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A mixing operator T for which (T, T^2) is not disjoint transitive

by

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Abstract. Using a result from ergodic Ramsey theory, we answer a question posed by Bès, Martin, Peris and Shkarin by exhibiting a mixing operator T on a Hilbert space such that the tuple (T, T^2) is not disjoint transitive.

1. Introduction. Let X be a separable topological vector space. Denote by $\mathcal{L}(X)$ the set of bounded linear operators on X. From now on, unless otherwise specified, an *operator* is a member of $\mathcal{L}(X)$. An operator T is called *hypercyclic* provided that there exists a vector $x \in X$ such that its orbit $\{T^n x : n \geq 0\}$ is dense in X; then x is called a *hypercyclic vector* for T. Hypercyclic operators are one of the most studied objects in linear dynamics; see [9] and [1] for further information concerning concepts, results and a detailed account on this subject. More generally, a tuple $(T_1, \ldots, T_N)$ of operators is said to be *disjoint hypercyclic* (*d-hypercyclic* for short) if

$$\{(T_1^n x, \ldots, T_N^n x) : n \in \mathbb{N}\}$$

is dense in X^N for some $x \in X$.

If X is an F-space, then thanks to Birkhoff's theorem [1], T is hypercyclic if and only if T is *topologically transitive*, i.e. for any non-empty open sets U, V in X, the return set $N(U, V) := \{n \geq 0 : T^n(U) \cap V \neq \emptyset\}$ is non-empty. If $N(U, V)$ is cofinite for any non-empty open sets U and V, then T is said to be *mixing*.

The notion of disjoint transitivity, a strengthening of transitivity, is defined as follows: a tuple $(T_1, \ldots, T_N)$ of operators is *disjoint transitive* (*d-transitive* for short) if for any $(N+1)$-tuple $(U_i)_{i=0}^N$ of non-empty open sets in X,

$$N_{U_1,\ldots,U_N;U_0} := \{k \geq 0 : T_1^{-k}(U_1) \cap \cdots \cap T_N^{-k}(U_N) \cap U_0 \neq \emptyset\}$$

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is non-empty. In particular, if $N_{U_1,\dots,U_N;U_0}$ happens to be cofinite for any $(U_i)_{i=0}^N$ as above, then $(T_1,\dots,T_N)$ is said to be *disjoint mixing* (*d-mixing* for short). A connection between d-hypercyclicity and d-transitivity can be found in [8].

Bès, Martin, Peris and Shkarin [5] showed the following: if T is an operator on X satisfying the Original Kitai Criterion, then the tuple $(T,\dots,T^r)$ is d-mixing for every $r\in\mathbb{N}$. As a consequence, a bilateral weighted shift T on $l^p(\mathbb{Z})$ $(1\le p<\infty)$ or $c_0(\mathbb{Z})$ is mixing if and only if $(T,\dots,T^r)$ is d-mixing, for any $r\in\mathbb{N}$. Nevertheless, they remarked that this phenomenon does not occur beyond the weighted shift context, by providing an example of a mixing Hilbert space operator T such that (T,T^2) is not d-mixing. This is a partial answer to the following question posed in [5].

Question 1.1. *Does there exist a mixing continuous linear operator T on a separable Banach space such that (T,T^2) is not d-transitive?*

Our aim is to give a positive answer to Question 1.1 (Theorem 1.6 below).

1.1. Preliminaries and main results. For $A\subseteq\mathbb{N}$, $|A|$ stands for the cardinality of A. Let $\mathscr{F}$ be a set of subsets of $\mathbb{N}$. We say that $\mathscr{F}$ is a *family* on $\mathbb{N}$ provided (I) $|A|=\infty$ for any $A\in\mathscr{F}$ and (II) $A\in\mathscr{F}$ and $A\subset B$ implies $B\in\mathscr{F}$. From now on $\mathscr{F}$ will be a family on $\mathbb{N}$.

In a natural way we generalize the notion of disjoint transitivity by introducing what we call *$\mathscr{F}$-disjoint transitivity* (or *d-$\mathscr{F}$* for short).

Definition 1.2. A tuple $(T_{1,n_k},\dots,T_{N,n_k})_k$ of sequences of operators is said to be *d-$\mathscr{F}$* if for any $(N+1)$-tuple $(U_i)_{i=0}^N$ of non-empty open sets we have

$$\{k\ge 0: T_{1,n_k}^{-1}(U_1)\cap\dots\cap T_{N,n_k}^{-1}(U_N)\cap U_0\ne\emptyset\}\in\mathscr{F}.$$

In particular, if $T_{i,n_k}=T_i^k$ for any $k\in\mathbb{N}$ and $1\le i\le N$ in the above definition, then the N-tuple $(T_1,\dots,T_N)$ of operators is said to be *d-$\mathscr{F}$*.

Observe that in particular when $\mathscr{F}$ is the family of non-empty sets or the family of cofinite sets, we obtain the notion of disjoint transitivity and disjoint mixing respectively. On the other hand, if $N=1$ we obtain $\mathscr{F}$-transitivity: an operator T is called *$\mathscr{F}$-transitive* (or an *$\mathscr{F}$-operator* for short) if $N(U,V):=\{n\ge 0: T^n(U)\cap V\ne\emptyset\}\in\mathscr{F}$ for any open U,V. This notion was introduced and studied in [7].

Recall that an operator T is said to be *chaotic* if it is hypercyclic and has a dense set of periodic points ($x\in X$ is a *periodic point* of T if $T^kx=x$ for some $k\ge 1$).

An operator T is said to be *reiteratively hypercyclic* if there exists $x\in X$ such that for every non-empty open set U in X, the set $N(x,U)=\{n\ge 0: T^nx\in U\}$ has positive upper Banach density, where the *upper Banach* *density* of a set $A \subset \mathbb{N}$ is given by

$$\overline{\mathrm{Bd}}(A) = \lim_{s\to\infty} \frac{\alpha^s}{s},$$

and $\alpha^s = \limsup_{k\to\infty} |A \cap [k+1, k+s]|$ for any $s \geq 1$. Reiteratively hypercyclic operators have been studied in [6] and [12].

It is known that there exists a reiteratively hypercyclic operator which is not chaotic (see [1]). However, concerning the converse we have the following result due to Menet.

THEOREM 1.3 ([11, Theorem 1.1]). *Any chaotic operator is reiteratively hypercyclic.*

Recall that a set $A \subseteq \mathbb{N}$ is *syndetic* if it has bounded gaps, i.e. if A is increasingly enumerated as $(x_n)_n = A$, then $\max_n(x_{n+1} - x_n) < M$ for some $M > 0$.

In [5], the authors show that there exists a mixing operator T on a Hilbert space such that (T, T^2) is not d-mixing. We show that the same operator has more specific properties.

THEOREM 1.4. *There exists $T \in \mathcal{L}(l^2)$ such that T is mixing, chaotic and (T, T^2) is not d-syndetic.*

So, our result improves the result of [5] already mentioned but still does not answer Question 1.1. In answering that question, Szemerédi's famous theorem will unexpectedly play an important role. Indeed, using a result of ergodic Ramsey theory due to Bergelson and McCutcheon [4], which is in fact a kind of Szemerédi's theorem for generalized polynomials, we obtain the following result.

THEOREM 1.5. *Let $r \in \mathbb{N}$. If T is reiteratively hypercyclic then $(T, \dots, T^r)$ is d-syndetic or not d-transitive.*

Now, from Theorems 1.3–1.5 we can deduce our main result which gives a positive answer to Question 1.1.

THEOREM 1.6. *There exists a mixing and chaotic operator T in $\mathcal{L}(l^2)$ such that (T, T^2) is not d-transitive.*

2. Proof of Theorem 1.6. As already mentioned, to prove Theorem 1.6 it is enough to prove Theorems 1.4 and 1.5.

2.1. Proof of Theorem 1.4. In [5, Theorem 3.8] the authors give an example of a mixing Hilbert space operator T such that (T, T^2) is not d-mixing. We will show that in addition T is chaotic and (T, T^2) is not d-syndetic. So in particular it is not d-mixing. We follow the proof of [5, Theorem 3.8] with minor modifications. Nevertheless, we describe all the details for completeness.

Let $1 \le p < \infty, -\infty < a < b < \infty$ and $k \in \mathbb{N}$. Recall that the Sobolev space $W^{k,p}[a,b]$ is the space of functions $f \in C^{k-1}[a,b]$ such that $f^{(k-1)}$ is absolutely continuous and $f^{(k)} \in L^p[a,b]$. The space $W^{k,p}[a,b]$ endowed with the norm

$$\|f\|_{W^{k,p}[a,b]} = \Big(\int_a^b \Big(\sum_{j=0}^k |f^{(j)}(x)|^p\Big)\,dx\Big)^{1/p}$$

is a Banach space isomorphic to $L^p[0,1]$. Now, $W^{k,2}[a,b]$ is a separable infinite-dimensional Hilbert space for each $k \in \mathbb{N}$. The family of operators to be considered lives on separable complex Hilbert spaces and is built from a single operator. Let $M \in \mathcal{L}(W^{2,2}[-\pi,\pi])$ be defined by the formula

$$M : W^{2,2}[-\pi,\pi] \to W^{2,2}[-\pi,\pi], \qquad Mf(x) = e^{ix}f(x). \tag{2.1}$$

Denote $\mathscr{H} = W^{2,2}[-\pi,\pi]$ and let M^* be the dual operator of M. Then $M^* \in \mathcal{L}(\mathscr{H}^*)$. For each $t \in [-\pi,\pi]$ we have $\delta_t \in \mathscr{H}^*$, where $\delta_t : \mathscr{H} \to \mathbb{C}$, $\delta_t(f) = f(t)$. Furthermore, the map $t \mapsto \delta_t$ from $[-\pi,\pi]$ to $\mathscr{H}^*$ is norm-continuous. For a non-empty compact subset K of $[-\pi,\pi]$, denote

$$X_K = \overline{\operatorname{span}\{\delta_t : t \in K\}}$$

where the closure is taken with respect to the norm of $\mathscr{H}^*$.

Now, the functionals δ_t are linearly independent, X_K is always a separable Hilbert space, and X_K is infinite-dimensional if and only if K is infinite. Moreover,

$$M^*\delta_t = e^{it}\delta_t \quad \text{for each } t \in [-\pi,\pi].$$

Hence, each X_K is an invariant subspace for M^*, which allows us to consider the operator

$$Q_K \in \mathcal{L}(X_K), \qquad Q_K = M^*|_{X_K}.$$

The following is taken from [5] and tells us when Q_K is mixing or non-transitive; we omit the proof.

Proposition 2.1 ([5, Proposition 3.9]). *Let K be a non-empty compact subset of $[-\pi,\pi]$. If K has no isolated points, then Q_K is mixing. If K has an isolated point, then Q_K is non-transitive.*

Now, consider the set

$$K = \Big\{\sum_{n=1}^{\infty} 2\pi\epsilon_n \cdot \frac{1}{2^{6^n}} : \epsilon \in \{0,1\}^{\mathbb{N}}\Big\}. \tag{2.2}$$

Then, as pointed out in [5], the operator $Q_K \in \mathcal{L}(X_K)$ is mixing, but (Q_K, Q_K^2) is not d-mixing. In addition, we will show that Q_K is chaotic and (Q_K, Q_K^2) is not d-syndetic.

Lemma 2.2. *Let K be the compact subset of $[-\pi,\pi]$ defined in (2.2). Then Q_K is chaotic.*

Proof. Since Q_K is mixing by Proposition 2.1, it remains to show that it has a dense set of periodic points. Denote by $\mathrm{Per}(Q_K)$ the set of periodic points of Q_K.

Recall that $Q_K^n\delta_t = e^{int}\delta_t$ for any $n \in \mathbb{Z}_+$ and $t \in K$; the details can be found in [5, proof of Proposition 3.9].

Consider the set $A = \{\sum_{n=1}^k 2\pi\epsilon_n/2^{6^n} : \epsilon \in \{0,1\}^{\{1,\dots,k\}},\ k \in \mathbb{N}\}$.

Observe that $\sum_{n=1}^k 2\pi\epsilon_n/2^{6^n} = 2\pi m/2^{6^k}$ for some m and any ϵ in $\{0,1\}^{\{1,\dots,k\}}$. So clearly $\{\delta_t : t \in A\} \subseteq \mathrm{Per}(Q_K)$. Moreover, if $r_1 = 2\pi m_1/2^{6^{n_1}}$ and $r_2 = 2\pi m_2/2^{6^{n_2}}$ are in A, then $Q_K^{2^{6^{n_1}}2^{6^{n_2}}}(\alpha_1\delta_{r_1} + \alpha_2\delta_{r_2}) = \alpha_1\delta_{r_1} + \alpha_2\delta_{r_2}$ for any $\alpha_1, \alpha_2 \in \mathbb{C}$, so $\mathrm{span}\{\delta_t : t \in A\} \subseteq \mathrm{Per}(Q_K)$.

On the other hand, since A is dense in K, we deduce that $\overline{\{\delta_t : t \in A\}} = \{\delta_t : t \in K\}$. Indeed, for any $r \in K$ there exists a sequence $(r_n)_n \subseteq A$ such that r_n tends to r. Hence, $\|\delta_r - \delta_{r_n}\| = \sup_{\|f\|=1}|f(r) - f(r_n)|$ tends to 0.

Thus, $X_K = \overline{\mathrm{span}\{\delta_t : t \in K\}} = \overline{\mathrm{span}\overline{\{\delta_t : t \in A\}}} = \overline{\mathrm{span}\{\delta_t : t \in A\}} \subseteq \overline{\mathrm{Per}(Q_K)}$. So, $\mathrm{Per}(Q_K)$ is dense in X_K. ∎

A set $A \subset \mathbb{N}$ is *thick* if it contains arbitrarily long intervals, i.e. for every $L > 0$ there exists $n \geq 1$ such that $\{n, n+1, \dots, n+L\} \subset A$.

Now, in order to obtain a mixing operator T such that (T, T^2) is not d-syndetic, it will be enough to show that the sequence $(2Q_K^{a_n} - Q_K^{2a_n})_n$ of operators is non-transitive along a thick set $A = (a_n)$. We have the following result.

PROPOSITION 2.3. *Let K be the compact subset of $[-\pi,\pi]$ defined in* (2.2). *Then the sequence $(2Q_K^{k_{n,r}} - Q_K^{2k_{n,r}})_{n\in\mathbb{N},\,0\leq r\leq n}$ of continuous linear operators on X_K is non-transitive, where $k_{n,r} = 2^{6^n} - r$ with $0 \leq r \leq n$, $n \in \mathbb{N}$.*

Now we are in a position to prove Theorem 1.4.

We follow [5, proof of Theorem 3.8], but still we give all the details. We need to exhibit a mixing and chaotic operator T such that (T, T^2) is not d-syndetic.

Let K be the compact set defined in (2.2). By Proposition 2.1 and Lemma 2.2, Q_K is mixing and chaotic on the separable infinite-dimensional Hilbert space X_K. On the other hand, by Proposition 2.3, $(2Q_K^{a_n} - Q_K^{2a_n})_{n\in\mathbb{N}}$ is non-transitive for some thick set A written increasingly as $A = (a_n)_n$. Hence, there exist non-empty open sets U, V in X_K such that $(2Q_K^{a_n} - Q_K^{2a_n})(U) \cap V = \emptyset$ for any $n \in \mathbb{N}$. In other words,

$$\{n \in \mathbb{N} : (2Q_K^n - Q_K^{2n})(U) \cap V \neq \emptyset\} \cap A = \emptyset,$$

i.e. the set $\{n \in \mathbb{N} : (2Q_K^n - Q_K^{2n})(U) \cap V \neq \emptyset\}$ cannot be syndetic. In particular, (Q_K, Q_K^2) is not d-syndetic. Indeed, pick a non-empty open set V_0 such that $2V_0 - V_0 \subseteq V$ (denote by $B(x;r)$ the open ball centered at x

in X_K with radius r; pick $x \in X_K$ and $r \in \mathbb{R}_+$ such that $B(x;r) \subset V$; then set $V_0 := B(x; r/3)$). Hence,

$$\{n \in \mathbb{N} : U \cap Q_K^{-n}(V_0) \cap Q_K^{-2n}(V_0) \neq \emptyset\} \subseteq \{n \in \mathbb{N} : (2Q_K^n - Q_K^{2n})(U) \cap V \neq \emptyset\}.$$

Consequently, $\{n \in \mathbb{N} : U \cap Q_K^{-n}(V_0) \cap Q_K^{-2n}(V_0) \neq \emptyset\}$ cannot be a syndetic set and so (Q_K, Q_K^2) is not d-syndetic. Since all separable infinite-dimensional Hilbert spaces are isomorphic to l^2, there is a mixing and chaotic $T \in \mathcal{L}(l^2)$ such that (T, T^2) is not d-syndetic. This concludes the proof of Theorem 1.4.

In order to close this subsection, we need to prove Proposition 2.3. We follow [5, proof of Proposition 3.10], except that instead of [5, Lemma A.3], we use Lemma 2.6 below.

To prove Lemma 2.6 we need another two lemmas proved in [5] that we state without proof.

Lemma 2.4 ([5, Lemma A.1]). *Let $f \in W^{2,2}[-\pi,\pi]$, $f(-\pi) = f(\pi)$, $f'(-\pi) = f'(\pi)$, $c_0 = \|f\|_{L^\infty[-\pi,\pi]}$ and $c_1 = \|f''\|_{L^2[-\pi,\pi]}$. Then $\|f\|_{W^2[-\pi,\pi]} \leq \sqrt{3c_1^2 + c_0^2}$.*

Lemma 2.5 ([5, Lemma A.2]). *Let $-\infty < \alpha < \beta < \infty$ and $a_0, a_1, b_0, b_1 \in \mathbb{C}$. Then there exists $f \in C^2[\alpha, \beta]$ such that*

$$f(\alpha) = a_0, \quad f'(\alpha) = a_1, \quad f(\beta) = b_0, \quad f'(\beta) = b_1,$$
$$\|f\|_{L^\infty[\alpha,\beta]} \leq |a_0 + b_0|/2 + |a_0 - b_0|/2 + (\beta - \alpha)(|a_1| + |b_1|)/5,$$
$$\|f''\|^2_{L^2[\alpha,\beta]} \leq \frac{24|a_0 - b_0|^2}{(\beta - \alpha)^3} + \frac{12(|a_1|^2 + |b_1|^2)}{\beta - \alpha}.$$

Lemma 2.6. *There exists a sequence $(f_{2^{6^n}-r})_{n\in\mathbb{N},\, 0\leq r\leq n}$ of 2π-periodic functions on $\mathbb{R}$ such that $f_{2^{6^n}-r}|_{[-\pi,\pi]} \in W^{2,2}[-\pi,\pi]$, the sequence $(\|f_{2^{6^n}-r}\|_{W^{2,2}[-\pi,\pi]})_{n,r}$ is bounded and*

$$f_{2^{6^n}-r}(x) = 2e^{i(2^{6^n}-r)x} - e^{2i(2^{6^n}-r)x}$$

whenever $|x - 2\pi m/2^{6^n}| \leq 2/(2^{6^n})^5$ for some $m \in \mathbb{Z}$ and every $n \in \mathbb{N}$ and $0 \leq r \leq n$.

Proof. We slightly modify the proof of Lemma A.3 in [5].

For $n \in \mathbb{N}$ and $0 \leq r \leq n$, let

$$k_{n,r} = 2^{6^n} - r \quad \text{and} \quad h_{k_{n,r}} = 2e^{ik_{n,r}x} - e^{2ik_{n,r}x}.$$

Note that $h_{k_{n,r}}$ is $2\pi/k_{n,r}$-periodic. Let also

$$\alpha_{n,r} = 2/(2^{6^n})^5 - 2\pi/2^{6^n} \quad \text{and} \quad \beta_{n,r} = -2/(2^{6^n})^5.$$

By Lemma 2.5, there is $g_{k_{n,r}} \in C^2[\alpha_{n,r}, \beta_{n,r}]$ such that

$$(2.3)\qquad \begin{aligned} g_{k_{n,r}}(\alpha_{n,r}) &= h_{k_{n,r}}(2/(2^{6^n})^5), \quad g_{k_{n,r}}(\beta_{n,r}) = h_{k_{n,r}}(-2/(2^{6^n})^5),\\ g'_{k_{n,r}}(\alpha_{n,r}) &= h'_{k_{n,r}}(2/(2^{6^n})^5), \quad g'_{k_{n,r}}(\beta_{n,r}) = h'_{k_{n,r}}(-2/(2^{6^n})^5), \end{aligned}$$

$$(2.4)\qquad \begin{aligned} \|g_{k_{n,r}}\|_{L^\infty_{[\alpha_{n,r},\beta_{n,r}]}} &\le \max\{|h_{k_{n,r}}(2/(2^{6^n})^5)|, |h_{k_{n,r}}(-2/(2^{6^n})^5)|\}\\ &\quad + \frac{(\beta_{n,r}-\alpha_{n,r})}{5}(|h'_{k_{n,r}}(2/(2^{6^n})^5)| + |h'_{k_{n,r}}(-2/(2^{6^n})^5)|), \end{aligned}$$

$$(2.5)\qquad \begin{aligned} \|g''_{k_{n,r}}\|^2_{L^2_{[\alpha_{n,r},\beta_{n,r}]}} &\le \frac{24|h_{k_{n,r}}(2/(2^{6^n})^5) - h_{k_{n,r}}(-2/(2^{6^n})^5)|^2}{(\beta_{n,r}-\alpha_{n,r})^3}\\ &\quad + 12\frac{|h'_{k_{n,r}}(2/(2^{6^n})^5)|^2 + |h'_{k_{n,r}}(-2/(2^{6^n})^5)|^2}{(\beta_{n,r}-\alpha_{n,r})}. \end{aligned}$$

The equalities (2.3) imply that there is a unique $f_{k_{n,r}} \in C^1(\mathbb{R})$ such that $f_{k_{n,r}}$ is $2\pi/2^{6^n}$-periodic,

$$f_{k_{n,r}}|_{[\alpha_{n,r},\beta_{n,r}]} = g_{k_{n,r}} \quad \text{and} \quad f_{k_{n,r}}|_{[\beta_{n,r},\alpha_{n,r}+2\pi/2^{6^n}]} = h_{k_{n,r}}.$$

$2\pi/2^{6^n}$-periodicity of $f_{k_{n,r}}$ and the equality $f_{k_{n,r}}|_{[\beta_{n,r},\alpha_{n,r}+2\pi/2^{6^n}]} = h_{k_{n,r}}$ imply that $f_{k_{n,r}}(x) = 2e^{i(2^{6^n}-r)x} - e^{2i(2^{6^n}-r)x}$ whenever $|x - 2\pi m/2^{6^n}| \le 2/(2^{6^n})^5$, for every $m \in \mathbb{Z}$ with $|2m| \le 2^{6^n}$ and all $n \in \mathbb{N}$ and $0 \le r \le n$. Since $f_{k_{n,r}}$ is piecewise C^2, we have $f_{k_{n,r}}|_{[-\pi,\pi]} \in W^{2,2}[-\pi,\pi]$. It remains to verify that the sequence $(\|f_{k_{n,r}}\|_{W^{2,2}[-\pi,\pi]})_{n,r}$ is bounded.

Using the inequality $|e^{it} - e^{is}| \le |t-s|$ for $t, s \in \mathbb{R}$, we get

$$|h'_{k_{n,r}}(2/(2^{6^n})^5)| = |h'_{k_{n,r}}(-2/(2^{6^n})^5)| \le 2(2^{6^n}-r)^2 \cdot 2/(2^{6^n})^5.$$

Hence by (2.4),

$$\|f_{k_{n,r}}\|_{L^\infty_{[\alpha_{n,r},\beta_{n,r}]}} \le 3 + 5^{-1}\left(\frac{2\pi}{2^{6^n}} - \frac{4}{(2^{6^n})^5}\right) \cdot 8\frac{(2^{6^n}-r)^2}{(2^{6^n})^5} < 9.$$

Since $\|h_{k_{n,r}}\|_{L^\infty_{[\beta_{n,r},\alpha_{n,r}+2\pi/2^{6^n}]}} \le 3$ and $f_{k_{n,r}}$ is $2\pi/2^{6^n}$-periodic, we obtain

$$(2.6)\qquad \|f_{k_{n,r}}\|_{L^\infty_{[-\pi,\pi]}} \le \max\{3, 9\} = 9.$$

Next,

$$\begin{aligned} &|h_{k_{n,r}}(2/(2^{6^n})^5) - h_{k_{n,r}}(-2/(2^{6^n})^5)|\\ &\quad = \left|2(e^{i(2^{6^n}-r)\frac{2}{(2^{6^n})^5}} - e^{i(2^{6^n}-r)\frac{-2}{(2^{6^n})^5}}) - (e^{2i(2^{6^n}-r)\frac{2}{(2^{6^n})^5}} - e^{2i(2^{6^n}-r)\frac{-2}{(2^{6^n})^5}})\right|\\ &\quad = \left|4\sin\left(2 \cdot \frac{2^{6^n}-r}{(2^{6^n})^5}\right) - 2\sin\left(4 \cdot \frac{2^{6^n}-r}{(2^{6^n})^5}\right)\right| \end{aligned}$$

$$= 4\sin\Big(2\cdot\frac{2^{6^n}-r}{(2^{6^n})^5}\Big)\Big(1-\cos\Big(2\cdot\frac{2^{6^n}-r}{(2^{6^n})^5}\Big)\Big)$$
$$= 16\sin^3\Big(\frac{2^{6^n}-r}{(2^{6^n})^5}\Big)\cos\Big(\frac{2^{6^n}-r}{(2^{6^n})^5}\Big) \le 16\Big(\frac{2^{6^n}-r}{(2^{6^n})^5}\Big)^3 \le \frac{16}{(2^{6^n})^{12}}.$$

On the other hand,

$$\frac{|h'_{k_{n,r}}(2/(2^{6^n})^5)|^2+|h'_{k_{n,r}}(-2/(2^{6^n})^5)|^2}{\beta_{n,r}-\alpha_{n,r}} \le \frac{32\frac{(2^{6^n}-r)^4}{(2^{6^n})^{10}}}{\frac{2\pi}{2^{6^n}}-\frac{4}{(2^{6^n})^5}}$$
$$\le \frac{\frac{32}{(2^{6^n})^6}}{\frac{2\pi}{2^{6^n}}-\frac{4}{(2^{6^n})^5}} = \frac{32}{2\pi(2^{6^n})^5-4\cdot 2^{6^n}} \le \frac{32}{2\pi(2^{6^n})^5-4(2^{6^n})^5} \le \frac{16}{(2^{6^n})^5}.$$

Hence by (2.5),

$$\|f''_{k_{n,r}}\|^2_{L^2_{[\alpha_{n,r},\beta_{n,r}]}} \le 24\cdot\frac{\big(\frac{16}{(2^{6^n})^{12}}\big)^2}{\big(\frac{2\pi}{2^{6^n}}-\frac{4}{(2^{6^n})^5}\big)^3} + 12\cdot\frac{16}{(2^{6^n})^5}$$
$$\le 24\cdot\frac{16^2\cdot(2^{6^n})^{-24}}{\big(\frac{2\pi}{2^{6^n}}-\frac{4}{2^{6^n}}\big)^3} + 12\cdot\frac{16}{(2^{6^n})^5}$$
$$\le \frac{24\cdot 16^2}{8\cdot(2^{6^n})^{21}} + \frac{12\cdot 16}{(2^{6^n})^5} \le \frac{960}{(2^{6^n})^5}.$$

Since $|h''_{k_{n,r}}(x)| \le 6(k_{n,r})^2$ for $x\in[\beta_{n,r},\alpha_{n,r}+2\pi/2^{6^n}]$, we have

$$\|f''_{k_{n,r}}\|^2_{L^2_{[\beta_{n,r},\alpha_{n,r}+2\pi/2^{6^n}]}} \le 36\cdot(2^{6^n}-r)^4\cdot\frac{4}{(2^{6^n})^5} \le \frac{144}{2^{6^n}}.$$

Hence,

$$\|f''_{k_{n,r}}\|^2_{L^2_{[\alpha_{n,r},\alpha_{n,r}+2\pi/2^{6^n}]}} \le \frac{960}{(2^{6^n})^5}+\frac{144}{2^{6^n}} \le \frac{1104}{2^{6^n}}.$$

Since $f''_{k_{n,r}}$ is $2\pi/2^{6^n}$-periodic, we find that

$$\|f''_{k_{n,r}}\|^2_{L^2_{[-\pi,\pi]}} = 2^{6^n}\cdot\|f''_{k_{n,r}}\|^2_{L^2_{[\alpha_{n,r},\alpha_{n,r}+2\pi/2^{6^n}]}} \le 1104. \tag{2.7}$$

Now, by Lemma 2.4 and using (2.7) and (2.6) we obtain

$$\|f_{k_{n,r}}\|_{W^{2,2}[-\pi,\pi]} \le \sqrt{3\cdot 1104+9^2} < 64$$

for each $n\in\mathbb{N}$ and $0\le r\le n$. ∎

2.2. Proof of Theorem 1.5. The main ingredient of the proof of Theorem 1.5 is a result due to Bergelson and McCutcheon concerning essential idempotents of $\beta\mathbb{N}$ (the Stone–Čech compactification of $\mathbb{N}$), and Szemerédi's theorem for generalized polynomials [4]. So, we first need some background on $\beta\mathbb{N}$.

Recall that a *filter* is a family $\mathscr{F}$ invariant by finite intersections, i.e. $A, B \in \mathscr{F}$ implies $A \cap B \in \mathscr{F}$. The collection of all maximal filters (in the sense of inclusion) is denoted by $\beta\mathbb{N}$. Elements of $\beta\mathbb{N}$ are known as *ultrafilters*; endowed with an appropiate topology, $\beta\mathbb{N}$ becomes the Stone–Čech compactification of $\mathbb{N}$. Each point $i \in \mathbb{N}$ is identified with the principal ultrafilter $\mathfrak{U}_i := \{A \subseteq \mathbb{N} : i \in A\}$, which yields an embedding of $\mathbb{N}$ into $\beta\mathbb{N}$. For any $A \subseteq \mathbb{N}$ and $p \in \beta\mathbb{N}$, the closure $\operatorname{cl} A$ of A in $\beta\mathbb{N}$ is defined as follows: $p \in \operatorname{cl} A$ if and only if $A \in p$.

Given $p, q \in \beta\mathbb{N}$ and $A \subseteq \mathbb{N}$, the operation $(\mathbb{N}, +)$ can be extended to $\beta\mathbb{N}$ so as to make $(\beta\mathbb{N}, +)$ a compact right topological semigroup. The extended operation can be defined by: $A \in p+q$ if and only if $\{n \in \mathbb{N} : -n+A \in q\} \in p$. Now, according to a famous theorem of Ellis, idempotents (with respect to $+$) exist. Let $E(\mathbb{N}) = \{p \in \beta\mathbb{N} : p = p+p\}$ be the collection of idempotents in $\beta\mathbb{N}$. For further details see [10]. Given a family $\mathscr{F}$, the *dual* family $\mathscr{F}^*$ consists of all sets A such that $A \cap F \neq \emptyset$ for every $F \in \mathscr{F}$. The following is a well-known result.

Lemma 2.7.

(1) *If $\mathscr{F}$ is an ultrafilter, then $\mathscr{F}^* = \mathscr{F}$.*
(2) *If $\mathscr{F} = \bigcup_\alpha \mathscr{F}_\alpha$, then $\mathscr{F}^* = \bigcap_\alpha \mathscr{F}_\alpha^*$.*

In particular, whenever $\mathscr{F}$ is a union of some collection of ultrafilters, then $\mathscr{F}^*$ is the intersection of the same collection.

The collection of essential idempotents is commonly denoted by $\mathcal{D}$.

The collection $\mathcal{D}$ (of D-sets) is the union of all idempotents $p \in \beta\mathbb{N}$ such that every member of p has positive upper Banach density. Accordingly, $\mathcal{D}^*$ is the intersection of all such idempotents.

The following result of ergodic Ramsey theory is due to Bergelson and McCutcheon [4]. It is indeed a sort of Szemerédi's theorem stated originally for generalized polynomials, and it will be crucial for proving Theorem 1.5.

Theorem 2.8 ([4, Theorem 1.25]). *Let $F \subset \mathbb{N}$ have positive upper Banach density and $g_1 \dots, g_r$ be polynomials. Then*

$$\big\{k \in \mathbb{N} : \overline{\mathrm{Bd}}\big(F \cap (F - g_1(k)) \cap \dots \cap (F - g_r(k))\big) > 0\big\} \in \mathcal{D}^*.$$

We can now prove Theorem 1.5.

Fix $r \in \mathbb{N}$. Let T be reiteratively hypercyclic. Then there exists $x \in X$ such that $\overline{\mathrm{Bd}}(N(x, U)) > 0$ for any non-empty open set U in X. First, let us see that

$$N_T(\underbrace{U, \dots, U}_{r}; U) = \{k \geq 0 : T^{-k}U \cap \dots \cap T^{-rk}U \cap U \neq \emptyset\} \in \mathcal{D}^* \tag{2.8}$$

for any non-empty open set U in X. We will show that

$$A_U := \{k \geq 0 : \overline{\mathrm{Bd}}(N(x,U) \cap (N(x,U)-k) \cap \cdots \cap (N(x,U)-rk)) > 0\}$$
$$\subseteq \{k \geq 0 : T^{-k}U \cap \cdots \cap T^{-rk}U \cap U \neq \emptyset\}.$$

In fact, let $k \in A_U$. Then there exists a set A with positive upper Banach density such that for any $n \in A$ we have $T^{n+ik}x \in U$ for any $i \in \{0,\ldots,r\}$. Consequently, $T^n x \in T^{-k}U \cap \cdots \cap T^{-rk}U \cap U$. Now, by Theorem 2.8, it follows that $A_U \in \mathcal{D}^*$. Thus condition (2.8) holds.

Next, let $(U_j)_{j=0}^r$ be a finite sequence of non-empty open sets in X. Suppose that $(T,\ldots,T^r)$ is d-transitive. We must show that $N_T(U_1,\ldots,U_r;U_0)$ is a syndetic set. In fact, there exists $n \in \mathbb{N}$ such that

$$V_n := T^{-n}U_1 \cap \cdots \cap T^{-rn}U_r \cap U_0 \neq \emptyset.$$

Thus V_n is open. Pick non-empty open sets O_1, O_2 such that $O_1 + O_2 \subset V_n$. Then

$$T^{jn}(O_1+O_2) \subset U_j \quad \text{for any } j \in \{0,\ldots,r\}. \tag{2.9}$$

It is known that $\mathcal{D}^*$ is a filter. Now, by (2.8) we have

$$A := N_T(\underbrace{O_1,\ldots,O_1}_{r};O_1) \cap N_T(\underbrace{O_2,\ldots,O_2}_{r};O_2) \in \mathcal{D}^*.$$

In addition, it is well known that each set in $\mathcal{D}^*$ is syndetic [2]. Hence, A is syndetic. Let us show that $A+n \subseteq N(U_1,\ldots,U_r;U_0)$, then we are done because $A+n$ is syndetic, since the collection of syndetic sets is shift invariant.

In fact, let $t \in A+n$. Then $t-n \in A$, which means that

$$T^{-t}T^n(O_1) \cap \cdots \cap T^{-rt}T^{rn}(O_1) \cap O_1 \neq \emptyset,$$
$$T^{-t}T^n(O_2) \cap \cdots \cap T^{-rt}T^{rn}(O_2) \cap O_2 \neq \emptyset.$$

By the linearity of T we obtain

$$T^{-t}(T^n(O_1+O_2)) \cap \cdots \cap T^{-rt}(T^{rn}(O_1+O_2)) \cap (O_1+O_2) \neq \emptyset.$$

Then by (2.9) we conclude that

$$T^{-t}U_1 \cap \cdots \cap T^{-rt}U_r \cap U_0 \neq \emptyset.$$

This concludes the proof of Theorem 1.5.

3. Tuple of powers of a weighted shift. In linear dynamics recurrence properties are frequently studied first in the context of weighted backward shifts.

Each bilateral bounded weight $w = (w_k)_{k\in\mathbb{Z}}$ induces a *bilateral weighted backward shift* B_w on $X = c_0(\mathbb{Z})$ or $l^p(\mathbb{Z})$ $(1 \leq p < \infty)$, given by $B_w e_k := w_k e_{k-1}$, where $(e_k)_{k\in\mathbb{Z}}$ denotes the canonical basis of X.

Analogously, each unilateral bounded weight $w = (w_n)_{n\in\mathbb{Z}_+}$ induces a *unilateral weighted backward shift* B_w on $X = c_0(\mathbb{Z}_+)$ or $l^p(\mathbb{Z}_+)$ $(1 \le p < \infty)$, given by $B_w e_n := w_n e_{n-1}$, $n \ge 1$, with $B_w e_0 := 0$, where $(e_n)_{n\in\mathbb{Z}_+}$ denotes the canonical basis of X.

As previously mentioned, the authors of [5] proved that for any weighted shift B_w, the following holds: B_w is mixing if and only if $(B_w, \dots, B_w^r)$ is d-mixing for all $r \in \mathbb{N}$. The aim of this section is to show that this result extends to some families on $\mathbb{N}$ frequently studied in Ramsey theory.

Let us recall some such families:

- $\mathcal{I} = \{A \subseteq \mathbb{N} : A \text{ is infinite}\}$;
- $\Delta = \{A \subseteq \mathbb{N} : B - B \subseteq A \text{ for some infinite set } B\}$;
- $\mathcal{IP} = \{A \subseteq \mathbb{N} : \exists (x_n)_n \subseteq \mathbb{N}, \sum_{n\in F} x_n \in A \text{ for any finite set } F\}$;
- the set A is *piecewise syndetic* ($A \in \mathcal{PS}$ for short) if A can be written as the intersection of a thick set and a syndetic set.

It is known that $\mathcal{I}^*$ (the family of cofinite sets), Δ^*, $\mathcal{IP}^*$ and $\mathcal{PS}^*$ are filters. In addition, $\mathcal{I}^* \subsetneq \Delta^* \subsetneq \mathcal{IP}^* \subsetneq \mathcal{S}$ and $\mathcal{I}^* \subsetneq \mathcal{PS}^* \subsetneq \mathcal{S}$, where $\mathcal{S}$ denotes the family of syndetic sets. For a rich source of information on this subject we refer the reader to [10].

The main result of this section is the following.

THEOREM 3.1. *Let $\mathscr{F}$ be the family Δ^*, $\mathcal{IP}^*$, $\mathcal{PS}^*$ or $\mathcal{S}$. Then for any $r \in \mathbb{N}$ the following are equivalent:*

(i) *T is an $\mathscr{F}$-operator;*
(ii) *$T \oplus \cdots \oplus T^r$ is an $\mathscr{F}$-operator on X^r.*

In particular, a bilateral (or unilateral) weighted backward shift B_w on c_0 or l^p $(1 \le p < \infty)$ is an $\mathscr{F}$-operator if and only if $(B_w, \dots, B_w^r)$ is d-$\mathscr{F}$.

REMARK 3.2. Obviously, mixing operators are Δ^*-operators, but the converse is not true, as exhibited in [7], and the example is a weighted shift. Therefore, the conclusion of Theorem 3.1 concerning weighted shifts does not necessarily follow from the statement: B_w is mixing if and only if $(B_w, \dots, B_w^r)$ is d-mixing, for any $r \in \mathbb{N}$, shown in [5].

In order to prove Theorem 3.1 we will need the following results.

Recall that any tuple of powers of a fixed backward weighted shift on c_0 or l^p is d-transitive if and only if it is d-hypercyclic. This follows from [8, Theorem 2.7] and [8, Theorem 4.1]. Now, combining [8, Theorem 4.1] and [13, Theorem 2.5] in its bilateral (or unilateral) version, we obtain the following two propositions.

PROPOSITION 3.3. *Let $X = c_0(\mathbb{Z})$ or $l^p(\mathbb{Z})$ $(1 \le p < \infty)$, $w = (w_j)_{j\in\mathbb{Z}}$ a bounded bilateral weight sequence, $\mathscr{F}$ a filter on $\mathbb{N}$ and $r_0 = 0 < 1 \le r_1 < \cdots < r_N$. Then the following are equivalent:*

(i) $(B_w^{r_1}, \dots, B_w^{r_N})$ *is d-*$\mathscr{F}$,
(ii) $\bigoplus_{0\le s<l\le N} B_w^{(r_l-r_s)}$ *is an* $\mathscr{F}$*-operator on* $X^{N(N+1)/2}$,
(iii) *for any* $M > 0$, $j \in \mathbb{Z}$ *and* $0 \le s < l \le N$,

$$\Big\{m \in \mathbb{N} : \prod_{i=j+1}^{j+m(r_l-r_s)} |w_i| > M\Big\} \in \mathscr{F},$$

$$\Big\{m \in \mathbb{N} : \frac{1}{\prod_{i=j-m(r_l-r_s)+1}^{j} |w_i|} > M\Big\} \in \mathscr{F}.$$

PROPOSITION 3.4. *Let* $X = c_0(\mathbb{Z}_+)$ *or* $l^p(\mathbb{Z}_+)$ $(1 \le p < \infty)$, $w = (w_n)_{n\in\mathbb{Z}_+}$ *a bounded unilateral weight sequence,* $\mathscr{F}$ *a filter on* $\mathbb{N}$ *and* $r_0 = 0 < 1 \le r_1 < \dots < r_N$. *Then the following are equivalent:*

(i) $(B_w^{r_1}, \dots, B_w^{r_N})$ *is d-*$\mathscr{F}$,
(ii) $\bigoplus_{0\le s<l\le N} B_w^{(r_l-r_s)}$ *is an* $\mathscr{F}$*-operator on* $X^{N(N+1)/2}$,
(iii) *for any* $M > 0$, $j \in \mathbb{Z}_+$ *and* $0 \le s < l \le N$,

$$\Big\{m \in \mathbb{N} : \prod_{i=j+1}^{j+m(r_l-r_s)} |w_i| > M\Big\} \in \mathscr{F}.$$

The following results of Ramsey theory concern preservation of certain notions of largeness in products.

PROPOSITION 3.5 ([3, Corollary 2.3]). *Let* $l \in \mathbb{N}$ *and* I *be a subsemigroup of* $\mathbb{N}^l$.

(a) *If* B *is an* $\mathcal{IP}^*$*-set in* $\mathbb{N}$*, then* $B^l \cap I$ *is an* $\mathcal{IP}^*$*-set in* I.
(b) *If* B *is a* Δ^**-set in* $\mathbb{N}$*, then* $B^l \cap I$ *is a* Δ^**-set in* I.

PROPOSITION 3.6 ([3, Corollary 2.7]). *Let* $l \in \mathbb{N}$ *and* I *be a subsemigroup of* $\mathbb{N}^l$. *If* B *is a* $\mathcal{PS}^*$*-set in* $\mathbb{N}$*, then* $B^l \cap I$ *is a* $\mathcal{PS}^*$*-set in* I.

We are now finally able to prove Theorem 3.1.

Proof of Theorem 3.1. If $T \oplus \dots \oplus T^r$ is an $\mathscr{F}$-operator on X^r for some $r \in \mathbb{N}$, then obviously T is an $\mathscr{F}$-operator. Conversely, let T be an $\mathscr{F}$-operator, $r \in \mathbb{N}$ and U, V non-empty open sets. We need to show that $N(U,V) \in t\mathscr{F}$ for any $t = 1, \dots, r$.

Denote

$$A = \{m, 2m, \dots, rm : m \in \mathbb{N}\} \cap (\underbrace{N(U,V) \times \dots \times N(U,V)}_{r \text{ times}}). \tag{3.1}$$

By Proposition 3.5, if $N(U,V)$ is an $\mathcal{IP}^*$-set [Δ^*-set] in $\mathbb{N}$, then A is an $\mathcal{IP}^*$-set [Δ^*-set] in $\{m, 2m, \dots, rm : m \in \mathbb{N}\}$. Analogously, by Proposition 3.6, if $N(U,V)$ is a $\mathcal{PS}^*$-set in $\mathbb{N}$, then A is a $\mathcal{PS}^*$-set in $\{m, 2m, \dots, rm : m \in \mathbb{N}\}$.

Denote by Π_i the projection onto the ith coordinate. It is not difficult to see that $\Pi_1(A) \in \mathscr{F}$ for $\mathscr{F} = \Delta^*, \mathcal{IP}^*, \mathcal{PS}^*$, which is equivalent to

$$B = \{m \in \mathbb{N} : tm \in N(U,V)\} \in \mathscr{F}$$

for any $t = 1, \dots, r$.

Hence, $tB \subseteq N(U,V)$ and $B \in \mathscr{F}$. Then $N(U,V) \in t\mathscr{F}$ for any $t = 1, \dots, r$. Since $\mathscr{F} = \Delta^*, \mathcal{IP}^*, \mathcal{PS}^*$, it is a filter, and it is not difficult to see that $T \oplus \cdots \oplus T^r$ is indeed an $\mathscr{F}$-operator on X^r.

If B_w is a weighted shift on c_0 or l^p and $\mathscr{F} = \Delta^*, \mathcal{IP}^*, \mathcal{PS}^*$, then by Proposition 3.3 (or Proposition 3.4), we deduce that B_w is an $\mathscr{F}$-operator if and only if $(B_w, \dots, B_w^r)$ is d-$\mathscr{F}$ for any $r \in \mathbb{N}$.

Finally, let $\mathscr{F}$ be the family of syndetic sets. Just recall that T is syndetic if and only if T is a $\mathcal{PS}^*$-operator [7]. Hence T is syndetic if and only if $T \oplus \cdots \oplus T^r$ is a $\mathcal{PS}^*$-operator on X^r, for any $r \in \mathbb{N}$. If B_w is a weighted shift, then B_w is syndetic if and only if $(B_w, \dots, B_w^r)$ is d-$\mathcal{PS}^*$, for any $r \in \mathbb{N}$. This concludes the proof of Theorem 3.1.

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