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A NOTE ON A PAPER BY PISZCZEK ON JORDAN DECOMPOSITION IN NONCOMMUTATIVE SCHWARTZ SPACE

BY

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Abstract. The positive cone in a closed *-subalgebra of the noncommutative Schwartz algebra of smooth operators is normal, which immediately implies decomposition of a continuous self-adjoint functional as a difference of two positive functionals. A decomposition as a difference of two representable positive functionals holds precisely for self-adjoint functionals continuous in the C^*-operator norm.

Introduction. Let $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let s be the Schwartz space of rapidly decreasing scalar sequences, $s = \{\xi = (\xi_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : |\xi|_k^2 := \sum_{j=1}^{\infty} |\xi_j|^2 j^{2k} < \infty$ for all $k \in \mathbb{N}_0\}$, a Fréchet space with the topology defined by the sequence $\{|\cdot|_k\}$ of norms. Its topological dual is the space of slowly increasing sequences given by $s' = \{\eta = (\eta_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : |\eta|_k'^2 := \sum_{j=1}^{\infty} |\eta_j|^2 j^{-2k} < \infty$ for some $k \in \mathbb{N}_0\}$ with the inductive topology, s' being the union of a sequence of Banach spaces.

Let $L(s', s)$ be the Fréchet space of all continuous linear operators from s' to s with the topology of uniform convergence on bounded sets defined by the seminorms $\|x\|_k = \sup\{|x\xi|_k : |\xi|_k' \le 1\}$. It is a *-algebra, with multiplication $x \cdot y := x \circ j \circ y$, $j : s \to s'$ being the inclusion, and with the involution $x^* = \Phi^{-1}(\Phi(x)^*)$, where $\Phi : L(s', s) \to \mathcal{S}$ is the isomorphism onto the matrix algebra $\mathcal{S}$ of rapidly decreasing matrices, $\mathcal{S} := \{x = [x_{ij}]_{i,j\in\mathbb{N}} : \|x\|_n^2 := \sum_{i,j=1}^{\infty} |x_{ij}|^2 i^{2n} j^{2n} < \infty$ for all $n \in \mathbb{N}_0\}$, $\Phi(x) := [\langle xe_k, e_j\rangle]$, $\Phi(x)^* = [\langle xe_j, e_k\rangle]^- = [\langle x^*e_k, e_j\rangle]$. The algebra $L(s', s)$ is a Fréchet *-algebra with norms satisfying $\|xy\|_k \le \|x\|_k\|y\|_k$, $\|x^*\|_k = \|x\|_k$. The operators in $L(s', s)$ are called *smooth operators*; and $L(s', s)$ is called the *noncommutative Schwartz space*.

The Fréchet algebra $L(s', s)$ has several nice representations [2], including one as the algebra of rapidly decreasing matrices [4], or as the algebra

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of operators from the space of distributions on a compact manifold M to the space of smooth functions on M. It also appears in [9]. The spectral representation of a normal smooth operator and subsequent functional calculus have been developed in [1]. Recently Piszczek [6] has shown that every continuous self-adjoint linear functional on $L(s', s)$ can be decomposed as a difference of two positive linear functionals in a minimal way. The present note supplements [6] in the following respects.

(a) A transparent argument involving the order structure shows that the positive cone in a closed *-subalgebra $\mathcal{A}$ of $L(s', s)$ is normal. This immediately gives Jordan decomposition in $\mathcal{A}$ (in particular in $L(s', s)$). This compares with the highly constructive proof in [6] for $L(s', s)$.

(b) The Fréchet algebra $L(s', s)$ is a dense nonunital *-subalgebra of the C^*-algebra $\mathcal{K}$ of compact operators on ℓ^2. For a nonunital algebra, a notion more appropriate than positive linear functionals is that of a representable positive linear functional [3, Chapter III, Section 14]. We describe representable positive functionals on $L(s', s)$, and show that not every continuous self-adjoint functional on $L(s', s)$ can be decomposed as a difference of two representable positive linear functionals. On a Fréchet *-algebra with a bounded approximate identity (bai), all positive functionals are shown to be representable; it follows that $L(s', s)$ cannot have a bai. The nonexistence of a bai in $L(s', s)$ has also been shown in [7, Proposition 2].

1. Normality of positive cone. We follow [8] for order structure. Let L be an ordered vector space with an order defined by a convex cone C of vertex 0 with $0 \in C$. For $(x, y) \in L \times L$, let $[x, y] = (x + C) \cap (y - C) = \{z \in L : x \leq z \leq y\}$. For $A \subset L$, let $[A] = (A + C) \cap (A - C) = \bigcup\{[x, y] : x \in A, y \in A\}$; and A is called *saturated* if $A = [A]$. If L is a topological vector space in which the topology has a base consisting of saturated neighborhoods of 0, then C is called a *normal cone*. We shall use Theorem 3.1 of [8, Chapter V] stating that if L is a real locally convex ordered vector space with a positive cone C, then C is normal provided there exists a family $\mathcal{P}$ of seminorms determining the topology of L such that for all $p \in \mathcal{P}$, $p(x) \leq p(x + y)$ for all $x, y \in C$.

Let $\mathcal{A}$ be a closed *-subalgebra of the Fréchet algebra $L(s', s)$. Let $\mathcal{U}$ be the C^*-algebra obtained by completing $\mathcal{A}$ in the operator C^*-norm $\|\cdot\|_0$. By [2], $L(s', s)$ is a Q-algebra in the sense that the set of all quasi-invertible elements of $L(s', s)$ is open in the topology of $L(s', s)$, which implies that $L(s', s)$ is spectrally invariant in $\mathcal{K}$. Then $\mathcal{A}$ is a Q-subalgebra of the C^*-normed algebra $(\mathcal{U}, \|\cdot\|_0)$, since for any $x \in \mathcal{A}$, the spectral radius satisfies $r_{\mathcal{A}}(x) = \sup_q \lim_{n\to\infty} \|x^n\|_q^{1/n} = r_{L(s',s)}(x) = r_{\mathcal{K}}(x) \leq \|x\|_0$. Further, $\mathcal{A}$ is spectrally invariant in $L(s', s)$. Indeed, for any $x \in \mathcal{A}$, we have $\mathrm{Sp}_{L(s',s)}(x) = \mathrm{Sp}_K(x) = \mathrm{Sp}_{\mathcal{U}}(x) = \mathrm{Sp}_{\mathcal{A}}(x)$.

Let $\mathcal{A}_h = \mathcal{A} \cap L(s', s)_h = \{x \in \mathcal{A} : x = x^*\}$ be the real subspace of all self-adjoint elements of $\mathcal{A}$. Let $\mathcal{A}_+ = \{x \in \mathcal{A} : x = x^*, \operatorname{Sp}_{\mathcal{A}}(x) \subset [0, \infty)\}$ be the cone of positive elements of $\mathcal{A}$. Then $\mathcal{A}_+ = \mathcal{A} \cap \mathcal{K}_+$. We briefly describe from [1] the functional calculus in $\mathcal{A}$. Let $x \in \mathcal{A}$ be normal, i.e. $x^*x = xx^*$. Then $\operatorname{Sp}(x)$ consists of nonzero eigenvalues λ_n with 0 as a possible limit point. Let $\operatorname{alg}(x)$ be the closed *-subalgebra of $\mathcal{A}$ generated by x. Then $\operatorname{alg}(x)$ is a commutative Fréchet *-algebra. By [1, Theorem 3.1], x admits a spectral representation $x = \sum \lambda_n P_n$, where (λ_n) is a decreasing (in modulus) sequence in s of nonzero pairwise different elements, (P_n) is a sequence of nonzero pairwise orthogonal finite-dimensional projections belonging to $\operatorname{alg}(x)$ (see [1, Proposition 4.2]) and the series converges absolutely in $\mathcal{A}$. Let $C_s(\operatorname{Sp}(x))$ consist of functions $f : \operatorname{Sp}(x) \to \mathbb{C}$ satisfying $f(0) = 0$ and $|f(\lambda_n)| \, \|P_n\|_q < \infty$ for all q. Then the map $\Psi : C_s(\operatorname{Sp}(x)) \to \operatorname{alg}(x)$, $\Psi(f) = f(x) = \sum_{n=1}^{\infty} f(\lambda_n) P_n$, gives an isomorphism such that $\Psi(\mathrm{id}) = x$, $\Psi(\bar{f}) = \Psi(f)^*$. Thus $|x| := (x^*x)^{1/2} \in \mathcal{A}$; and any $x = x^* \in \mathcal{A}$ can be expressed as $x = x^+ - x^-$ with $x^+, x^- \in \mathcal{A}_+$ satisfying $x^+x^- = 0 = x^-x^+$. Indeed, $x^+ = (|x| + x)/2$, $x^- = (|x| - x)/2$. Thus $\mathcal{A}_h = \mathcal{A}_+ - \mathcal{A}_+$; and [8, Theorem 5.5, p. 228] implies that every positive linear functional on $\mathcal{A}$ (in particular on $L(s', s)$) is continuous.

Theorem 1.1. *The positive cone $\mathcal{A}_+$ in the Fréchet algebra $\mathcal{A}$ is normal.*

Proof. The Fréchet algebra $\mathcal{A}$ is identified, via the isomorphism Φ, with a closed *-subalgebra of the Fréchet algebra $\mathcal{S}$. Since $\mathcal{S}$ is nuclear, its Fréchet topology is also determined by the family $\{\|\!|\cdot|\!\|_q : q \in \mathbb{N}\}$ of seminorms, where, for $x = [x_{ij}]$, $\|\!|x|\!\|_q = \sup_{i,j} i^q j^q |x_{ij}|$. Now let $x = [x_{ij}]$, $y = [y_{ij}]$ in $\mathcal{A}$ with $0 \le x \le y$. By [7, Corollary 6], $\|\!|x|\!\|_q \le \|\!|y|\!\|_q$ for all $q \in \mathbb{N}_0$. It follows from [8, Theorem 3.1, p. 215] that $\mathcal{A}_+$ is normal in $\mathcal{A}$. ∎

[8, Corollary 1, p. 219] gives the following.

Corollary 1.2. *Let E be an equicontinuous set of self-adjoint linear functionals on $\mathcal{A}$. Then there exists an equicontinuous set B of positive linear functionals on $\mathcal{A}$ such that $E \subset B - B$.*

Thus, taking $\mathcal{A} = L(s', s)$, it follows that every continuous self-adjoint linear functional on $L(s', s)$ can be decomposed as a difference of two positive linear functionals.

2. Jordan decomposition with representable positive functionals. For nonunital algebras, the Jordan decomposition involves decomposition of a self-adjoint continuous linear functional as a difference of representable positive linear functionals. A linear functional f on an involutive algebra is *representable* [3, p. 175] if there exists a *-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of $\mathcal{A}$ into the C^*-algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on a

Hilbert space $\mathcal{H}$ admitting a cyclic vector ξ in $\mathcal{H}$ such that $f(x) = \langle \pi(x)\xi, \xi\rangle$ for all x in $\mathcal{A}$. By a theorem of Sebestyén [10], f is representable if and only if there exist positive constants k_1 and k_2 and a C^*- seminorm p on $\mathcal{A}$ such that $|f(x)|^2 \le k_1 f(x^*x)$ and $|f(x)| \le k_2 p(x)$ for all $x \in \mathcal{A}$.

LEMMA 2.1. *Let f be a positive linear functional on an involutive algebra $\mathcal{A}$. Then f is representable if and only if there exist a scalar $k > 0$ and a C^*-seminorm p on $\mathcal{A}$ such that $|f(x)| \le kp(x)$ for all $x \in \mathcal{A}$.*

Proof. Assume that f is representable. Then there exists a *-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ with a cyclic vector $\xi \in \mathcal{H}$ such that $f(x) = \langle \pi(x)\xi, \xi\rangle$ for all $x \in \mathcal{A}$. Then $|f(x)| = |\langle \pi(x)\xi, \xi\rangle| \le \|\pi(x)\xi\|\,\|\xi\| \le \|\pi(x)\|\,\|\xi\|^2$ where $p(x) := \|\pi(x)\|$, $x \in \mathcal{A}$, is the required C^*-seminorm.

Conversely, assume there exists a C^*-seminorm p on $\mathcal{A}$ and a positive scalar $k > 0$ such that $|f(x)| \le kp(x)$ for all $x \in \mathcal{A}$. Let $N_p = \{x \in \mathcal{A} : p(x) = 0\}$, a *-ideal in $\mathcal{A}$. Let $\mathcal{A}_p$ be the C^*-algebra obtained by completing the C^*-normed algebra $\mathcal{A}/N_p$ in the C^*-norm $\|x + N_p\| := p(x)$ $(x \in \mathcal{A})$. Then $\tilde{f}(x + N_p) := f(x)$ is a well defined positive linear functional on $\mathcal{A}/N_p$ satisfying $|\tilde{f}(z)| \le k\|z\|$ $(z \in \mathcal{A}/N_p)$, hence continuous. Thus by extension, $\tilde{f}$ defines a positive linear functional on the C^*-algebra $\mathcal{A}_p$ satisfying $|\tilde{f}(z)| \le k\|z\|$ for all $z \in \mathcal{A}_p$. Since every positive linear functional on a C^*-algebra is representable [11, Lemma 9.11], it follows that for some $c > 0$, $|\tilde{f}(z)|^2 \le c\tilde{f}(z^*z)$ holds for all $z \in \mathcal{A}_p$. Thus $|f(x)|^2 \le cf(x^*x)$ for all $x \in \mathcal{A}$. Now the result of Sebestyén stated above implies that f is representable on $\mathcal{A}$. ∎

LEMMA 2.2. *The operator norm $\|\cdot\|_0$ is the greatest C^*-seminorm on any closed *-subalgebra $\mathcal{A}$ of $L(s', s)$.*

Proof. Let p be any C^*-seminorm on $\mathcal{A}$. Let the C^*-algebra $\mathcal{A}_p$ be realized as a closed *-subalgebra of $\mathcal{B}(\mathcal{H}_p)$ for a Hilbert space $\mathcal{H}_p$. Let $\pi_p : \mathcal{A} \to \mathcal{B}(\mathcal{H}_p)$ be the *-homomorphism $\pi_p(x) = x + N_p$. Then for all $x \in \mathcal{A}$, we have $\mathrm{Sp}(\pi_p(x)) \subset \mathrm{Sp}(x)$ and

$$\begin{aligned}\|\pi_p(x)\|^2 = \|\pi_p(x^*x)\| &= r_{\mathcal{A}_p}(\pi_p(x^*x)) \le r_{\pi_p(\mathcal{A})}(\pi_p(x^*x)) \\ &\le r_{\mathcal{A}}(x^*x) = r_{\mathcal{K}}(x^*x) = \|x^*x\|_0 = \|x\|_0^2.\end{aligned}$$

Thus $p(x) \le \|x\|_0$ for all $x \in \mathcal{A}$. ∎

Let $\mathcal{N}$ be the Banach space of all trace class operators on ℓ^2 with the trace norm $\mathcal{N} \ni x \mapsto \mathrm{tr}(|x|)$. The well known dualities $\mathcal{K}' = \mathcal{N}$ and $(\mathcal{N}(\ell^2))' = \mathcal{B}(\ell^2)$ are given by the trace functional $\langle x, y\rangle = \mathrm{tr}(x\bar{y})$. This carries over to the noncommutative Schwartz space [6]. Let $L(s, s')$ be the space of all continuous linear operators from s to s' realized as the space of slowly growing matrices, $\mathcal{S}' := \{y = [y_{ij}] : \sup |y_{ij}| i^{-k} j^{-k} < \infty$ for some $k \in \mathbb{N}_0\}$. Then $\langle x, y\rangle = \sum_{i,j=1}^{\infty} x_{ij}\bar{y}_{ij}$.

THEOREM 2.3. *Let f be a self-adjoint positive linear functional on the Fréchet *-algebra $L(s', s)$. Then f is representable if and only if there exists a positive trace class operator y in $\mathcal{N}$ such that $f(x) = \operatorname{tr}(x\bar{y})$ for all x in $L(s', s)$.*

Proof. By the duality $\langle L(s', s), L(s, s')\rangle$, there exists $y \in L(s, s')$ such that $f(x) = \operatorname{tr}(x\bar{y})$ for all $x \in L(s', s)$. Notice that $y \geq 0$ in $L(s, s')$. Indeed, for any n, and for ξ in ℓ^2 of the form $\xi = (\xi_1, \ldots, \xi_n, 0, 0, \ldots)$, the matrix $x = [\bar{\xi}_i \xi_j] \in \mathcal{S}$ is positive definite, so $f(x) \geq 0$. Thus $\sum_{i,k=1}^{n} \bar{\xi}_i y_{ik} \xi_k \geq 0$. In particular the truncation matrix $u_n y u_n$ is ≥ 0 for all n, where u_n is the $n \times n$ identity matrix embedded in $\mathcal{S}$ by making all other entries zero. Hence $y \geq 0$ by [6, Proposition 3.1]. If f is representable, then by Lemmas 2.1 and 2.2, f is continuous in the operator norm $\|\cdot\|_0$. Since $L(s', s)$ is dense in $\mathcal{K}$, f uniquely extends to a $\|\cdot\|_0$-continuous positive linear functional on the C^*-algebra $\mathcal{K}$. In view of the duality $\langle \mathcal{K}, \mathcal{N}\rangle$, the operator y satisfies $y(s) \subset \ell^2$ and y extends to ℓ^2 so as to be in $\mathcal{N}$. The converse is clear in the light of Lemma 2.1. ∎

COROLLARY 2.4. *A continuous self-adjoint linear function f on $L(s', s)$ can be expressed as a difference of two representable positive linear functionals if and only if f is continuous in the operator C^*-norm.*

Note that a positive linear functional f on a Fréchet *-algebra $\mathcal{A}$ with a bai is representable. Indeed, let (e_α) be a bai for $\mathcal{A}$. By a theorem due to Dixon [3, Theorem 15.5, p. 191], f is continuous; hence there exist a continuous submultiplicative *-seminorm p on $\mathcal{A}$ and a scalar $k_1 > 0$ such that $|f(x)| \leq k_1 p(x)$ for all $x \in \mathcal{A}$. By the Cauchy–Schwarz inequality, for any $x \in \mathcal{A}$,

$$\begin{aligned}|f(x)|^2 &= |f(\lim e_\alpha x)|^2 = \lim |f(e_\alpha x)|^2 \\ &\leq \{\limsup f(e_\alpha e_\alpha^*)\} f(x^* x) = k_2 f(x^* x)\end{aligned}$$

where $k_2 = \limsup f(e_\alpha e_\alpha^*) < \infty$ by the continuity of f and boundedness of $(e_\alpha e_\alpha^*)$ in $\mathcal{A}$. It follows from a theorem of Sebestyén [3, Theorem 14.13, p. 175] that f is representable. Thus $L(s', s)$ fails to admit a bai; and this is crucial for nondecomposability in $L(s', s)$ of a continuous functional as a difference of representable positive functionals. In fact, a Jordan decomposition involving representable positive functionals is a highly nonunital C^*-like phenomenon. A result due to Grothendieck [5] states that a hermitian Banach *-algebra in which every self-adjoint continuous functional admits Jordan decomposition in terms of representable functionals is necessarily a C^*-algebra with an equivalent norm; and the Banach *-algebra $\mathcal{N}$ shows that the representability condition cannot be omitted.

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