$$E := \{x \in l_\infty : \mu(x) = \mu(X) \text{ for some } X \in \mathcal{E}\},$$
$$\|x\|_E := \|X\|_{\mathcal{E}}, \quad x \in E,$$

is a complete symmetric Δ-normed sequence space.

(ii) *If the algebra $\mathcal{M}$ is atomless and $\tau(\mathbf{1}) = \mathbf{1}$, then the pair $(E, \|\cdot\|_E)$ defined by setting*

$$E := \{x \in L_0(0,1) : \mu(x) = \mu(X) \text{ for some } X \in \mathcal{E}\},$$
$$\|x\|_E := \|X\|_{\mathcal{E}}, \quad x \in E,$$

is a complete symmetric Δ-normed function space.

(iii) *If the algebra $\mathcal{M}$ is atomless and $\tau(\mathbf{1}) = \infty$, then the pair $(E, \|\cdot\|_E)$ defined by setting*

$$E := \{x \in S : \mu(x) = \mu(X) \text{ for some } X \in \mathcal{E}\},$$
$$\|x\|_E := \|X\|_{\mathcal{E}}, \quad x \in E,$$

is a complete symmetric Δ-normed function space.

Proof. The proof is identical to that of [LSZ, Theorem 2.5.3], and therefore is omitted. ∎

Theorems 3.8 and 3.9 imply the following

COROLLARY 3.10. *If $\mathcal{M}$ is an atomless (or atomic) von Neumann algebra, then the Calkin correspondence for complete Δ-normed spaces*

$$(E, \|\cdot\|_E) \leftrightarrow (E(\mathcal{M},\tau), \|\cdot\|_{E(\mathcal{M},\tau)})$$

is one-to-one.

4. Pietsch correspondence. Shift spaces were introduced by A. Pietsch [Pi1] and have a variety of applications (see e.g. [SSUZ, Pi2]). Pietsch [Pi2] considered quasi-normed shift-monotone ideals. Below we shall extend this notion to the Δ-normed case.

Let $o_k(a) = \sup_{h\geq k} |a_h|$ be the *ordering number* of a sequence $a := (a_h)_{h=0}^{\infty}$. Note that the sequence $(o_k(a))$ is also the decreasing envelope of the sequence $|a|$. A linear subset $\mathfrak{z}$ of $c_0(\mathbb{N}_0)$ is called a *Δ-normed monotone sequence ideal* if $(\mathfrak{z}, \|\cdot\|_{\mathfrak{z}})$ is a Δ-normed space with constant $C_{\mathfrak{z}}$ such that

(1) $e_0 \in \mathfrak{z}$ and $\|e_0\|_{\mathfrak{z}} = 1$,
(2) $x \in \mathfrak{z}$ and $\|x\|_{\mathfrak{z}} \leq \|a\|_{\mathfrak{z}}$ whenever $o_k(x) \leq o_k(a)$,

for $e_0 = (1, 0, 0, \ldots)$, $a, b \in \mathfrak{z}$, $x \in c_0$, $\lambda \in \mathbb{K}$ (scalar field) and $k = 0, 1, 2, \ldots$ (see e.g. [Pi2, LPSZ]).

Every monotone sequence ideal is invariant under the (backward) shift

$$S_- : (a_0, a_1, a_2, \ldots) \mapsto (a_1, a_2, \ldots).$$

A Δ-normed monotone sequence ideal is said to be *shift-monotone* if it remains invariant under the forward shift

$$S_+ : (a_0, a_1, a_2, \ldots) \mapsto (0, a_0, a_1, a_2, \ldots).$$

Additionally, we assume that S_+ is a bounded mapping on $(\mathfrak{z}, \|\cdot\|_{\mathfrak{z}})$ and $\|S_+\|_\infty \geq 1$, where $\|S_+\|_\infty$ denotes the operator norm of S_+ from $(\mathfrak{z}, \|\cdot\|_{\mathfrak{z}})$ into itself.

A. Pietsch [Pi2, Theorem 4.6] proved that there is a one-to-one correspondence between the collection of all symmetric sequence ideals and the collection of all shift-monotone sequence ideals (called the *Pietsch correspondence*). In a recent paper [LPSZ], G. Levitina et al. have shown that the completeness of a quasi-normed shift-monotone sequence ideal implies the completeness of the corresponding quasi-normed symmetric operator ideal.

We now extend this correspondence to Δ-normed ideals. Firstly, we need some technical lemmas. The first assertion of the first lemma can be found in [Pi2, Lemma 4.5].

LEMMA 4.1 (cf. [Pi2, Lemma 7.1]). *Let E be a Δ-normed symmetric sequence ideal. Let $x_1 \geq x_2 \geq \cdots \geq 0$. Then*

$$(x_n^{\sim}) \in E \;\Leftrightarrow\; (x_n) \in E \;\Leftrightarrow\; (x_{2n-1}^{\sim}) \in E.$$

Here, $x_n^{\sim} := x_{2^k}$ *if* $2^k \le n < 2^{k+1}$. *Moreover,*

$$\|(x_{2n-1}^{\sim})\|_E \le \|(x_n)\|_E \le \|(x_n^{\sim})\|_E \le 2C_E\|(x_{2n-1}^{\sim})\|_E.$$

Proof. Choose k such that $2^k \le 2n-1 < 2^{k+1}$. It follows from $n \le 2^k$ that $x_{2n-1}^{\sim} = x_{2^k} \le x_n$, which implies

$$\|(x_{2n-1}^{\sim})\|_E \le \|(x_n)\|_E \le \|(x_n^{\sim})\|_E.$$

Assume that $x_{2n-1}^{\sim} \in E$. We set

$$u = (x_1^{\sim}, 0, x_3^{\sim}, 0, \ldots) \quad \text{and} \quad v = (0, x_2^{\sim}, 0, x_4^{\sim}, \ldots).$$

Then $\|v\|_E \le \|u\|_E = \|(x_{2n-1}^{\sim})\|_E$. Therefore

$$\|(x_n^{\sim})\|_E = \|u+v\|_E \le C_E(\|u\|_E + \|v\|_E) \le 2C_E\|(x_{2n-1}^{\sim})\|_E. \ \blacksquare$$

Given any $a \in c_0$, we define the stretched sequence

$$d_a := (a_0, a_1, a_1, \ldots, \overbrace{a_h, \ldots, a_h}^{2^h \text{ terms}}, \ldots) \in c_0. \tag{4.1}$$

Proposition 4.2. *Given a Δ-normed symmetric sequence ideal E, the set*

$$\mathfrak{z} := \{a \in c_0 : d_a \in E\} \quad \textit{with} \quad \|a\|_{\mathfrak{z}} := \|d_a\|_E$$

is a Δ-normed shift-monotone sequence ideal.

Proof. Since the other statements are obvious, we check only the triangle inequality for $\|\cdot\|_{\mathfrak{z}}$ and that $\|b\|_{\mathfrak{z}} \le \|a\|_{\mathfrak{z}}$ whenever $o_k(b) \le o_k(a)$. By the definition of $\|\cdot\|_{\mathfrak{z}}$ and since $\|\cdot\|_E$ is a Δ-norm, we have

$$\|x+y\|_{\mathfrak{z}} = \|d_{x+y}\|_E \le C_E(\|d_x\|_E + \|d_y\|_E) = C_E(\|x\|_{\mathfrak{z}} + \|y\|_{\mathfrak{z}}).$$

Assume that $a \in \mathfrak{z}$, $b \in c_0$ and $o_k(b) \le o_k(a)$. By [Pi2, Lemma 4.1], $\mu(2^k, d_x) = o_k(x)$ for every $x \in l_\infty$. Hence, $\mu(d_a) \ge \mu(d_b)$, and therefore $\|a\|_{\mathfrak{z}} = \|d_a\|_E \ge \|d_b\|_E = \|b\|_{\mathfrak{z}}$. ■

The following result is a converse of the preceding assertion.

Proposition 4.3. *Every Δ-normed shift-monotone sequence ideal* $\mathfrak{z}$ *determines a Δ-normed symmetric sequence ideal*

$$E := \{a \in c_0 : (\mu(2^k, a))_k \in \mathfrak{z}\} \quad \textit{with} \quad \|a\|_E := \|(\mu(2^k, a))_k\|_{\mathfrak{z}}.$$

Proof. Similar to the preceding proposition, we check only the quasi-triangle inequality and the symmetry of $\|\cdot\|_E$. Firstly,

$$\begin{aligned}\|x+y\|_E &= \|(\mu(2^k, x+y))_k\|_{\mathfrak{z}} \le \|\sigma_2(\mu(2^k, x) + \mu(2^k, y))\|_{\mathfrak{z}} \\ &\le 2C_E^2(\|x\|_E + \|y\|_E).\end{aligned}$$

Further, assume that $a \in E$, $b \in c_0$ and $\mu(b) \le \mu(a)$. By [Pi2, Lemma 4.1], $o_k((\mu(2^k, x))_k) = \mu(2^k, d_{(\mu(2^k,x))_k}) = \mu(2^k, x)$. Hence, $\|a\|_E = \|(\mu(2^k, a))_k\|_{\mathfrak{z}} \ge \|(\mu(2^k, b))_k\|_{\mathfrak{z}} = \|b\|_E$. ■

Sequence ideals E and $\mathfrak{z}$ related to each other in this way (Propositions 4.2 and 4.3) are said to be *associated*.

Combining Propositions 4.2 and 4.3 we obtain a correspondence between Δ-normed shift-monotone sequence ideals and Δ-normed symmetric sequence ideals. This correspondence can be extended to Δ-normed symmetric operator ideals in the following way.

Let $(\mathfrak{z}, \|\cdot\|_{\mathfrak{z}})$ be a Δ-normed shift-monotone sequence ideal and let $(E, \|\cdot\|_E)$ be the corresponding Δ-normed symmetric sequence space. We associate with $(\mathfrak{z}, \|\cdot\|_{\mathfrak{z}})$ the operator ideal (see [Pi2])

$$\mathfrak{D}_{\mathfrak{z}} := \{X \in \mathcal{C}_0(H) : (\mu(2^k, X))_k \in \mathfrak{z}\} \text{ with } \|X\|_{\mathfrak{D}_{\mathfrak{z}}} := \|(\mu(2^k, X))_k\|_{\mathfrak{z}},$$

where $\mathcal{C}_0(H)$ denotes the ideal of compact operators on H.

By Calkin correspondence (see Corollary 3.10) we can associate with $(E, \|\cdot\|_E)$ the Δ-normed symmetric operator ideal $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$. By [Pi2, Theorem 4.7], we have $\mathcal{E} = \mathfrak{D}_{\mathfrak{z}}$. Employing now Lemma 4.1 one can show that the Δ-norms $\|\cdot\|_{\mathcal{E}}$ and $\|\cdot\|_{\mathfrak{D}_{\mathfrak{z}}}$ are equivalent. Thus, we arrive at the following result.

Corollary 4.4. *Let $(\mathfrak{z}, \|\cdot\|_{\mathfrak{z}})$ be a Δ-normed shift-monotone sequence ideal. Then the following diagram commutes:*

$$\begin{array}{ccc} (\mathfrak{z}, \|\cdot\|_{\mathfrak{z}}) & \longleftrightarrow & (E, \|\cdot\|_E) \\ \updownarrow & \swarrow\!\!\nearrow & \\ (\mathfrak{D}_{\mathfrak{z}}, \|\cdot\|_{\mathfrak{D}_{\mathfrak{z}}}) = (\mathcal{E}, \|\cdot\|_{\mathcal{E}}) & & \end{array}$$

The following theorem is the second main result of this paper. It establishes that the correspondence in Corollary 4.4 preserves completeness. In particular, this theorem provides an alternative proof of Theorem 3.8 for the special case when $\mathcal{M} = B(H)$.

Theorem 4.5. *If at least one of the associated ideals $(\mathfrak{z}, \|\cdot\|_{\mathfrak{z}})$, $(E, \|\cdot\|_E)$, $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ or $(\mathfrak{D}_{\mathfrak{z}}, \|\cdot\|_{\mathfrak{D}_{\mathfrak{z}}})$ is complete, then so are the others.*

Proof. (1) We know from Corollary 4.4 that $\mathcal{E} = \mathfrak{D}_{\mathfrak{z}}$ and the Δ-norms $\|\cdot\|_{\mathcal{E}}$ and $\|\cdot\|_{\mathfrak{D}_{\mathfrak{z}}}$ are equivalent. Hence, $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ is complete if and only if $(\mathfrak{D}_{\mathfrak{z}}, \|\cdot\|_{\mathfrak{D}_{\mathfrak{z}}})$ is complete.

(2) Every $x = (x_m) \in c_0$ generates a diagonal operator $\operatorname{diag}(x) := x_n e_n \otimes e_n$. Since $\mu(x) = \mu(\operatorname{diag}(x))$, we have $\|x\|_E = \|\operatorname{diag}(x)\|_{\mathcal{E}}$. We conclude that completeness of $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ implies completeness of $(E, \|\cdot\|_E)$.

(3) Assume that $(E, \|\cdot\|_E)$ is a complete Δ-normed sequence ideal. Since E and $\mathfrak{z}$ are associated, by the definition, we have

$$\mathfrak{z} = \{a \in c_0 : d_a \in E\} \quad \text{with} \quad \|a\|_{\mathfrak{z}} = \|d_a\|_E.$$

Take an arbitrary Cauchy sequence $(x_n)_n \subset (\mathfrak{z}, \|\cdot\|_{\mathfrak{z}})$, $x_n = (x_h^{(n)})_{h\in\mathbb{N}_0}$. Then $(d_{x_n})_n \subset (E, \|\cdot\|_E)$ is a Cauchy sequence in E. Therefore, there exists $y \in E$ such that $d_{x_n} \xrightarrow{\|\cdot\|_E} y$ as $n \to \infty$. For every fixed $h \in \mathbb{N}_0$ we denote by d_h the characteristic sequence of the set $\{m \in \mathbb{N} : 2^h \le m < 2^{h+1}\}$. We have

$$x_h^{(n)} d_h = d_h d_{x^{(n)}} \to y d_h =: x_h d_h.$$

In addition, if $k \ne h, k \in \mathbb{N}_0$, then

$$0 = x_h^{(n)} d_h d_k = d_h d_{x^{(n)}} d_k \to y d_h d_k.$$

Consequently, $y = \sum_{h=0}^{\infty} x_h d_h$. Set $x = (x_h)$. Then $y = d_x$, and therefore $x \in \mathfrak{z}$ and by the definition of $\|\cdot\|_{\mathfrak{z}}$ we have

$$\|x_n - x\|_{\mathfrak{z}} = \|d_{x_n} - d_x\|_E = \|d_{x_n} - y\|_E \to 0.$$

Hence, the space $(\mathfrak{z}, \|\cdot\|_{\mathfrak{z}})$ is complete. Thus, completeness of $(E, \|\cdot\|_E)$ implies completeness of $(\mathfrak{z}, \|\cdot\|_{\mathfrak{z}})$.

(4) It now remains to show that completeness carries over from $(\mathfrak{z}, \|\cdot\|_{\mathfrak{z}})$ to $(\mathfrak{D}_{\mathfrak{z}}, \|\cdot\|_{\mathfrak{D}_{\mathfrak{z}}})$.

Since $\|\cdot\|_{\mathfrak{z}}$ is a Δ-norm and $\|S_+\|_\infty \ge 1$, for any $h \in \mathbb{N}_0$ there exists $n_h \ge h$ such that

$$\left\| \frac{1}{(2C_{\mathfrak{z}}\|S_+\|_\infty)^{n-1}} e_0 \right\|_{\mathfrak{z}} \le \frac{1}{(2C_{\mathfrak{z}}\|S_+\|_\infty)^h} \tag{4.2}$$

for all $n \ge n_h$ (recall $e_0 = (1, 0, 0, \ldots)$). Then we may assume that $n_h < n_{h+1}$ for every h.

Take an arbitrary Cauchy sequence $(Y_h)_{h\in\mathbb{N}_0} \subset (\mathfrak{D}_{\mathfrak{z}}, \|\cdot\|_{\mathfrak{D}_{\mathfrak{z}}})$. Lemma 2.4 implies that $(Y_h)_h$ is also a Cauchy sequence in $B(H)$. We choose $m_h \ge n_h$ such that

$$\max\{\|Y_{m_h} - Y_m\|_{\mathfrak{D}_{\mathfrak{z}}}, \|Y_{m_h} - Y_m\|_\infty\} \le (2C_{\mathfrak{z}}\|S_+\|_\infty)^{-n_h}$$

for all $m \ge m_h$ and set $X_h = Y_{m_{h+1}} - Y_{m_h}$, $h \in \mathbb{N}_0$. Then

$$\max\{\|X_h\|_{\mathfrak{D}_{\mathfrak{z}}}, \|X_h\|_\infty\} \le (2C_{\mathfrak{z}}\|S_+\|_\infty)^{-n_h}. \tag{4.3}$$

The estimate (4.3) implies that the series $\sum_{h=0}^{\infty} X_h$ converges in $B(H)$ to some operator $X \in B(H)$. By [LPSZ, Lemma 5.2],

$$\mu(2^k, X) \le \sum_{h=0}^{k-1} \mu(2^{k-h-1}, X_h) + \Big\| \sum_{h=k}^{\infty} X_h \Big\|_\infty, \tag{4.4}$$

where $\sum_{h=0}^{k-1} \mu(2^{k-h-1}, X_h) = 0$ when $k = 0$. We claim that $X \in \mathfrak{D}_{\mathfrak{z}}$ and the series $\sum_{h=0}^{\infty} X_h$ converges to X with respect to the Δ-norm $\|\cdot\|_{\mathfrak{D}_{\mathfrak{z}}}$.

Since $\mathfrak{z}$ is a shift-monotone sequence ideal, we deduce that the sequence $S_+^{h+1}(\mu(2^k, X_h))_k$ is in $\mathfrak{z}$ for all $h \in \mathbb{N}_0$. Consider the series

$$\sum_{h=0}^{\infty} C_{\mathfrak{z}}^{h+1} \| S_+^{h+1}(\mu(2^k, X_h))_k \|_{\mathfrak{z}}.$$

We have

$$\begin{aligned}(4.5)\qquad \sum_{h=0}^{\infty} C_{\mathfrak{z}}^{h+1} \| S_+^{h+1}(\mu(2^k, X_h))_k \|_{\mathfrak{z}} &\le \sum_{h=0}^{\infty} C_{\mathfrak{z}}^{h+1} \| S_+ \|_\infty^{h+1} \| (\mu(2^k, X_h))_k \|_{\mathfrak{z}} \\ = \sum_{h=0}^{\infty} C_{\mathfrak{z}}^{h+1} \| S_+ \|_\infty^{h+1} \| X_h \|_{\mathfrak{D}_{\mathfrak{z}}} &\overset{(4.3)}{\le} \sum_{h=0}^{\infty} C_{\mathfrak{z}}^{h+1} \| S_+ \|_\infty^{h+1} (2C_{\mathfrak{z}} \| S_+ \|_\infty)^{-n_h} \\ \le C_{\mathfrak{z}} \| S_+ \|_\infty \sum_{h=0}^{\infty} 2^{-h} < \infty, \end{aligned}$$

where we have used $n_h \ge h$. Since the space $(\mathfrak{z}, \|\cdot\|_{\mathfrak{z}})$ is complete, Lemma 3.7 implies that the series $\beta := \sum_{h=0}^{\infty} S_+^{h+1}(\mu(2^k, X_h))_k$ converges in $(\mathfrak{z}, \|\cdot\|_{\mathfrak{z}})$.

Set $\alpha_k := \| \sum_{h=k}^{\infty} X_h \|_\infty$. By (4.3), we have

$$\begin{aligned}(4.6)\qquad \Big\| \sum_{h=k}^{\infty} X_h \Big\|_\infty &\le \sum_{h=k}^{\infty} \| X_h \|_\infty \le \sum_{h=k}^{\infty} (2C_{\mathfrak{z}} \| S_+ \|_\infty)^{-n_h} \\ &\le \frac{2C_{\mathfrak{z}} \| S_+ \|_\infty}{2C_{\mathfrak{z}} \| S_+ \|_\infty - 1} (2C_{\mathfrak{z}} \| S_+ \|_\infty)^{-n_k} \le \frac{1}{(2C_{\mathfrak{z}} \| S_+ \|_\infty)^{n_k - 1}}. \end{aligned}$$

Therefore, by Lemma 3.7, we infer from (4.2) and (4.6) that $\alpha = (\alpha_k)$ is in $\mathfrak{z}$.

Since $\alpha, \beta \in \mathfrak{z}$, we also find that $\alpha + \beta \in \mathfrak{z}$. Inequality (4.4) implies that $o((\mu(2^k, X))_k) \le o(\alpha + \beta)$, and therefore $(\mu(2^k, X))_k \in \mathfrak{z}$ and $X \in \mathfrak{D}_{\mathfrak{z}}$. By (4.4), we have

$$\begin{aligned}(4.7)\qquad \Big\| \sum_{h=j}^{\infty} X_h \Big\|_{\mathfrak{D}_{\mathfrak{z}}} &= \Big\| \sum_{h=0}^{\infty} X_{h+j} \Big\|_{\mathfrak{D}_{\mathfrak{z}}} = \Big\| \Big(\mu\Big(2^k, \sum_{h=0}^{\infty} X_{h+j}\Big)\Big)_k \Big\|_{\mathfrak{z}} \\ &\le \Big\| \Big(\sum_{h=0}^{k-1} \mu(2^{k-h-1}, X_{h+j}) + \Big\| \sum_{h=k}^{\infty} X_{h+j} \Big\|_\infty \Big)_k \Big\|_{\mathfrak{z}} \\ &\le C_{\mathfrak{z}} \Big(\Big\| \Big(\sum_{h=0}^{k-1} \mu(2^{k-h-1}, X_{h+j}) \Big)_k \Big\|_{\mathfrak{z}} + \Big\| \Big(\Big\| \sum_{h=k}^{\infty} X_{h+j} \Big\|_\infty \Big)_k \Big\|_{\mathfrak{z}} \Big). \end{aligned}$$

By (4.2) and (4.6), we have

$$\Big\| \Big(\Big\| \sum_{h=k}^{\infty} X_{h+j} \Big\|_\infty \Big)_k \Big\|_{\mathfrak{z}} \to 0 \quad \text{as } j \to \infty.$$

Noticing that $(\sum_{h=0}^{k-1}\mu(2^{k-h-1},X_{h+j}))_k=\sum_{h=0}^{\infty}S_+^{h+1}(\mu(2^k,X_{h+j}))_k$, we obtain

$$\Big\|\Big(\sum_{h=0}^{k-1}\mu(2^{k-h-1},X_{h+j})\Big)_k\Big\|_{\mathfrak{z}}=\Big\|\sum_{h=0}^{\infty}S_+^{h+1}(\mu(2^k,X_{h+j}))_k\Big\|_{\mathfrak{z}}$$
$$\overset{(3.3)}{\le}\sum_{h=0}^{\infty}C_{\mathfrak{z}}^{h+1}\|S_+^{h+1}(\mu(2^k,X_{h+j}))_k\|_{\mathfrak{z}}\le\sum_{h=0}^{\infty}C_{\mathfrak{z}}^{h+1}\|S_+\|_\infty^{h+1}\|(\mu(2^k,X_{h+j}))_k\|_{\mathfrak{z}}$$
$$=\sum_{h=0}^{\infty}C_{\mathfrak{z}}^{h+1}\|S_+\|_\infty^{h+1}\|X_{h+j}\|_{\mathfrak{D}_{\mathfrak{z}}}\overset{(4.3)}{\le}\sum_{h=0}^{\infty}C_{\mathfrak{z}}^{h+1}\|S_+\|_\infty^{h+1}(2C_{\mathfrak{z}}\|S_+\|_\infty)^{-n_{h+j}}$$
$$\le C_{\mathfrak{z}}\|S_+\|_\infty\sum_{h=0}^{\infty}2^{-(h+j)}\to 0\quad\text{ as } j\to\infty.$$

Hence, (4.7) implies that $\|\sum_{h=j}^{\infty}X_h\|_{\mathfrak{D}_{\mathfrak{z}}}\to 0$ as $j\to\infty$, and therefore the series $\sum_{h=0}^{\infty}X_h$ converges to X in $(\mathfrak{D}_{\mathfrak{z}},\|\cdot\|_{\mathfrak{D}_{\mathfrak{z}}})$. Moreover, letting $Y=X+Y_{m_1}$, we have $\sum_{h=1}^{\infty}X_h=Y-Y_{m_1}$. Thus, $\lim_{n\to\infty}Y_n=Y\in\mathfrak{D}_{\mathfrak{z}}$, which completes the proof. ∎

5. Concluding remarks. In this section we consider an important example of symmetric Δ-normed function space. Set

$$\exp(L_1)=\{f\in L_0[0,1]:\log_+(|f|)\in L_1[0,1]\},$$

where $\log_+(\lambda)=\max\{\log(\lambda),0\}$, $\lambda>0$. It is clear that $\exp(L_1)$ is a subalgebra of $L_0[0,1]$. Introduce the family of neighbourhoods of zero

$$U(R,\varepsilon)=\Big\{f\in\exp(L_1):\int_0^1\log_+(R|f(x)|)\,dx\le\varepsilon\Big\},\qquad R,\varepsilon>0.$$

The fact that $U(n,1/n)$, $n\in\mathbb{N}$, gives a countable base of neighbourhoods (so the space satisfies the first axiom of countability) implies that $\exp(L_1)$ can be equipped with a Δ-norm $\|\cdot\|_{\exp(L_1)}$ (see e.g. [K, Ch. 3, Section 15.11]). It is easy to see that $(\exp(L_1),\|\cdot\|_{\exp(L_1)})$ is a complete symmetric Δ-normed function space. Therefore, we can introduce a noncommutative operator space on a type II_1 factor $\mathcal{M}$, by setting

$$\exp(L_1)(\mathcal{M},\tau)=\{X\in S(\mathcal{M},\tau):\mu(X)\in\exp(L_1)\},$$
$$\|X\|_{\exp(L_1)(\mathcal{M},\tau)}=\|\mu(X)\|_{\exp(L_1)}.$$

Using Theorem 3.8 we conclude that $(\exp(L_1)(\mathcal{M},\tau),\|\cdot\|_{\exp(L_1)(\mathcal{M},\tau)})$ is a complete symmetric Δ-normed space. This argument provides an alternative proof of [DSZ, Theorem A.5].

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