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# STRONG HOPF MODULES FOR WEAK HOPF QUASIGROUPS

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**Abstract.** This paper is a further step in the study of the theory of modules associated to a weak Hopf quasigroup $H$. We introduce the category of strong $H$-Hopf modules, and we prove that there exists an adjoint equivalence between this category and the category of right modules over the image of the target morphism of $H$. In the Hopf quasigroup setting every Hopf module is strong, and we recover the results of Brzeziński. Also, in the weak Hopf case, every Hopf module is strong, and we generalize the theorem proved by Böhm, Nill and Szlachányi that contains as a particular instance the categorical equivalence associated to the category of Hopf modules for a Hopf algebra $H$.

**1. Introduction.** Let $\mathbb{F}$ be a field and $\mathcal{C} = \mathbb{F}$-Vect. Let $M$ be a right $H$-module and a right $H$-comodule. If, for all $m \in M$ and $h \in H$, we write $m.h$ for the action and we use the Sweedler notation $\rho_M(m) = m_{[0]} \otimes m_{[1]}$ for the coaction, we will say that $M$ is a *Hopf module* if

$$\rho_M(m.h) = m_{[0]}.h_{(1)} \otimes m_{[1]}h_{(2)},$$

where $\delta_H(h) = h_{(1)} \otimes h_{(2)}$ is the coproduct of $H$ and $m_{[1]}h_{(2)}$ the product in $H$ of $m_{[1]}$ and $h_{(2)}$. A morphism between two Hopf modules is a $\mathbb{F}$-linear map that is $H$-linear and $H$-colinear. Hopf modules and morphisms of Hopf modules constitute the category of Hopf modules denoted by $\mathcal{M}_H^H$. In 1969 Larsson and Sweedler proved the Fundamental Theorem of Hopf Modules: If $M \in \mathcal{M}_H^H$ and $M^{\operatorname{co} H} = \{m \in M \mid \rho_M(m) = m \otimes 1_H\}$ are the coinvariants of $H$ in $M$, then $M$ is isomorphic to $M^{\operatorname{co} H} \otimes H$ as a Hopf module (see [8] and [12]). On the other hand, if $N$ is an $\mathbb{F}$-vector space, then the tensor product $N \otimes H$, with the action and coaction induced by the product and coproduct of $H$, is a Hopf module. This construction is functorial, so we have a functor, called the *induction functor*, $F = - \otimes H : \mathcal{C} \to \mathcal{M}_H^H$. Also,

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for all $M \in \mathcal{M}_H^H$, the construction of $M^{\operatorname{co} H}$ is functorial and we have a new functor, the functor of coinvariants, $G = (\ \ )^{\operatorname{co} H} : \mathcal{M}_H^H \to \mathcal{C}$, and $F \dashv G$. Moreover, $F$ and $G$ is a pair of inverse equivalences, and therefore $\mathcal{M}_H^H$ is equivalent to the category of $\mathbb{F}$-vector spaces.

The Fundamental Theorem of Hopf Modules also holds for weak Hopf algebras, as proved by Böhm, Nill and Szlachányi [4]. If $H$ is a weak Hopf algebra, the category of Hopf modules is defined in the same way as in the Hopf algebra setting. For $M \in \mathcal{M}_H^H$, the coinvariants of $H$ in $M$ are defined by $M^{\operatorname{co} H} = \{m \in M \mid \rho_M(m) = m_{[0]} \otimes \Pi_H^L(m_{[1]})\}$, where $\Pi_H^L$ is the target morphism associated to $H$. Böhm, Nill and Szlachányi proved that $M$ is isomorphic to $M^{\operatorname{co} H} \otimes_{H_L} H$ as Hopf modules, where $H_L$ is the image of $\Pi_H^L$. Moreover, if $\mathcal{C}_{H_L}$ is the category of right $H_L$-modules, then the functors $F = - \otimes_{H_L} H : \mathcal{C}_{H_L} \to \mathcal{M}_H^H$ and $G = (\ \ )^{\operatorname{co} H} : \mathcal{M}_H^H \to \mathcal{C}_{H_L}$ are such that $F$ is left adjoint of $G$ and they induce a pair of inverse equivalences (see [6]). Therefore, in the weak setting, $\mathcal{M}_H^H$ is equivalent to $\mathcal{C}_{H_L}$. In this case, the following property is relevant: the tensor product $M^{\operatorname{co} H} \otimes_{H_L} H$ is isomorphic as a Hopf module to $M^{\operatorname{co} H} \times H$ where $M^{\operatorname{co} H} \times H$ is the image of a suitable idempotent $\nabla_M : M^{\operatorname{co} H} \otimes H \to M^{\operatorname{co} H} \otimes H$. As a consequence, in the weak framework the Fundamental Theorem of Hopf Modules can be written using $M^{\operatorname{co} H} \times H$ instead of $M^{\operatorname{co} H} \otimes_{H_L} H$ (see [1]).

In the previous two paragraphs we spoke about associative algebraic structures like Hopf algebras and weak Hopf algebras. Recently Klim and Majid [7] introduced the notion of Hopf quasigroup as a generalization of Hopf algebras in the context of non-associative algebra, in order to understand the structure and relevant properties of the algebraic 7-sphere. A Hopf quasigroup is a particular instance of a unital coassociative $H$-bialgebra in the sense of Pérez Izquierdo [10]; an example is the enveloping algebra of a Malcev algebra, when the base ring has characteristic not equal to 2 or 3. In this sense Hopf quasigroups extend the notion of Hopf algebra in a parallel way to Malcev algebras extending Lie algebras. On the other hand, it also contains as an example the notion of quasigroup algebra of an I.P. loop. Therefore, Hopf quasigroups unify I.P. loops and Malcev algebras, just as Hopf algebras unify groups and Lie algebras. For these non-associative algebraic structures, Brzeziński [5] introduced the notion of Hopf module and he proved a version of the Fundamental Theorem of Hopf Modules. In this case, the main difference with the associative setting appears in the definition of the category of Hopf modules $\mathcal{M}_H^H$, because the notion of Hopf module reflects the non-associativity of the product defined on $H$, and the morphisms are $H$-quasilinear and $H$-colinear (see [5, Definition 3.4]). In [5, Lemma 3.5], it is shown that if $M \in \mathcal{M}_H^H$ and $M^{\operatorname{co} H}$ is defined as in the Hopf algebra setting, then $M$ is isomorphic to $M^{\operatorname{co} H} \otimes H$ as a Hopf module. Moreover, the functors $F = - \otimes H : \mathcal{C} \to \mathcal{M}_H^H$ and $G = (\ \ )^{\operatorname{co} H} : \mathcal{M}_H^H \to \mathcal{C}$ are such that $F \dashv G$, and they induce a pair of inverse equivalences. Therefore, in this non-associative context, $\mathcal{M}_H^H$ is equivalent to the category of $\mathbb{F}$-vector spaces as in the Hopf algebra ambit.

Therefore a question naturally arises: Is it possible to introduce a new notion that encompasses weak Hopf algebras and Hopf quasigroups? If true, any result obtained in this way would automatically lead to a double generalization, recalling the well-known results in the weak setting and showing the way to get ones for the non-associative context. Fortunately, we have got a positive answer to this question by weakening the (co)unitality and associativity conditions. The new notion, called weak Hopf quasigroup, was introduced in [2] in a monoidal setting. A family of non-trivial examples of this algebraic structure can be obtained by working with bigroupoids, i.e., bicategories where every 1-cell is an equivalence and every 2-cell is an isomorphism (see [2, Example 2.3]).

For a weak Hopf quasigroup $H$ in a braided monoidal category $\mathcal{C}$ with tensor product $\otimes$, using the ideas proposed by Brzeziński for Hopf quasigroups, in [2] we introduce the notion of Hopf module and the category $\mathcal{M}_H^H$ of Hopf modules. In this case, if we define $M^{\mathrm{co}\,H}$ in the same way as in the weak Hopf algebra setting, we obtain the following version of the Fundamental Theorem of Hopf Modules: every Hopf module $M$ is isomorphic to $M^{\mathrm{co}\,H} \times H$ as a Hopf module, where $M^{\mathrm{co}\,H} \times H$ is the image of the same idempotent $\nabla_M$ used for Hopf modules associated to a weak Hopf algebra. Moreover, in [3] we proved that $H_L$, the image of the target morphism, is a monoid, and so it is possible to take into consideration the category $\mathcal{C}_{H_L}$, to construct the tensor product $M^{\mathrm{co}\,H} \otimes_{H_L} H$, and if the functor $- \otimes H$ preserves coequalizers, to endow this object with a Hopf module structure. Unfortunately, it is not possible to ensure that $M^{\mathrm{co}\,H} \otimes_{H_L} H$ is isomorphic to $M^{\mathrm{co}\,H} \times H$ as in the weak Hopf algebra case.

In this paper we find conditions under which these objects are isomorphic in $\mathcal{M}_H^H$. As a consequence, we introduce the category of strong Hopf modules, denoted by $\mathcal{S}M_H^H$, and we show that there exist functors, $F = - \otimes_{H_L} H : \mathcal{C}_{H_L} \to \mathcal{S}M_H^H$ and $G = (\ \ )^{\mathrm{co}\,H} : \mathcal{S}M_H^H \to \mathcal{C}_{H_L}$, such that $F$ is left adjoint of $G$ and they induce a pair of inverse equivalences. In the Hopf quasigroup setting every Hopf module is strong, and we recover the results of Brzeziński [5]. Also, in the weak Hopf case, every Hopf module is strong, and we generalize the theorem proved by Böhm, Nill and Szlachányi [4].

**2. Weak Hopf quasigroups.** As in [2], throughout this paper $\mathcal{C}$ denotes a strict braided monoidal category with tensor product $\otimes$, unit object $K$ and braiding $c$. For each object $M$ in $\mathcal{C}$, we denote the identity morphism by $\mathrm{id}_M : M \to M$. For simplicity of notation, given objects $M$, $N$ and $P$

in $\mathcal{C}$ and a morphism $f : M \to N$, we write $P \otimes f$ for $\mathrm{id}_P \otimes f$ and $f \otimes P$ for $f \otimes \mathrm{id}_P$. We also assume that every idempotent morphism in $\mathcal{C}$ splits, i.e., if $\nabla : Y \to Y$ is such that $\nabla = \nabla \circ \nabla$, there exist an object $Z$, called the image of $p$, and morphisms $i : Z \to Y$ and $p : Y \to Z$ such that $\nabla = i \circ p$ and $p \circ i = \mathrm{id}_Z$.

As for prerequisites, the reader is expected to be familiar with the notions of unital magma, counital comagma, monoid, comonoid, morphism of unital magmas, and morphism of counital comagmas (see [2]). Moreover, for a comagma $D$, $\delta_{D\otimes D}$ denotes the usual coproduct in $D \otimes D$ defined by $\delta_{D\otimes D} = (D \otimes c_{D,D} \otimes D) \circ (\delta_D \otimes \delta_D)$, and if $D$ is a comagma and $A$ a magma, then given two morphisms $f, g : D \to A$, we will denote by $f * g$ their convolution product in $\mathcal{C}$, that is, $f * g = \mu_A \circ (f \otimes g) \circ \delta_D$.

Now we recall the notion of weak Hopf quasigroup in a braided monoidal category, introduced in [2]. In that paper the interested reader can find a complete list of properties of weak Hopf quasigroups that we need.

Definition 2.1. A *weak Hopf quasigroup* $H$ in the category $\mathcal{C}$ is a unital magma $(H, \eta_H, \mu_H)$ and a comonoid $(H, \varepsilon_H, \delta_H)$ such that the following axioms hold:

(a1) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ \delta_{H\otimes H}$.

(a2) $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = \varepsilon_H \circ \mu_H \circ (H \otimes \mu_H)$
$= ((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes \delta_H \otimes H)$
$= ((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes (c^{-1}_{H,H} \circ \delta_H) \otimes H)$.

(a3) $(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H))$
$= (H \otimes (\mu_H \circ c^{-1}_{H,H}) \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H))$.

(a4) There exists $\lambda_H : H \to H$ in $\mathcal{C}$ (called the *antipode* of $H$) such that if we denote $\mathrm{id}_H * \lambda_H$ by $\Pi^L_H$ (*target morphism*) and $\lambda_H * \mathrm{id}_H$ by $\Pi^R_H$ (the *source morphism*), then:

(a4-1) $\Pi^L_H = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$.

(a4-2) $\Pi^R_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$.

(a4-3) $\lambda_H * \Pi^L_H = \Pi^R_H * \lambda_H = \lambda_H$.

(a4-4) $\mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H) = \mu_H \circ (\Pi^R_H \otimes H)$.

(a4-5) $\mu_H \circ (H \otimes \mu_H) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H)$
$= \mu_H \circ (\Pi^L_H \otimes H)$.

(a4-6) $\mu_H \circ (\mu_H \otimes \lambda_H) \circ (H \otimes \delta_H) = \mu_H \circ (H \otimes \Pi^L_H)$.

(a4-7) $\mu_H \circ (\mu_H \otimes H) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H) = \mu_H \circ (H \otimes \Pi^R_H)$.

By [2, Proposition 3.2] we know that the antipode is unique, and satisfies $\lambda_H \circ \eta_H = \eta_H$ and $\varepsilon_H \circ \lambda_H = \varepsilon_H$. Also, by [2, Theorem 3.19], $\lambda_H$ is antimultiplicative and anticomultiplicative. Moreover, if we define the morphisms $\overline{\Pi}_H^L$ and $\overline{\Pi}_H^R$ by

$$\overline{\Pi}_H^L = (H\otimes(\varepsilon_H\circ\mu_H))\circ((\delta_H\circ\eta_H)\otimes H),$$
$$\overline{\Pi}_H^R = ((\varepsilon_H\circ\mu_H)\otimes H)\circ(H\otimes(\delta_H\circ\eta_H)),$$

we proved in [2, Proposition 3.4] that $\Pi_H^L$, $\Pi_H^R$, $\overline{\Pi}_H^L$ and $\overline{\Pi}_H^R$ are idempotent. On the other hand, if $H_L$ is the image of the idempotent morphism $\Pi_H^L$, and $p_L : H \to H_L$ and $i_L : H_L \to H$ are the morphisms such that $\Pi_H^L = i_L\circ p_L$ and $p_L\circ i_L = \mathrm{id}_{H_L}$, then by [2, Proposition 3.13], $i_L$ is the equalizer of $\delta_H$ and $(H\otimes\Pi_H^L)\circ\delta_H$, and $p_L$ is the coequalizer of $\mu_H$ and $\mu_H\circ(H\otimes\Pi_H^L)$. As a consequence, $(H_L, \eta_{H_L} = p_L\circ\eta_H, \mu_{H_L} = p_L\circ\mu_H\circ(i_L\otimes i_L))$ is a unital magma in $\mathcal{C}$, and $(H_L, \varepsilon_{H_L} = \varepsilon_H\circ i_L, \delta_H = (p_L\otimes p_L)\circ\delta_H\circ i_L)$ is a comonoid in $\mathcal{C}$ (see [2, Proposition 3.13]). Fortunately (see [3, Proposition 2.4]), the product $\mu_{H_L}$ is associative, and therefore the unital magma $H_L$ is a monoid in $\mathcal{C}$.

Finally, using the same proof of the similar result proved for weak braided Hopf algebras [1, Proposition 2.19], we conclude that the monoid $H_L$ is Frobenius separable, and by [11], if $\mathcal{C}$ is the category of vector spaces over a field $\mathbb{F}$, then $H_L$ is semisimple.

## 3. Hopf modules, strong Hopf modules and categorical equivalences.

The definition of right-right $H$-Hopf module for a weak Hopf quasigroup $H$ was introduced in [2]. If $H$ is a Hopf quasigroup and $\mathcal{C}$ is the symmetric monoidal category $\mathbb{F}$-Vect, we get the notion defined by Brzeziński [5] for Hopf quasigroups.

Definition 3.1. Let $H$ be a weak Hopf quasigroup and $M$ an object in $\mathcal{C}$. We say that $(M, \phi_M, \rho_M)$ is a *right-right $H$-Hopf module* if the following axioms hold:

- (b1) The pair $(M, \rho_M)$ is a right $H$-comodule, i.e. $\rho_M : M \to M\otimes H$ is a morphism such that $(M\otimes\varepsilon_H)\circ\rho_M = \mathrm{id}_M$ and $(\rho_M\otimes H)\circ\rho_M = (M\otimes\delta_H)\circ\rho_M$.
- (b2) The morphism $\phi_M : M\otimes H \to M$ satisfies:
  - (b2-1) $\phi_M\circ(M\otimes\eta_H) = \mathrm{id}_M$.
  - (b2-2) $\rho_M\circ\phi_M = (\phi_M\otimes\mu_H)\circ(M\otimes c_{H,H}\otimes H)\circ(\rho_M\otimes\delta_H)$, i.e. $\phi_M$ is a morphism of right $H$-comodules with the codiagonal coaction on $M\otimes H$.
- (b3) $\phi_M\circ(\phi_M\otimes\lambda_H)\circ(M\otimes\delta_H) = \phi_M\circ(M\otimes\Pi_H^L)$.
- (b4) $\phi_M\circ(\phi_M\otimes H)\circ(M\otimes\lambda_H\otimes H)\circ(M\otimes\delta_H) = \phi_M\circ(M\otimes\Pi_H^R)$.
- (b5) $\phi_M\circ(\phi_M\otimes H)\circ(M\otimes\Pi_H^L\otimes H)\circ(M\otimes\delta_H) = \phi_M$.

By [2, Proposition 4.7], if $(M, \phi_M, \rho_M)$ and $(N, \phi_N, \rho_N)$ are right-right $H$-Hopf modules, and there exists a right $H$-comodule isomorphism $\alpha$ :

$M \to N$, then the triple $(M, \phi_M^{\alpha} = \alpha^{-1} \circ \phi_N \circ (\alpha \otimes H), \rho_M)$ is a right-right $H$-Hopf module.

On the other hand, by [2, Proposition 4.3], for every right-right $H$-Hopf module $(M, \phi_M, \rho_M)$, the morphism $q_M := \phi_M \circ (M \otimes \lambda_H) \circ \rho_M : M \to M$ is idempotent. Moreover, if $M^{\mathrm{co}\,H}$ (the object of coinvariants) is the image of $q_M$, and $p_M : M \to M^{\mathrm{co}\,H}$ and $i_M : M^{\mathrm{co}\,H} \to M$ are the morphisms such that $q_M = i_M \circ p_M$ and $\mathrm{id}_{M^{\mathrm{co}\,H}} = p_M \circ i_M$, then $i_M$ is the equalizer of $\rho_M$ and $(M \otimes \overline{\Pi}_H^R) \circ \rho_M$. Equivalently (see [2, Remark 4.4]), $i_M$ is the equalizer of $\rho_M$ and $(M \otimes \Pi_H^L) \circ \rho_M$. Moreover (see [2, Remark 4.4]):

$$\phi_M \circ (q_M \otimes H) \circ \rho_M = \mathrm{id}_M, \tag{3.1}$$

$$\rho_M \circ \phi_M \circ (i_M \otimes H) = (\phi_M \otimes H) \circ (i_M \otimes \delta_H), \tag{3.2}$$

$$p_M \circ \phi_M \circ (i_M \otimes H) = p_M \circ \phi_M \circ (i_M \otimes \Pi_H^L). \tag{3.3}$$

The morphism

$$\nabla_M := (p_M \otimes H) \circ \rho_M \circ \phi_M \circ (i_M \otimes H) : M^{\mathrm{co}\,H} \otimes H \to M^{\mathrm{co}\,H} \otimes H$$

is idempotent and satisfies

$$\nabla_M = ((p_M \circ \phi_M) \otimes H) \circ (i_M \otimes \delta_H), \tag{3.4}$$

$$(M^{\mathrm{co}\,H} \otimes \delta_H) \circ \nabla_M = (\nabla_M \otimes H) \circ (M^{\mathrm{co}\,H} \otimes \delta_H) \tag{3.5}$$

(see [2, Proposition 4.5]). If we define $\omega_M : M^{\mathrm{co}\,H} \otimes H \to M$ and $\omega'_M : M \to M^{\mathrm{co}\,H} \otimes H$ by $\omega_M = \phi_M \circ (i_M \otimes H)$ and $\omega'_M = (p_M \otimes H) \circ \rho_M$, we have $\omega_M \circ \omega'_M = \mathrm{id}_M$ and $\nabla_M = \omega'_M \circ \omega_M$. If $M^{\mathrm{co}\,H} \times H$ denotes the image of $\nabla_M$ and $p_{M^{\mathrm{co}\,H} \otimes H}$ and $i_{M^{\mathrm{co}\,H} \otimes H}$ are the morphisms such that $p_{M^{\mathrm{co}\,H} \otimes H} \circ i_{M^{\mathrm{co}\,H} \otimes H} = \mathrm{id}_{M^{\mathrm{co}\,H} \times H}$ and $i_{M^{\mathrm{co}\,H} \otimes H} \circ p_{M^{\mathrm{co}\,H} \otimes H} = \nabla_M$, then $\alpha_M = p_{M^{\mathrm{co}\,H} \otimes H} \circ \omega'_M : M \to M^{\mathrm{co}\,H} \times H$ is an isomorphism of right $H$-modules (i.e. $\rho_{M^{\mathrm{co}\,H} \times H} \circ \alpha_M = (\alpha_M \otimes H) \circ \rho_M$) with inverse $\alpha_M^{-1} = \omega_M \circ i_{M^{\mathrm{co}\,H} \otimes H}$ (see [2, Remark 4.6]). The comodule structure of $M^{\mathrm{co}\,H} \times H$ is induced by the isomorphism $\alpha_M$ and it is equal to $\rho_{M^{\mathrm{co}\,H} \times H} = (p_{M^{\mathrm{co}\,H} \otimes H} \otimes H) \circ (M^{\mathrm{co}\,H} \otimes \delta_H) \circ i_{M^{\mathrm{co}\,H} \otimes H}$. As a consequence, the triple $(M^{\mathrm{co}\,H} \times H, \phi_{M^{\mathrm{co}\,H} \times H}, \rho_{M^{\mathrm{co}\,H} \times H})$ where $\phi_{M^{\mathrm{co}\,H} \times H} = p_{M^{\mathrm{co}\,H} \otimes H} \circ (M^{\mathrm{co}\,H} \otimes \mu_H) \circ (i_{M^{\mathrm{co}\,H} \otimes H} \otimes H)$, is a right-right $H$-Hopf module (see [2, Proposition 4.8]).

Finally, in view of [2, Proposition 4.9], the triple $(M, \phi_M^{\alpha_M}, \rho_M)$ is a right-right $H$-Hopf module with the same object of coinvariants of $(M, \phi_M, \rho_M)$ because

$$q_M^{\alpha_M} = q_M, \tag{3.6}$$

where $q_M^{\alpha_M}$ is the idempotent morphism associated to the Hopf module $(M, \phi_M^{\alpha_M}, \rho_M)$. Moreover, $\phi_M^{\alpha_M} = \phi_M \circ (q_M \otimes \mu_H) \circ (\rho_M \otimes H)$, and if $\nabla_M^{\alpha_M}$ denotes the idempotent morphism associated to the coinvariants of $(M, \phi_M^{\alpha_M}, \rho_M)$, we have $\nabla_M^{\alpha_M} = \nabla_M$ and so, for $(M, \phi_M^{\alpha_M}, \rho_M)$, the associated isomorphism of comodules is $\alpha_M$. Finally, $(\phi_M^{\alpha_M})^{\alpha_M} = \phi_M^{\alpha_M}$.

Definition 3.2. Let $H$ be a weak Hopf quasigroup and let $(M, \phi_M, \rho_M)$ and $(N, \phi_N, \rho_N)$ be right-right $H$-Hopf modules. A morphism $f : M \to N$ in $\mathcal{C}$ is said to be $H$-*quasilineal* if

$$\phi_N^{\alpha_N} \circ (f \otimes H) = f \circ \phi_M^{\alpha_M}. \tag{3.7}$$

A morphism of right-right $H$-Hopf modules between $M$ and $N$ is a morphism $f : M \to N$ in $\mathcal{C}$ that is both $H$-quasilineal and a morphism of right $H$-comodules. The collection of all right $H$-Hopf modules with their morphisms forms a category which will be denoted by $\mathcal{M}_H^H$.

If $(M, \phi_M, \rho_M)$ is an object in $\mathcal{M}_H^H$, for $(M^{\mathrm{co}\,H} \times H, \phi_{M^{\mathrm{co}\,H} \times H}, \rho_{M^{\mathrm{co}\,H} \times H})$ we have

$$\phi_{M^{\mathrm{co}\,H} \times H}^{\alpha_{M^{\mathrm{co}\,H} \times H}} = \phi_{M^{\mathrm{co}\,H} \times H} \tag{3.8}$$

(see [2, Proposition 4.12]). As a consequence, we can prove (see [2, Theorem 4.13])

Theorem 3.3 (Fundamental Theorem of Hopf Modules). *Let $H$ be a weak Hopf quasigroup and assume that $(M, \phi_M, \rho_M)$ is an object in the category $\mathcal{M}_H^H$. Then $(M, \phi_M, \rho_M)$ and $(M^{\mathrm{co}\,H} \times H, \phi_{M^{\mathrm{co}\,H} \times H}, \rho_{M^{\mathrm{co}\,H} \times H})$ are isomorphic in $\mathcal{M}_H^H$.*

The following lemma will be used to obtain the main result of this paper.

Lemma 3.4. *Let $H$ be a weak Hopf quasigroup and let $(M, \phi_M, \rho_M)$ be a right-right $H$-Hopf module. Then*

$$\phi_M \circ (i_M \otimes \mu_H) = \phi_M \circ (i_M \otimes \mu_H) \circ (\nabla_M \otimes H). \tag{3.9}$$

*Proof.* The equality holds because

$$\begin{aligned}
&\phi_M \circ (i_M \otimes \mu_H) \circ (\nabla_M \otimes H)\\
&\quad = \phi_M \circ (q_M \otimes \mu_H) \circ ((\rho_M \circ \phi_M \circ (i_M \otimes H)) \otimes H) \ \text{(by definition of } \nabla_M)\\
&\quad = \phi_M \circ (q_M \otimes \mu_H) \circ (((\phi_M \otimes H) \circ (i_M \otimes \delta_H)) \otimes H) \ \text{(by (3.2))}\\
&\quad = \phi_M \circ ((\phi_M \circ (M \otimes \lambda_H)) \otimes H) \circ ((\rho_M \circ \phi_M \circ (i_M \otimes H)) \otimes \mu_H)\\
&\qquad \circ (M^{\mathrm{co}H} \otimes \delta_H \otimes H) \ \text{(by definition of } q_M)\\
&\quad = \phi_M \circ ((\phi_M \circ (\phi_M \otimes \lambda_H) \circ (M \otimes \delta_H)) \otimes \mu_H) \circ (i_M \otimes \delta_H \otimes H) \ \text{(by (3.2))}\\
&\quad = \phi_M \circ ((\phi_M \circ (M \otimes \Pi_H^L)) \otimes \mu_H) \circ (i_M \otimes \delta_H \otimes H) \ \text{(by (b3) of Definition 3.1)}\\
&\quad = \phi_M \circ (\phi_M \otimes \mu_H) \circ (i_M \otimes (((\varepsilon_H \circ \mu_H) \otimes H \otimes H) \circ \delta_{H \otimes H} \circ (\eta_H \otimes H)) \otimes H)\\
&\qquad \text{(by definition of } \Pi_H^L)\\
&\quad = \phi_M \circ (\phi_M \otimes \mu_H) \circ (i_M \otimes ((H \otimes (\varepsilon_H \circ \mu_H) \otimes H) \circ ((c_{H,H} \circ \delta_H \circ \eta_H)\\
&\qquad \otimes \delta_H)) \otimes H) \ \text{(by the naturality of } c)
\end{aligned}$$

$$= \phi_M \circ (\phi_M \otimes (\mu_H \circ ((\mu_H \circ (\overline{\Pi}_H^R \otimes H)) \otimes H))) \circ (i_M \otimes (c_{H,H} \circ \delta_H) \otimes H \otimes H)$$

(by [2, (10)])

$$= \phi_M \circ (\phi_M \otimes (\mu_H \circ (\overline{\Pi}_H^R \otimes \mu_H))) \circ (i_M \otimes (c_{H,H} \circ \delta_H \circ \eta_H) \otimes H \otimes H)$$

(by [2, (33)] and [3, (32)])

$$= \phi_M \circ (\phi_M \otimes (((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes \delta_H))) \circ (i_M \otimes (c_{H,H} \circ \delta_H \circ \eta_H) \otimes \mu_H)$$

(by [2, (10)])

$$= \phi_M \circ (\phi_M \otimes H) \circ (M \otimes ((\Pi_H^L \otimes H) \circ \delta_H)) \circ (i_M \otimes \mu_H) \text{ (by the naturality of } c\text{)}$$

$$= \phi_M \circ (i_M \otimes \mu_H) \text{ (by (b5) of Definition 3.1).} \ \blacksquare$$

From now on we assume that $\mathcal{C}$ admits coequalizers. We denote by $\mathcal{C}_{H_L}$ the category of right $H_L$-modules, i.e., the category whose objects are pairs $(N, \psi_N)$ with $N$ an object in $\mathcal{C}$ and $\psi_N : N \otimes H_L \to N$ a morphism such that $\psi_N \circ (N \otimes \mu_{H_L}) = \psi_N \circ (\psi_N \otimes H_L)$ and $\psi_N \circ (N \otimes \eta_{H_L}) = \mathrm{id}_N$. A morphism $f : (N, \psi_N) \to (P, \psi_P)$ in $\mathcal{C}_{H_L}$ is a morphism $f : N \to P$ in $\mathcal{C}$ such that $\psi_P \circ (f \otimes H) = f \circ \psi_N$. Note that the pair $(H, \psi_H = \mu_H \circ (H \otimes i_L))$ is a right $H_L$-module.

Let $(N, \psi_N)$ be an object in $\mathcal{C}_{H_L}$ and consider the coequalizer diagram

$$\begin{tikzcd} N \otimes H_L \otimes H \arrow[r, shift left, "N \otimes \varphi_H"] \arrow[r, shift right, "\psi_N \otimes H"'] & N \otimes H \arrow[r, "n_N"] & N \otimes_{H_L} H \end{tikzcd} \tag{3.10}$$

where $\varphi_H = \mu_H \circ (i_L \otimes H)$. By [3, (35)] we have

$$\begin{aligned}(n_N \otimes H) \circ (\psi_N \otimes \delta_H) &= ((n_N \circ (N \otimes \varphi_H)) \otimes H) \circ (N \otimes H_L \otimes \delta_H) \\ &= (n_N \otimes H) \circ (N \otimes (\delta_H \circ \varphi_H)),\end{aligned}$$

and so there exists a unique morphism

$$\rho_{N \otimes_{H_L} H} : N \otimes_{H_L} H \to (N \otimes_{H_L} H) \otimes H$$

such that

$$\rho_{N \otimes_{H_L} H} \circ n_N = (n_N \otimes H) \circ (N \otimes \delta_H). \tag{3.11}$$

The pair $(N \otimes_{H_L} H, \rho_{N \otimes_{H_L} H})$ is a right $H$-comodule. Indeed, trivially, $((N \otimes_{H_L} H) \otimes \varepsilon_H) \circ \rho_{N \otimes_{H_L} H} = \mathrm{id}_{N \otimes_{H_L} H}$ because composing with $n_N$ we have $((N \otimes_{H_L} H) \otimes \varepsilon_H) \circ \rho_{N \otimes_{H_L} H} \circ n_N = (n_N \otimes \varepsilon_H) \circ (N \otimes \delta_H) = n_N$. Moreover, $(\rho_{N \otimes_{H_L} H} \otimes H) \circ \rho_{N \otimes_{H_L} H} = ((N \otimes_{H_L} H) \otimes \delta_H) \circ \rho_{N \otimes_{H_L} H}$ follows from

$$\begin{aligned}(\rho_{N \otimes_{H_L} H} \otimes H) \circ \rho_{N \otimes_{H_L} H} \circ n_N &= (n_N \otimes \delta_H) \circ (N \otimes \delta_H) \\ &= ((N \otimes_{H_L} H) \otimes \delta_H) \circ \rho_{N \otimes_{H_L} H} \circ n_N.\end{aligned}$$

On the other hand, by [3, (18)] we have

$$n_N \circ (\psi_N \otimes \mu_H) = n_N \circ (N \otimes (\mu_H \circ (i_L \otimes \mu_H))) = n_N \circ (N \otimes (\mu_H \circ (\varphi_H \otimes H))),$$

and so if the functor $-\otimes H$ preserves coequalizers, then there exists a unique morphism

$$\phi_{N\otimes_{H_L} H} : (N\otimes_{H_L} H)\otimes H \to N\otimes_{H_L} H$$

such that

$$\phi_{N\otimes_{H_L} H}\circ (n_N\otimes H) = n_N\circ (N\otimes \mu_H). \tag{3.12}$$

Trivially, $\phi_{N\otimes_{H_L} H}\circ ((N\otimes_{H_L} H)\otimes \eta_H) = \mathrm{id}_{N\otimes_{H_L} H}$ because

$$\phi_{N\otimes_{H_L} H}\circ (n_N\otimes \eta_H) = n_N\circ (N\otimes (\mu_H\circ (H\otimes \eta_H))) = n_N.$$

Thus, if the functor $-\otimes H$ preserves coequalizers, then the triple

$$(N\otimes_{H_L} H, \phi_{N\otimes_{H_L} H}, \rho_{N\otimes_{H_L} H})$$

is a right-right $H$-Hopf module. Indeed, by the previous reasoning, conditions (b1) and (b2-1) of Definition 3.1 hold. Composing with $n_N\otimes H$ and using (3.11), (3.12) and (a1) of Definition 2.1 we obtain

$$\begin{aligned}
&\rho_{N\otimes_{H_L} H}\circ \phi_{N\otimes_{H_L} H}\circ (n_N\otimes H)\\
&\quad = (n_N\otimes H)\circ (N\otimes (\delta_H\circ \mu_H)) = (n_N\otimes H)\circ (N\otimes ((\mu_H\otimes \mu_H)\circ \delta_{H\otimes H}))\\
&\quad = (\phi_{N\otimes_{H_L} H}\otimes \mu_H)\circ ((N\otimes_{H_L} H)\otimes c_{H,H}\otimes H)\circ ((\rho_{N\otimes_{H_L} H}\circ n_N)\otimes \delta_H),
\end{aligned}$$

and so (b2-2) of Definition 3.1 holds. Also, by (a4-6) of Definition 2.1 and (3.12) we obtain

$$\begin{aligned}
\phi_{N\otimes_{H_L} H}\circ (\phi_{N\otimes_{H_L} H}\otimes \lambda_H)\circ (n_N\otimes \delta_H) &= n_N\circ (N\otimes (\mu_H\circ (\mu_H\otimes \lambda_H)\circ (H\otimes \delta_H)))\\
&= n_N\circ (N\otimes (\mu_H\circ (H\otimes \Pi_H^L))) = \phi_{N\otimes_{H_L} H}\circ (n_N\otimes \Pi_H^L),
\end{aligned}$$

and thus (b3) of Definition 3.1 holds. Similarly, by (3.12) and (a4-7) of Definition 2.1 we get (b4) of Definition 3.1. The equality (b5) of that definition is a consequence of (3.12) and [2, (41)].

Note that, by (3.11), (3.12),

$$q_{N\otimes_{H_L} H}\circ n_N = n_N\circ (N\otimes \Pi_H^L). \tag{3.13}$$

Also,

$$\phi_{N\otimes_{H_L} H}^{\alpha_{N\otimes_{H_L} H}} = \phi_{N\otimes_{H_L} H} \tag{3.14}$$

because by (3.11), (3.12) and [2, (42)],

$$\begin{aligned}
\phi_{N\otimes_{H_L} H}^{\alpha_{N\otimes_{H_L} H}}\circ (n_N\otimes H) &= n_N\circ (N\otimes (\mu_H\circ (\Pi_H^L\otimes \mu_H)\circ (\delta_H\otimes H)))\\
&= n_N\circ (N\otimes \mu_H)\\
&= \phi_{N\otimes_{H_L} H}\circ (n_N\otimes H).
\end{aligned}$$

On the other hand, if $f : N\to P$ is a morphism in $\mathcal{C}_{H_L}$, we have

$$n_P\circ (f\otimes H)\circ (\psi_N\otimes H) = n_P\circ (f\otimes H)\circ (N\otimes \varphi_H)$$

and hence there exists a unique morphism

$$f\otimes_{H_L}H : N\otimes_{H_L}H\to P\otimes_{H_L}H$$

such that

$$n_P\circ(f\otimes H)=(f\otimes_{H_L}H)\circ n_N. \tag{3.15}$$

The morphism $f\otimes_{H_L}H$ is a morphism in $\mathcal{M}_H^H$ because by (3.11), (3.12), (3.14) and (3.15),

$$\rho_{P\otimes_{H_L}H}\circ(f\otimes_{H_L}H)\circ n_N=(n_P\otimes H)\circ(f\otimes\delta_H)=((f\otimes_{H_L}H)\otimes H)\circ\rho_{N\otimes_{H_L}H}\circ n_N$$

and

$$\begin{aligned}&\phi_{P\otimes_{H_L}H}^{\alpha_{P\otimes_{H_L}H}}\circ((f\otimes_{H_L}H)\otimes H)\circ(n_N\otimes H)\\&\quad=\phi_{P\otimes_{H_L}H}\circ((f\otimes_{H_L}H)\otimes H)\circ(n_N\otimes H)=n_P\circ(f\otimes\mu_H)\\&\quad=(f\otimes_{H_L}H)\circ\phi_{N\otimes_{H_L}H}\circ(n_N\otimes H)=(f\otimes_{H_L}H)\circ\phi_{N\otimes_{H_L}H}^{\alpha_{N\otimes_{H_L}H}}\circ(n_N\otimes H).\end{aligned}$$

Summarizing, we have the following proposition.

Proposition 3.5. *Let $H$ be a weak Hopf quasigroup such that the functor $-\otimes H$ preserves coequalizers. There exists a functor $F:\mathcal{C}_{H_L}\to\mathcal{M}_H^H$, called the induction functor, defined on objects by*

$$F((N,\psi_N))=(N\otimes_{H_L}H,\phi_{N\otimes_{H_L}H},\rho_{N\otimes_{H_L}H})$$

*and for morphisms by $F(f)=f\otimes_{H_L}H$.*

Definition 3.6. Let $H$ be a weak Hopf quasigroup. We denote by $\mathcal{S}M_H^H$ the full subcategory of $\mathcal{M}_H^H$ whose objects are the right-right $H$-Hopf modules $(M,\phi_M,\rho_M)$ such that

$$\phi_M\circ((\phi_M\circ(M\otimes i_L))\otimes H)=\phi_M\circ(M\otimes(\mu_H\circ(i_L\otimes H))). \tag{3.16}$$

The objects of $\mathcal{S}M_H^H$ will be called *right-right strong $H$-Hopf modules.*

By [3, (33)], $(H,\phi_H=\mu_H,\rho_H=\delta_H)$ is a right-right strong $H$-Hopf module. Note that if $H$ is a Hopf quasigroup, then (3.16) holds because $i_L=\eta_H$ (see [9, Theorem 1]). Thus in this particular setting $\mathcal{S}M_H^H=\mathcal{M}_H^H$. This equality also holds trivially for any Hopf module associated to a weak (braided) Hopf algebra (see [1, Section 3]).

Proposition 3.7. *Let $H$ be a weak Hopf quasigroup such that the functor $-\otimes H$ preserves coequalizers. The induction functor $F:\mathcal{C}_{H_L}\to\mathcal{M}_H^H$ factorizes through the category $\mathcal{S}M_H^H$.*

*Proof.* We must show that for any $(N,\psi_N)\in\mathcal{C}_{H_L}$, the triple

$$(N\otimes_{H_L}H,\phi_{N\otimes_{H_L}H},\rho_{N\otimes_{H_L}H})$$

is an object in $\mathcal{S}M_H^H$. First note that if $-\otimes H$ preserves coequalizers then so does $-\otimes H_L$, and (3.16) holds because by (3.12) and [3, (32)],

$$\begin{aligned}
&\phi_{N\otimes_{H_L}H}\circ((\phi_{N\otimes_{H_L}H}\circ(n_N\otimes i_L))\otimes H)\\
&\qquad\qquad = n_N\circ(N\otimes(\mu_H\circ((\mu_H\circ(H\otimes i_L))\otimes H)))\\
&\qquad\qquad = n_N\circ(N\otimes(\mu_H\circ(H\otimes(\mu_H\circ(i_L\otimes H)))))\\
&\qquad\qquad = \phi_{N\otimes_{H_L}H}\circ(n_N\otimes(\mu_H\circ(i_L\otimes H))). \quad \blacksquare
\end{aligned}$$

Let $(M,\phi_M,\rho_M)$ be a right-right $H$-Hopf module. If $M$ is strong, the pair $(M^{\mathrm{co}\,H},\psi_{M^{\mathrm{co}\,H}}=p_M\circ\phi_M\circ(i_M\otimes i_L))$ is a right $H_L$-module. Indeed, trivially $\psi_{M^{\mathrm{co}\,H}}\circ(M^{\mathrm{co}\,H}\otimes\eta_{H_L})=\mathrm{id}_{M^{\mathrm{co}\,H}}$. Moreover,

$$\begin{aligned}
&\psi_{M^{\mathrm{co}\,H}}\circ(\psi_{M^{\mathrm{co}\,H}}\otimes H_L)\\
&\qquad = p_M\circ\phi_M\circ((\phi_M\circ(\phi_M\otimes\lambda_H)\circ(i_M\otimes(\delta_H\circ i_L)))\otimes i_L) \ \text{(by (3.2))}\\
&\qquad = p_M\circ\phi_M\circ((\phi_M\circ(i_M\otimes i_L))\otimes i_L) \ \text{(by (b3) of Definition 3.1)}\\
&\qquad = p_M\circ\phi_M\circ(i_M\otimes(\mu_H\circ(i_L\otimes i_L))) \ \text{(by (3.16))}\\
&\qquad = \psi_{M^{\mathrm{co}\,H}}\circ(M^{\mathrm{co}\,H}\otimes\mu_{H_L}) \ \text{(by the properties of } \mu_{H_L}\text{)}.
\end{aligned}$$

Let $g:M\to T$ be a morphism in $\mathcal{S}M_H^H$. Using the comodule morphism condition we obtain $\rho_T\circ g\circ i_M=(T\otimes\overline{\Pi}_H^R)\circ\rho_T\circ g\circ i_M$, and this implies that there exists a unique morphism $g^{\mathrm{co}\,H}:M^{\mathrm{co}\,H}\to T^{\mathrm{co}\,H}$ such that

$$i_T\circ g^{\mathrm{co}\,H}=g\circ i_M. \tag{3.17}$$

Then, by (3.17) and (3.6), $i_T\circ g^{\mathrm{co}\,H}\circ p_M=g\circ q_M=g\circ q_M^{\alpha_M}=q_T^{\alpha_T}\circ g=q_T\circ g$, and so

$$g^{\mathrm{co}\,H}\circ p_M=p_T\circ g. \tag{3.18}$$

On the other hand, for any right-right $H$-Hopf module $M$, by (3.6) we know that $\nabla_M=\nabla_M^{\alpha_M}$. Composing with $\phi_M\circ(i_M\otimes H)$ and using (3.1) we get

$$\phi_M\circ(i_M\otimes H)=\phi_M^{\alpha_M}\circ(i_M\otimes H). \tag{3.19}$$

By (3.17)–(3.19), $g^{\mathrm{co}\,H}$ is a morphism of right $H_L$-modules because

$$\begin{aligned}
g^{\mathrm{co}\,H}\circ\psi_{M^{\mathrm{co}\,H}} &= p_T\circ g\circ\phi_M\circ(i_M\otimes i_L)\\
&= p_T\circ g\circ\phi_M^{\alpha_M}\circ(i_M\otimes i_L)=p_T\circ\phi_T^{\alpha_T}\circ((g\circ i_M)\otimes i_L)\\
= p_T\circ\phi_T^{\alpha_T}\circ((i_T\circ g^{\mathrm{co}\,H})\otimes i_L) &= p_T\circ\phi_T\circ((i_T\circ g^{\mathrm{co}\,H})\otimes i_L)=\psi_{T^{\mathrm{co}\,H}}\circ(g^{\mathrm{co}\,H}\otimes H_L).
\end{aligned}$$

Thus, in this setting we have the following result.

PROPOSITION 3.8. *Let $H$ be a weak Hopf quasigroup. There exists a functor $G:\mathcal{S}M_H^H\to\mathcal{C}_{H_L}$, called the functor of coinvariants, defined on objects by $G((M,\phi_M,\rho_M))=(M^{\mathrm{co}\,H},\psi_{M^{\mathrm{co}\,H}})$ and for morphisms by $G(g)=g^{\mathrm{co}\,H}$.*

Proposition 3.9. *Let $H$ be a weak Hopf quasigroup such that the functor $-\otimes H$ preserves coequalizers. For any $(M,\phi_M,\rho_M)\in \mathcal{S}M_H^H$, the objects $M^{\mathrm{co}\,H}\otimes_{H_L} H$ and $M^{\mathrm{co}\,H}\times H$ are isomorphic right-right $H$-Hopf modules.*

*Proof.* First note that $p_{M^{\mathrm{co}\,H}\otimes H}\circ(\psi_{M^{\mathrm{co}\,H}}\otimes H) = p_{M^{\mathrm{co}\,H}\otimes H}\circ(M^{\mathrm{co}\,H}\otimes\varphi_H)$ because

$$\nabla_M\circ(\psi_{M^{\mathrm{co}\,H}}\otimes H)$$
$$= (p_M\otimes H)\circ\rho_M\circ\phi_M\circ((q_M\circ\phi_M\circ(i_M\otimes i_L))\otimes H) \text{ (by the definition of } \nabla_M)$$
$$= (p_M\otimes H)\circ\rho_M\circ\phi_M\circ((\phi_M\circ(\phi_M\otimes\lambda_H)\circ(i_M\otimes(\delta_H\circ i_L)))\otimes H)$$
(by (3.2))
$$= (p_M\otimes H)\circ\rho_M\circ\phi_M\circ((\phi_M\circ(i_M\otimes i_L))\otimes H) \text{ (by (b3) of Definition 3.1 and}$$
by the properties of $\mathit{\Pi}_H^L$)
$$= (p_M\otimes H)\circ\rho_M\circ\phi_M\circ(i_M\otimes(\mu_H\circ(i_L\otimes H))) \text{ (by (3.16))}$$
$$= \nabla_M\circ(M^{\mathrm{co}\,H}\otimes\varphi_H) \text{ (by the definition of } \nabla_M).$$

Let $t : M^{\mathrm{co}\,H}\otimes H\to P$ be a morphism such that $t\circ(\psi_{M^{\mathrm{co}\,H}}\otimes H) = t\circ(M^{\mathrm{co}\,H}\otimes\varphi_H)$. Define $t' : M^{\mathrm{co}\,H}\times H\to P$ by $t' = t\circ i_{M^{\mathrm{co}\,H}\otimes H}$. Then $t'\circ p_{M^{\mathrm{co}\,H}\otimes H} = t\circ\nabla_M = t$ because

$$t\circ\nabla_M$$
$$= t\circ((p_M\circ\phi_M)\otimes H)\circ(i_M\otimes\delta_H) \text{ (by (3.4))}$$
$$= t\circ((p_M\circ\phi_M)\otimes H)\circ(i_M\otimes((\mathit{\Pi}_H^L\otimes H)\circ\delta_H)) \text{ (by (3.3))}$$
$$= t\circ(\psi_{M^{\mathrm{co}\,H}}\otimes H)\circ(M^{\mathrm{co}\,H}\otimes((p_L\otimes H)\circ\delta_H)) \text{ (by the definition of } \psi_{M^{\mathrm{co}\,H}})$$
$$= t\circ(M^{\mathrm{co}\,H}\otimes(\mathit{\Pi}_H^L * \mathrm{id}_H)) \text{ (by the properties of } t)$$
$$= t \text{ (by [2, (4)])}.$$

The morphism $t'$ is the unique such that $t'\circ p_{M^{\mathrm{co}\,H}\otimes H} = t$, because if $r : M^{\mathrm{co}\,H}\times H\to P$ satisfies $r\circ p_{M^{\mathrm{co}\,H}\otimes H} = t$, then composing with $i_{M^{\mathrm{co}\,H}\otimes H}$, we obtain $r = t\circ i_{M^{\mathrm{co}\,H}\otimes H} = t'$. Therefore,

$$\begin{tikzcd} M^{\mathrm{co}\,H}\otimes H_L\otimes H \arrow[r, shift left, "M^{\mathrm{co}\,H}\otimes\varphi_H"] \arrow[r, shift right, "\psi_{M^{\mathrm{co}\,H}}\otimes H"'] & M^{\mathrm{co}\,H}\otimes H \arrow[r, "p_{M^{\mathrm{co}\,H}\otimes H}"] & M^{\mathrm{co}\,H}\times H \end{tikzcd}$$

is a coequalizer diagram, and consequently there exists an isomorphism $s_M : M^{\mathrm{co}\,H}\otimes_{H_L} H\to M^{\mathrm{co}\,H}\times H$ such that

$$s_M\circ n_{M^{\mathrm{co}\,H}} = p_{M^{\mathrm{co}\,H}\otimes H}. \tag{3.20}$$

The morphism $s_M$ belongs to the category of right-right $H$-Hopf modules. Indeed, it is a morphism of right $H$-comodules because composing with $n_{M^{\mathrm{co}\,H}}$ and using (3.20), (3.5) we obtain

$$\begin{aligned}
&\rho_{M^{\mathrm{co}\,H}\times H}\circ s_M\circ n_{M^{\mathrm{co}\,H}}=\rho_{M^{\mathrm{co}\,H}\times H}\circ p_{M^{\mathrm{co}\,H}\otimes H}\\
&\quad=(p_{M^{\mathrm{co}\,H}\otimes H}\otimes H)\circ(M^{\mathrm{co}\,H}\otimes\delta_H)\circ\nabla_M=(p_{M^{\mathrm{co}\,H}\otimes H}\otimes H)\circ(M^{\mathrm{co}\,H}\otimes\delta_H)\\
&\quad=((s_M\circ n_{M^{\mathrm{co}\,H}})\otimes H)\circ(M^{\mathrm{co}\,H}\otimes\delta_H)=(s_M\otimes H)\circ\rho_{M^{\mathrm{co}\,H}\otimes_{H_L}H}\circ n_{M^{\mathrm{co}\,H}}.
\end{aligned}$$

Moreover, by (3.8) and (3.14) we know that $\phi^{\alpha_{M^{\mathrm{co}\,H}\times H}}_{M^{\mathrm{co}\,H}\times H}=\phi_{M^{\mathrm{co}\,H}\times H}$ and $\phi^{\alpha_{M^{\mathrm{co}\,H}\otimes_{H_L}H}}_{M^{\mathrm{co}\,H}\otimes_{H_L}H}=\phi_{M^{\mathrm{co}\,H}\otimes_{H_L}H}$. As a consequence, $s_M$ is $H$-quasilineal because composing with the coequalizer $n_{M^{\mathrm{co}\,H}}\otimes H$ and the equalizer $i_{M^{\mathrm{co}\,H}\otimes H}$ we obtain

$$\begin{aligned}
&i_{M^{\mathrm{co}\,H}\otimes H}\circ s_M\circ\phi_{M^{\mathrm{co}\,H}\otimes_{H_L}H}\circ(n_{M^{\mathrm{co}\,H}}\otimes H)\\
&\qquad=\omega'_M\circ\phi_M\circ(i_M\otimes\mu_H)\ \text{(by (3.12))}\\
&\qquad=\omega'_M\circ\phi_M\circ(i_M\otimes\mu_H)\circ(\nabla_M\otimes H)\ \text{(by (3.9))}\\
&\qquad=i_{M^{\mathrm{co}\,H}\otimes H}\circ\phi_{M^{\mathrm{co}\,H}\times H}\circ((s_M\circ n_{M^{\mathrm{co}\,H}})\otimes H)\ \text{(by (3.20))}.\ \blacksquare
\end{aligned}$$

MAIN THEOREM 3.10. *For any Hopf quasigroup $H$ such the functor $-\otimes H$ preserves coequalizers, the category $\mathcal{S}M^H_H$ is equivalent to $\mathcal{C}_{H_L}$.*

*Proof.* To prove the theorem we will show that the induction functor $F$ is left adjoint to the coinvariants functor $G$ and that the unit and counit associated to this adjunction are natural isomorphisms. We divide the proof into three steps.

STEP 1. In this step we will define the unit of the adjunction. For any right $H_L$-module $(N,\psi_N)$ define $u_N:N\to GF(N)=(N\otimes_{H_L}H)^{\mathrm{co}\,H}$ as the unique morphism such that

$$i_{N\otimes_{H_L}H}\circ u_N=n_N\circ(N\otimes\eta_H).\tag{3.21}$$

This morphism exists and is unique because by (3.11) and [2, (18)] we have

$$\begin{aligned}
((N\otimes_{H_L}H)\otimes\overline{\varPi}^R_H)\circ\rho_{N\otimes_{H_L}H}\circ n_N\circ(N\otimes\eta_H)&=(n_N\otimes\overline{\varPi}^R_H)\circ(N\otimes(\delta_H\circ\eta_H))\\
=(n_N\otimes H)\circ(N\otimes(\delta_H\circ\eta_H))&=\rho_{N\otimes_{H_L}H}\circ n_N\circ(N\otimes\eta_H).
\end{aligned}$$

Also, it is a morphism in $\mathcal{C}_{H_L}$. Indeed, composing with the equalizer $i_{N\otimes_{H_L}H}$ we have

$$\begin{aligned}
&i_{N\otimes_{H_L}H}\circ\psi_{N\otimes_{H_L}H}\circ(u_N\otimes H_L)\\
&\qquad=q_{N\otimes_{H_L}H}\circ\phi_{N\otimes_{H_L}H}\circ((n_N\circ(N\otimes\eta_H))\otimes i_L)\ \text{(by (3.21))}\\
&\qquad=q_{N\otimes_{H_L}H}\circ n_N\circ(N\otimes(\mu_H\circ(\eta_H\otimes i_L)))\ \text{(by (3.12))}\\
&\qquad=q_{N\otimes_{H_L}H}\circ n_N\circ(N\otimes i_L)\ \text{(by the unit properties)}\\
&\qquad=n_N\circ(N\otimes(\varPi^L_H\circ i_L))\ \text{(by (3.13))}
\end{aligned}$$

$$
\begin{aligned}
&= n_N \circ (N \otimes i_L) \text{ (by the properties of } \Pi_H^L) \\
&= n_N \circ (N \otimes (\mu_H \circ (i_L \otimes \eta_H))) \text{ (by the unit properties)} \\
&= n_N \circ (\psi_N \otimes \eta_H) \text{ (by the definition of } N \otimes_{H_L} H) \\
&= i_{N \otimes_{H_L} H} \circ u_N \circ \psi_N \text{ (by (3.21))}.
\end{aligned}
$$

The morphism $u_N$ is natural in $N$ because if $f : N \to P$ is a morphism in $\mathcal{C}_{H_L}$, then by (3.17), (3.21) and (3.15) we have

$$
\begin{aligned}
i_{P \otimes_{H_L} H} \circ (f \otimes_{H_L} H)^{\mathrm{co}\, H} \circ u_N &= (f \otimes_{H_L} H) \circ i_{N \otimes_{H_L} H} \circ u_N \\
&= (f \otimes_{H_L} H) \circ n_N \circ (N \otimes \eta_H) = n_P \circ (f \otimes \eta_H) = i_{P \otimes_{H_L} H} \circ u_P \circ f,
\end{aligned}
$$

and thus $(f \otimes_{H_L} H)^{\mathrm{co}\, H} \circ u_N = u_P \circ f$.

Finally, we prove that $u_N$ is an isomorphism for all right $H_L$-modules $N$. First note that $\psi_N \circ (\psi_N \otimes p_L) = \psi_N \circ (N \otimes (p_L \circ \varphi_H))$, and so there exists a unique morphism $m_N : N \otimes_{H_L} H \to N$ such that

$$
m_N \circ n_N = \psi_N \circ (N \otimes p_L). \tag{3.22}
$$

Define $x_N = m_N \circ i_{N \otimes_{H_L} H} : (N \otimes_{H_L} H)^{\mathrm{co}\, H} \to N$. Composing with $i_{N \otimes_{H_L} H}$ and $p_{N \otimes_{H_L} H} \circ n_N$ and using (3.13), (3.22), (3.21) and the properties of $\Pi_H^L$ we obtain

$$
\begin{aligned}
i_{N \otimes_{H_L} H} \circ u_N \circ x_N \circ p_{N \otimes_{H_L} H} \circ n_N &= i_{N \otimes_{H_L} H} \circ u_N \circ m_N \circ q_{N \otimes_{H_L} H} \circ n_N \\
= i_{N \otimes_{H_L} H} \circ u_N \circ \psi_N \circ (N \otimes (p_L \circ \Pi_H^L)) &= n_N \circ ((\psi_N \circ (N \otimes p_L)) \otimes \eta_H) \\
&= n_N \circ (N \otimes (\mu_H \circ (\Pi_H^L \otimes \eta_H))) = q_{N \otimes_{H_L} H} \circ n_N.
\end{aligned}
$$

Therefore, $u_N \circ x_N = \mathrm{id}_{(N \otimes_{H_L} H)^{\mathrm{co}\, H}}$. Moreover, by (3.21) and (3.22), we have $x_N \circ u_N = \psi_N \circ (N \otimes (p_L \circ \eta_H)) = \mathrm{id}_N$.

Step 2. For any $(M, \phi_M, \rho_M) \in \mathcal{S}M_H^H$ the counit is defined by $v_M = \alpha_M^{-1} \circ s_M : M^{\mathrm{co}\, H} \otimes_{H_L} H \to M$, where $\alpha_M^{-1} = \omega_M \circ i_{M^{\mathrm{co}\, H} \otimes H}$ is the inverse of the isomorphism $\alpha_M$ defined in Theorem 3.3, and $s_M$ the isomorphism defined in Proposition 3.9. Note that $\alpha_M^{-1}$ and $s_M$ are isomorphisms in $\mathcal{S}M_H^H$, and so $v_M$ is an isomorphism in $\mathcal{S}M_H^H$. Also, $v_M$ is the unique morphism such that

$$
v_M \circ n_{M^{\mathrm{co}\, H}} = \phi_M \circ (i_M \otimes H), \tag{3.23}
$$

because by (3.2), (b3) of Definition 3.1, the properties of $\Pi_H^L$ and (3.16),

$$
\begin{aligned}
\phi_M \circ ((i_M \circ \psi_{M^{\mathrm{co}\, H}}) \otimes H) &= \phi_M \circ ((\phi_M \circ (\phi_M \otimes \lambda_H) \circ (i_M \otimes (\delta_H \circ i_L))) \otimes H) \\
= \phi_M \circ ((\phi_M \circ (i_M \otimes (\Pi_H^L \circ i_L))) \otimes H) &= \phi_M \circ ((\phi_M \circ (i_M \otimes i_L)) \otimes H) \\
&= \phi_M \circ (i_M \otimes \varphi_H),
\end{aligned}
$$

and, on the other hand, by (3.20), $v_M \circ n_{M^{\mathrm{co}\, H}} = \alpha_M^{-1} \circ s_M \circ n_{M^{\mathrm{co}\, H}} = \omega_M \circ \omega'_M \circ \omega_M = \omega_M = \phi_M \circ (i_M \otimes H)$.

Step 3. Now we prove the triangular identities for the unit and the counit that we defined previously. Indeed, the first triangular identity holds because composing with $n_N$ we have

$$\begin{aligned} v_{N\otimes_{H_L}H}\circ(u_N\otimes_{H_L}H)\circ n_N &= v_{N\otimes_{H_L}H}\circ n_{(N\otimes_{H_L}H)^{\mathrm{co}\,H}}\circ(u_N\otimes H) \text{ (by (3.15))}\\ &= \phi_{N\otimes_{H_L}H}\circ((i_{N\otimes_{H_L}H}\circ u_N)\otimes H) \text{ (by (3.23))}\\ &= \phi_{N\otimes_{H_L}H}\circ((n_N\circ(N\otimes\eta_H))\otimes H) \text{ (by (3.21))}\\ &= n_N\circ(N\otimes(\mu_H\circ(\eta_H\otimes H))) \text{ (by (3.12))}\\ &= n_N \text{ (by the properties of the unit)}. \end{aligned}$$

Finally, if we compose with $i_M$, applying (3.17), (3.21) and (3.23) we obtain

$$\begin{aligned} i_M\circ v_M^{\mathrm{co}\,H}\circ u_{M^{\mathrm{co}\,H}} = v_M\circ i_{N\otimes_{H_L}H}\circ u_{M^{\mathrm{co}\,H}} &= v_M\circ n_{M^{\mathrm{co}\,H}}\circ(M^{\mathrm{co}\,H}\otimes\eta_H)\\ &= \phi_M\circ(i_M\otimes\eta_H) = i_M, \end{aligned}$$

and so $v_M^{\mathrm{co}\,H}\circ u_{M^{\mathrm{co}\,H}} = \mathrm{id}_{M^{\mathrm{co}\,H}}$. ■

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## REFERENCES

[1] J. N. Alonso Álvarez, J. M. Fernández Vilaboa and R. González Rodríguez, *Weak braided Hopf algebras*, Indiana Univ. Math. J. 57 (2008), 2423–2458.

[2] J. N. Alonso Álvarez, J. M. Fernández Vilaboa and R. González Rodríguez, *Weak Hopf quasigroups*, Asian J. Math. 20 (2016), 665–694.

[3] J. N. Alonso Álvarez, J. M. Fernández Vilaboa and R. González Rodríguez, *Cleft and Galois extensions associated to a weak Hopf quasigroup*, J. Pure Appl. Algebra 220 (2016), 1002–1034.

[4] G. Böhm, F. Nill and K. Szlachányi, *Weak Hopf algebras, I. Integral theory and $C^*$-structure*, J. Algebra 221 (1999), 385–438.

[5] T. Brzeziński, *Hopf modules and the fundamental theorem for Hopf (co)quasigroups*, Int. Electron. J. Algebra 8 (2010), 114–128.

[6] H. Henker, *Module categories over quasi-Hopf algebras and weak Hopf algebras and the projectivity of Hopf modules*, Thesis Dissertation, LMU Munich, 2011; http://edoc.ub.uni-muenchen.de/13148/.

[7] J. Klim and S. Majid, *Hopf quasigroups and the algebraic 7-sphere*, J. Algebra 323 (2010), 3067–3110.

[8] R. G. Larson and M. E. Sweedler, *An associative orthogonal bilinear form for Hopf algebras*, Amer. J. Math. 91 (1969), 75–93.

[9] M. P. López and E. Villanueva, *The antipode and the (co)invariants of a finite Hopf (co)quasigroup*, Appl. Categor. Structures 21 (2013), 237–247.

[10] J. M. Pérez-Izquierdo, *Algebras, hyperalgebras, nonassociative bialgebras and loops*, Adv. Math. 208 (2007), 834–876.

[11] R. S. Pierce, *Associative Algebras*, Grad. Texts in Math. 88, Springer, New York, 1982.
[12] M. E. Sweedler, *Hopf Algebras*, Benjamin, New York, 1969.

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