COLLOQUIUM MATHEMATICUM

VOL. 148 2017 NO. 2

AN ASYMPTOTIC FORMULA FOR GOLDBACH'S CONJECTURE WITH MONIC POLYNOMIALS IN $\mathbb{Z}[\theta][x]$

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Abstract. Let $k \geq 2$ be a squarefree integer, and

$$\theta = \begin{cases} \sqrt{-k} & \text{if } -k \not\equiv 1 \pmod 4, \\ (\sqrt{-k}+1)/2 & \text{if } -k \equiv 1 \pmod 4. \end{cases}$$

We prove that the number $R(y)$ of representations of a monic polynomial $f(x) \in \mathbb{Z}[\theta][x]$, of degree $d \geq 1$, as a sum of two monic irreducible polynomials $g(x)$ and $h(x)$ in $\mathbb{Z}[\theta][x]$, with the coefficients of $g(x)$ and $h(x)$ bounded in modulus by y, is asymptotic to $(4y)^{2d-2}$.

1. Introduction. In 1965, Hayes [H1] showed that Goldbach's conjecture is considerably simpler for polynomials with integer coefficients.

THEOREM 1.1. *If $f(x)$ is a monic polynomial in $\mathbb{Z}[x]$ of degree $d > 1$, then there are monic irreducible polynomials $g(x)$ and $h(x)$ in $\mathbb{Z}[x]$ with $f(x) = g(x) + h(x)$.*

In a recent note, Saidak [S], improving on a result of Hayes, obtained a Chebyshev-type estimate for the number $R(y) = R_f(y)$ of representations of the monic polynomial $f(x) \in \mathbb{Z}[x]$ of degree $d > 1$ as a sum of two irreducible monics $g(x)$ and $h(x)$ in $\mathbb{Z}[x]$, with the coefficients of $g(x)$ and $h(x)$ bounded in absolute value by y. Saidak's argument with slight modifications shows that, for y sufficiently large,

$$c_1 y^{d-1} < R(y) < c_2 y^{d-1},$$

where c_1 and c_2 are constants that depend on the degree and coefficients of the polynomial $f(x)$.

Recently, Kozek [K] proved that $R(y)$ is asymptotic to $(2y)^{d-1}$, i.e.,

$$\lim_{y\to\infty} \frac{R(y)}{(2y)^{d-1}} = 1.$$

2010 *Mathematics Subject Classification*: Primary 11R09; Secondary 11C08.
Key words and phrases: irreducible polynomial, Goldbach's problem for polynomials.
Received 11 April 2016; revised 5 July 2016.
Published online 9 March 2017.

DOI: 10.4064/cm6948-7-2016

His approach implies that there is a constant c_3, depending only on d, such that if y is sufficiently large, then

$$R(y) = (2y)^{d-1} + E, \quad \text{where} \quad |E| \le c_3 y^{d-2} \ln y.$$

In 2011, Dubickas [D1] proved a more general result for the number of representations of f as a sum of r monic irreducible (over $\mathbb{Q}$) integer polynomials $f_1, \dots, f_r$ of height at most y, i.e.,

$$f(x) = f_1(x) + \cdots + f_r(x).$$

For $r = 2$, he proved that

$$R(y) = (2y)^{d-1} + O(y^{d-2}) \tag{1}$$

for $d \ge 4$,

$$R(y) = (2y)^2 + O(y \ln(y)) \tag{2}$$

for $d = 3$, and

$$R(y) = 2y + O(\sqrt{y}) \tag{3}$$

for $d = 2$. Moreover, for each $d \ge 4$, the error term in (1) is the best possible for some f. Note that these results improve the error term proved by Kozek [K].

In 2013, Dubickas [D2] proved a necessary and sufficient condition on the list of nonzero integers $u_1, \dots, u_r$, $r \ge 2$, under which a monic polynomial $f(x) \in \mathbb{Z}[x]$ is expressible by a linear form $u_1 f_1(x) + \cdots + u_r f_r(x)$ in monic polynomials $f_1(x), \dots, f_r(x) \in \mathbb{Z}[x]$.

We say that a ring D has *property* (GC) if

every element of $D[x]$ of degree $d \ge 1$
can be written as the sum of two irreducibles in $D[x]$.

If D is the ring of integers, then Theorem 1.1 implies that $\mathbb{Z}$ has property GC.

Pollack proved the following:

PROPOSITION 1.1 ([P2, Theorem 1]). *Suppose that D is an integral domain which is Noetherian and has infinitely many maximal ideals. Then D has property* (GC).

COROLLARY 1.1 ([P2, Theorem 2]). *If S is any integral domain, then $D = S[x]$ has property* (GC).

When $D = \mathbb{F}_q$ is a finite field (note that in this case, the assumptions of Proposition 1.1 do not hold), Hayes [H2], in 1966, gave an asymptotic formula for the number $R(f)$ of representations of an odd polynomial $f(x) \in \mathbb{F}_q[x]$ as a sum $\alpha g(x) + \beta h(x) + \gamma t(x) = f(x)$ with three monic irreducible polynomials $g(x)$, $h(x)$ and $t(x)$ in $\mathbb{F}_q[x]$, where α, β, γ in $\mathbb{F}_q^*$ are such that

$\alpha + \beta + \gamma = f_r$ and f_r is the leading coefficient of $f(x)$. Hayes' formula is

$$R(f) = r^{-3}q^{2r}S(f) + O(q^{1/4}q^{(5/4+\epsilon)r}),$$

where $r = \deg f$, $S(f)$ is the singular series (see [H2, (6.10)]), and the constant implied in O is independent of both f and q. The number ϵ is defined as $\max\{1/2, \epsilon^*\}$, where ϵ^* is the least upper bound of the real parts of the zeros of certain L-functions. Advances in this direction can be found in Pollack [P1, P3], Webb [W] and Car [C1, C2, C3].

In this paper, we do not distinguish the sum $g(x)+h(x)$ from $h(x)+g(x)$ and we recall that a monic polynomial in $\mathbb{Z}[x]$ is irreducible over $\mathbb{Z}$ if and only if it is irreducible over $\mathbb{Q}$.

Following the results of Hayes and Pollack with $D = \mathbb{Z}[\theta]$, where θ has the properties described in Lemma 2.1 below, and following the ideas of Kozek [K], we prove that the number $R(y)$ of representations of a monic polynomial $f(x) \in \mathbb{Z}[\theta][x]$ as a sum of two monic irreducible polynomials $g(x)$ and $h(x)$ in $\mathbb{Z}[\theta][x]$, with the coefficients of $g(x)$ and $h(x)$ bounded in modulus by y, is asymptotic to $(4y)^{2d-2}$.

2. Notation and preliminary results. Some well known facts are presented below. Let $f(x) = \sum_{i=0}^{d} f_i x^i$ be a polynomial in $\mathbb{C}[x]$. Set

$$H(f) = \max_{0\le i\le d} |f_i| \quad \text{and} \quad M(f) = \exp\Big(\int_0^1 \ln|f(e^{2\pi i t})|\,dt\Big).$$

The expressions $H(f)$ and $M(f)$ are known as the *height* and *Mahler's measure* of f, respectively (see [M1, M2]). Mahler [M2] showed that for $0 \le i \le n$, $|f_i| \le \binom{n}{i} M(f)$, and that $M(f)$ is multiplicative. Another result of Mahler is

$$\frac{M(f)}{\sqrt{d+1}} \le H(f) \le 2^{d-1}M(f). \tag{4}$$

An important property of Mahler's measure is

$$1 \le M(f) \le \Big(\sum_{i=0}^{d} |f_i|^2\Big)^{1/2}, \tag{5}$$

which was proved by Landau [L].

Assume that $f(x) = a(x)b(x)$ with $\deg a = d_1$ and $\deg b = d_2$, i.e., $d = d_1 + d_2$. A direct application of Jensen's formula (see [M1]) results in $H(f) = H(ab) \le (1 + d_1)H(a)H(b)$ if $d_1 \le d_2$. To see this, let

$$f(x) = \sum_{i=0}^{d} f_i x^i, \qquad a(x) = \sum_{k=0}^{d_1} a_k x^k, \qquad b(x) = \sum_{t=0}^{d_2} b_t x^t.$$

Then

$$|f_i| = \Big|\sum_{j=0}^{d_1} a_j b_{i-j}\Big| \le (1+d_1)\max|a_k|\max|b_t| = (1+d_1)H(a)H(b).$$

Moreover,

$$H(ab) \ge \frac{H(a)H(b)}{2^{d-2}\sqrt{d+1}}, \tag{6}$$

which follows from the fact that $M(f)$ is multiplicative and from (4) for $a(x)$ and $b(x)$, i.e., $H(a) \le 2^{d_1-1}M(a)$, $H(b) \le 2^{d_2-1}M(b)$ and $M(ab) \le H(ab)\sqrt{d+1}$.

Using the above, we prove the result below.

LEMMA 2.1. *Let $f(x) = a(x)b(x)$ be a polynomial of degree d in $\mathbb{Z}[\theta][x]$, where $a(x), b(x) \in \mathbb{Z}[\theta][x]$ and*

$$\theta = \begin{cases} \sqrt{-k} & \text{if } -k \not\equiv 1 \pmod 4, \\ (\sqrt{-k}+1)/2 & \text{if } -k \equiv 1 \pmod 4, \end{cases} \tag{7}$$

where k is a squarefree integer and $k \ge 2$. Let

$$a(x) = \sum_{i=0}^{m} a_i x^i \quad \text{and} \quad f(x) = \sum_{i=0}^{d} f_i x^i.$$

Then, for $0 \le l \le m$,

$$|a_l| \le 2^{2d-2}\sqrt{d+1}\Big(\sum_{i=0}^{d}|f_i|^2\Big)^{1/2}.$$

Proof. First, we observe that $|r+s\theta| \ge 1/2$ for any θ as above and $r, s \in \mathbb{Z}$ with $r \ne 0$ or $s \ne 0$. Indeed,

$$|r+s\theta| = \begin{cases} \sqrt{r^2+s^2k} & \text{if } -k \not\equiv 1 \pmod 4, \\ \frac{1}{2}\sqrt{r_1^2+s^2k} & \text{if } -k \equiv 1 \pmod 4, \end{cases}$$

where $r_1 = 2r+s$. Since r and s are integers with $r \ne 0$ or $s \ne 0$, and $k > 1$, we have $\sqrt{r^2+s^2k} \ge 1$ and $\sqrt{r_1^2+s^2k} \ge 1$. Consequently, $H(b) \ge 1/2$.

From (4) and (6), we have

$$H(a)H(b) \le 2^{2d-3}\sqrt{d+1}\,M(f).$$

Now, using (5), it follows that

$$\frac{1}{2}|a_l| \le \frac{1}{2}H(a) \le H(a)H(b) \le 2^{2d-3}\sqrt{d+1}\,\Big(\sum_{i=0}^{d}|f_i|^2\Big)^{1/2}.$$

Consequently,

$$|a_l| \le 2^{2d-2}\sqrt{d+1}\,\Big(\sum_{i=0}^{d}|f_i|^2\Big)^{1/2}. \ \blacksquare$$

Before we state the next lemma, we recall the big O notation. For two functions $r(y)$ and $\phi(y)$, we write $r(y) = O(\phi(y))$ as $y \to \infty$ if there is a y_0 and a $C > 0$ such that $|r(y)| \le C\phi(y)$ for all $y > y_0$. In the event that the constant C depends only on a value s, we write $|r(y)| \le C_s\phi(y)$, and also $r(y) = O_s(\phi(y))$. If C depends on the coefficients and the degree of a polynomial $f(x)$, we use O_f instead.

LEMMA 2.2. *Let $d > 1$ be an integer and let $g_{d-1} \in \mathbb{Z}[\theta]$ be fixed, with θ as in Lemma 2.1. For each $y \ge 2$, let r_y denote the number of d-tuples $(g_{d-1}, g_{d-2}, \ldots, g_1, g_0)$ of elements in $\mathbb{Z}[\theta]$ satisfying $|g_i| \le y$ for each i with $0 \le i \le d-1$ such that the polynomial*

$$g(x) = \sum_{i=0}^{d-1} g_i x^i + x^d$$

is reducible. Then $r_y = O_g(y^{2d-4}\ln y)$. In particular, $r_y = 0$ if $y < |g_{d-1}|$.

Proof. Let $g(x) \in \mathbb{Z}[\theta](x)$ be as above with g_{d-1} fixed. Then there are monic polynomials $a(x), b(x) \in \mathbb{Z}[\theta][x]$ of degree ≥ 1 such that $g(x) = a(x)b(x)$. Let

$$\deg(a) = m \ge n = \deg b,$$

where $m + n = d$. We write

$$a(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_1x + a_0,$$
$$b(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_1x + b_0.$$

We assert that the number of monic polynomials $g(x)$ that we are considering with $g_0 = 0$ is $O_g(y^{2d-4})$. Indeed, denoting by $g_j = g_{j,1} + i\sqrt{k}\,g_{j,2}$, we have $|g_j| = \sqrt{g_{j,1}^2 + kg_{j,2}^2} \le y$. Therefore, the number of possibilities for g_j is bounded by

(8)
$$(2y+1)\Big(2\frac{y}{\sqrt{k}}+1\Big) - 2\frac{y}{\sqrt{k}} - \Big[2\frac{y}{\sqrt{k}} - 2\frac{\sqrt{2y-1}}{\sqrt{k}}\Big] - \cdots - \Big[2\frac{y}{\sqrt{k}} - 2\frac{\sqrt{y^2-1}}{\sqrt{k}}\Big],$$

with the sum having y terms, i.e., (8) is $O_k(y^2)$, and the assertion follows.

The argument above is sufficient to show that the number of d-tuples

$$(a_{m-1}, a_{m-2}, \ldots, a_1, a_0, b_{n-1}, b_{n-2}, \ldots, b_1, b_0)$$

as above with $a_0b_0 \ne 0$ is equal to $O_g(y^{2d-4}\ln y)$.

We consider $a(x)$, which has degree $m \le d-1$. A similar argument applies to $b(x)$. For $1 \le l \le m-1$, Lemma 2.1 implies

$$|a_l| \le 2^{2d-2}\sqrt{d+1}\Big(\sum_{i=0}^{d}|g_i|^2\Big)^{1/2} \le 2^{2d-2}\sqrt{d+1}((d+1)y^2)^{1/2} = C_d y,$$

where C_d depends only on d. Thus, the number of $(d-4)$-tuples

$$(a_{m-2},\dots,a_1,b_{n-2},\dots,b_1)$$

is $O_g(y^{2m-4}y^{2n-4}) = O_g(y^{2d-8})$.

Observe that when we multiply $a(x)$ and $b(x)$, the value of the coefficient g_{d-1} is $a_{m-1}+b_{n-1}$, since $a(x)$ and $b(x)$ are both monic. Also, since g_{d-1} is fixed, when a_{m-1} is determined, b_{n-1} is determined as well. Hence, the number of pairs (a_{m-1},b_{n-1}) is $O_k(y^2)$.

Since $a_0b_0 = g_0$, we have $1 \le |a_0b_0| \le y$, i.e.,

$$1 \le (a_{0,1}^2 + ka_{0,2}^2)(b_{0,1}^2 + kb_{0,2}^2) \le y^2.$$

Thus, the number of pairs (a_0,b_0) is bounded by

$$\begin{aligned} 16\sum_{q\le y^2}\sum_{\delta\mid y^2} 1 = 16\sum_{\delta\le y^2}\sum_{\substack{q\le y^2\\ \delta\mid q}} 1 &\le 16\sum_{\delta\le y^2}\frac{y^2}{\delta} \\ &\le 16y^2\sum_{\delta\le y^2}\frac{1}{\delta} \le 16y^2\Big(1+\int_1^{y^2}\frac{1}{t}\,dt\Big) \\ &= O(y^2\ln y^2) = O(y^2\ln y), \end{aligned}$$

where 16 appears because each term of a_0 and b_0 may be either positive or negative.

Finally, for an integer $d>1$ and a fixed $g_{d-1}\in\mathbb{Z}[\theta]$, the number of d-tuples

$$(a_0,a_1,\dots,a_{m-1},b_0,b_1,\dots,b_{n-1}),$$

corresponding to the coefficients of two monic polynomials $a(x)$ and $b(x)$ in $\mathbb{Z}[\theta][x]$ of degrees $m,n\ge 1$ such that $g(x)=a(x)b(x)$ and the coefficients of $g(x)$ are bounded in absolute value by y is

$$r_y = O_g(y^{2m-4}y^{2n-4}y^2(\ln y)y^2) = O_g(y^{2d-4}\ln y) \quad \text{as } y\to\infty,$$

with the constants depending only on d and k. ∎

Remark 2.1. If we replace (7) by

$$\theta = \begin{cases} \sqrt{k} & \text{if } k \not\equiv 1 \pmod 4, \\ (\sqrt{k}+1)/2 & \text{if } k\equiv 1 \pmod 4, \end{cases}$$

where $k\ge 2$ is a squarefree integer and we consider $\mathbb{Z}[\theta]$ with the usual norm induced by $\mathbb{Q}$, our argument cannot be applied because, for y large enough, there are infinitely many possibilities for $g_j = g_{j,1} + \sqrt{k}\, g_{j,2}$ with $g_{j,1}, g_{j,2} \in \mathbb{Z}$ and $|g_j| \le y$. Therefore, we could not achieve the limitation (8).

Remark 2.2. If we remove the condition in Lemma 2.2 that g_{d-1} is fixed, then $r_y = O_g(y^{2d-2} \ln y)$. This is a direct consequence of the same arguments used to prove (8). If the degree of g is $d-1$, and in this case g_{d-2} is not fixed, then $r_y = O_g(y^{2d-4} \ln y)$.

Lemma 2.3. *Let $f(x)$ be a monic polynomial in $\mathbb{Z}[\theta][x]$ of degree $d > 1$. Consider pairs $(g(x), h(x))$ of monic polynomials such that $f(x) = g(x) + h(x)$, where $g(x)$ or $h(x)$ is reducible with* $\deg g = d$ *and* $1 \le \deg h \le d-1$, *and the coefficients of $g(x)$ and $h(x)$ are bounded in modulus by y. The number of such pairs is $O_f(y^{2d-4} \ln y)$.*

Proof. Write

$$(9) \quad f(x) = x^d + \sum_{j=0}^{d-1} f_j x^j, \quad g(x) = x^d + \sum_{j=0}^{d-1} g_j x^j, \quad h(x) = x^n + \sum_{j=0}^{n-1} h_j x^j,$$

where $f_j = g_j + h_j$ and $1 \le n \le d-1$.

Suppose at least one of $g(x)$ or $h(x)$ is reducible. Once $g(x)$ or $h(x)$ is fixed, it determines the other. Thus, we can count separately when $g(x)$ is reducible and when $h(x)$ is reducible. We count the ways that $g(x)$ might be reducible. Since $f(x) = g(x) + h(x)$, by (9), we have either $g_{d-1} = f_{d-1}$ or $g_{d-1} = f_{d-1} - 1$. Therefore, in any case, g_{d-1} is fixed. Now, the coefficients of g are bounded in modulus by y, and thus by Lemma 2.2, the number of monic reducible polynomials $g(x)$ is $O_f(y^{2d-4} \ln y)$. Now, we count the ways that $g(x)$ might be reducible. If $\deg h = d-1$, then by Remark 2.2, the number of monic reducible polynomials $h(x)$ is $O_f(y^{2d-4} \ln y)$, and if $\deg h < d-1$, this number is smaller, also by Remark 2.2. ∎

3. Main result

Theorem 3.1. *Let $f(x)$ be a monic polynomial in $\mathbb{Z}[\theta][x]$ of degree $d > 1$ and with θ as in Lemma 2.1. The number $R(y)$ of representations of $f(x)$ as a sum of two monic irreducible $g(x)$ and $h(x)$ in $\mathbb{Z}[\theta][x]$, with the coefficients of $g(x)$ and $h(x)$ bounded in modulus by y, is asymptotic to $(4y)^{2d-2}$.*

Proof. Let

$$f(x) = x^d + \sum_{j=0}^{d-1} f_j x^j,$$

where $f_j = f_{j,1} + i\sqrt{k}\, f_{j,2}$. We are looking for pairs of monic polynomials $g(x), h(x) \in \mathbb{Z}[\theta][x]$ such that $f(x) = g(x) + h(x)$ and the coefficients of $g(x)$ and $h(x)$ are bounded in modulus by y. Without loss of generality, let

$\deg g > \deg h$, and observe that $\deg g = d$ and $1 \leq \deg h \leq d-1$. In this case,

$$g(x) = x^d + \sum_{j=0}^{d-1} g_j x^j, \qquad h(x) = x^n + \sum_{j=0}^{n-1} h_j x^j,$$

where $g_j = g_{j,1} + i\sqrt{k}\, g_{j,2}$, $h_j = h_{j,1} + i\sqrt{k}\, h_{j,2}$, $f_j = g_j + h_j$ and $1 \leq n \leq d-1$.

If $y \geq 1 + \{|f_0|, |f_1|, \ldots, |f_{d-1}|\}$, then the total number of pairs of monic (not necessarily irreducible) polynomials $g(x), h(x)$ is

$$\sum_{S=0}^{d-2} \prod_{j=0}^{S} (2\lfloor y \rfloor + 1 - |f_{j,1}|)(2\lfloor y \rfloor + 1 - |f_{j,2}|) = (4y)^{2d-2} + O_f(y^{2d-4}),$$

since $f_{j,1} + i\sqrt{k}\, f_{j,2} = g_{j,1} + h_{j,1} + i\sqrt{k}\,(g_{j,2} + h_{j,2})$.

By Lemma 2.3, almost all of these pairs of monic polynomials $g(x)$, $h(x)$ are irreducible. In fact, the number of pairs $(g(x), h(x))$ where $g(x)$ or $h(x)$ is reducible is $O_f(y^{2d-4} \ln y)$. Thus,

$$\begin{aligned} R(y) &= \sum_{S=0}^{d-2} \prod_{j=0}^{S} (2\lfloor y \rfloor + 1 - |f_{j,1}|)(2\lfloor y \rfloor + 1 - |f_{j,2}|) + O_f(y^{2d-4} \ln y) \\ &= ((4y)^{2d-2} + O_f(y^{2d-4} \ln y)) + O_f(y^{2d-4} \ln y) \\ &= (4y)^{2d-2} + O_f(y^{2d-4} \ln y). \end{aligned}$$

Since any constant depending only on the coefficients and degree of $f(x)$ is small compared to $\ln y$ when y is sufficiently large, we get

$$R(y) \sim (4y)^{2d-2}. \ \blacksquare$$

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