ANNALES
POLONICI MATHEMATICI
119.1 (2017)

Whitney's extension theorem in o-minimal structures

Athipat Thamrongthanyalak (Bangkok and Columbus, OH)

Abstract. In 1934, H. Whitney gave a necessary and sufficient condition on a jet of order m on a closed subset of E of $\mathbb{R}^n$ to be the jet of order m of a C^m-function. Later, K. Kurdyka and W. Pawłucki proposed a subanalytic version of this theorem. In this paper, we work in an o-minimal expansion of a real closed field and prove a definable version of Whitney's Extension Theorem.

Throughout, we fix an o-minimal expansion $\boldsymbol{R}$ of a real closed ordered field R in a language extending the language of ordered fields. As usual, "definable" means "definable in $\boldsymbol{R}$ possibly with parameters" unless indicated otherwise. We assume that the reader is familiar with the basic definitions and facts concerning o-minimal structures (see, e.g., [1, 2]). Whitney's Extension Theorem, which can be regarded as a partial converse of Taylor's Theorem, was proved by H. Whitney in 1934. (See [9, 12] for the proof, and [13, 14] for related problems.) It roughly says that a continuous function on a closed subset of $\mathbb{R}^n$ which can be approximated by Taylor polynomials of degree m in a certain uniform way is the restriction of a C^m-function. A collection of functions which encodes the relevant data for such an approximation is called a C^m-Whitney field. Later, K. Kurdyka and W. Pawłucki [7] proposed a version of Whitney's Extension Theorem in the category of subanalytic functions. The question on Whitney's Extension Theorem in o-minimal structures was raised by C. Miller in early 2000s.

In this paper, we prove a definable version of Whitney's Extension Theorem:

Theorem A. *Suppose $E \subseteq R^n$ is definable and closed. Let $m, q \in \mathbb{N}$. Then every definable C^m-Whitney field on E has a definable C^m-extension which is C^q outside E.*

2010 *Mathematics Subject Classification*: Primary 03C64; Secondary 14P10, 32B20.
Key words and phrases: Whitney's Extension Theorem, o-minimal structures.
Received 24 February 2016; revised 23 February 2017.
Published online 20 March 2017.

DOI: 10.4064/ap3940-2-2017

Note that this theorem was independently proved by K. Kurdyka and W. Pawłucki [8]. Due to the differences in the approaches, the author believes this article is of some interest.

Let us make precise what we mean by a definable C^m-Whitney field and an extension of such a Whitney field. Let $E \subseteq R^n$ be definable. A (definable) *jet of order* m on E is a family $F = (F^\alpha)_{|\alpha| \le m}$ where each $F^\alpha \colon E \to R$ is a definable continuous function. If F is a jet of order m on E and $E' \subseteq E$ is definable, then $F \restriction E' := (F^\alpha \restriction E')_{|\alpha| \le m}$ is a jet of order m on E'. If E is open, then for each definable C^m-function $f \colon E \to R$, we obtain a jet $J^m(f) = (D^\alpha f)_{|\alpha| \le m}$ of order m on E. Here, $\alpha = (\alpha_1, \dots, \alpha_n)$ ranges over $\mathbb{N}^n$, and we let $D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$ and $|\alpha| := \alpha_1 + \dots + \alpha_n$. Now for every $x \in R^n$, $a \in E$, and F a jet of order m on E, set

$$T_a^m F(x) = \sum_{|\alpha| \le m} F^\alpha(a) \frac{(x-a)^\alpha}{\alpha!},$$

$$R_a^m F(x) = F - J^m(T_a^m F(x)).$$

We say that a jet F of order m is a *definable* C^m*-Whitney field on* E ($F \in \mathscr{E}^m_{\mathrm{def}}(E)$) if, for all $x_0 \in E$ and $|\alpha| \le m$, we have

$$(R_x^m F)^\alpha(y) = o(\|x-y\|^{m-|\alpha|}) \qquad \text{as } E \ni x, y \to x_0;$$

equivalently, if for all for $x_0 \in E$ and $z \in R^n$,

$$|T_x^m F(z) - T_y^m F(z)| = o(\|x-z\|^m + \|y-z\|^m) \qquad \text{as } E \ni x, y \to x_0.$$

(See [9].) Note that if $F \in \mathscr{E}^m_{\mathrm{def}}(E)$ and $E' \subseteq E$ is definable, then $F \restriction E' \in \mathscr{E}^m_{\mathrm{def}}(E')$. Also, if E is open and $f \colon E \to R$ is a definable C^m-function, then $J^m(f)$ is a C^m-Whitney field, by Taylor's Theorem. Given $F \in \mathscr{E}^m_{\mathrm{def}}(E)$, we say that a definable C^m-function $f \colon R^n \to R$ is an *extension* of F if $J^m(f) \restriction E = F$.

An immediate consequence of the theorem above is the following:

Corollary. *Suppose that E is regularly closed (i.e., E equals the closure of its interior). Let $f \colon E \to R$ be a definable function such that for each $x \in E$ there is an open neighborhood U of x in R^n and an extension of $f \restriction (E \cap U)$ to a definable C^m-function $U \to R$. Then f extends to a definable C^m-function $R^n \to R$.*

One of the key ingredients in the construction of Kurdyka and Pawłucki [7] is a partition of unity, which is not generally available in o-minimal expansions of real closed fields. In [11], Pawłucki introduced a new algorithm to extend C^m-Whitney fields on $E \subseteq \mathbb{R}^n$. However, this new construction does not preserve definability in a given o-minimal expansion of $\mathbb{R}$, due to its use of integration. In this paper, we still follow Pawłucki's five-step strategy from [11], while combining it with Λ^m-regular Stratification Theorem from [6, 3].

Conventions and notation. Throughout this paper, d, k, m, n, and q will range over the set $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers. Given a map $f\colon X \to Y$ we write

$$\Gamma(f) = \{(x, f(x)) : x \in X\} \subseteq X \times Y$$

for the graph of f. For any set X, we also consider $+\infty$ and $-\infty$ as constant functions on X. For $f, g\colon X \to R \cup \{\pm\infty\}$, we write $f < g$ if $f(x) < g(x)$ for all $x \in X$, and in this case we set

$$(f, g) := \{(x, r) \in X \times R : f(x) < r < g(x)\}.$$

Similarly an interval in R is a set of the form

$$(a, b) := \{r \in R : a < r < b\} \quad \text{where } a, b \in R \cup \{-\infty, +\infty\} \text{ and } a < b.$$

For a set $S \subseteq R^n$ we denote by $\mathrm{cl}(S)$ its closure and by $\partial S := \mathrm{cl}(S) \setminus S$ its frontier. We denote the Euclidean norm on R^n by $\|\cdot\|$ and the associated metric by $(x, y) \mapsto d(x, y) := \|x - y\|$.

Given $x \in R^n$, for a non-empty definable set $S \subseteq R^n$ let $d(x, S) := \inf_{y \in S} d(x, y) \in R^{\geq 0}$ be the distance between x and S, and $d(x, \emptyset) := +\infty$. Given a collection $\mathscr{C}$ of subsets of R^n, we let $\mathscr{C}^o := \{C \in \mathscr{C} : C \text{ is open}\}$.

1. Preliminaries. The style of the proof of Theorem A will be analogous to the approach to the C^p-zero set problem (see [2] for more information). When dealing with the C^p-zero set problem, we split the domain into "smaller" or "nicer" pieces and work on each new piece separately; then we glue them up to obtain the desired extension. In this section, we introduce notation, terminology, and basic facts which will serve the purposes mentioned above.

Definition 1.1. For every subset E of R^n, let $\dim(E)$ denote the largest integer k such that, after some permutation of coordinates, the projection of E onto the first k coordinates has non-empty interior.

Let $X \subseteq E$ be subsets of R^n. We say that X is a *small* subset of E if $\dim(X) < \dim(E)$.

1.1. Λ^m-stratifications. One of our main tools is the Λ^m-Stratification Theorem (see [6] and [3]). To properly introduce this theorem and some of its modifications, first more definitions will be introduced. In the following, we assume $m \geq 1$.

Definition 1.2. Let $f = (f_1, \dots, f_n)\colon \Omega \to R^n$ be a C^m-map, where Ω is a non-empty open subset of R^d with $d \geq 1$. We say that f is *Λ^m-regular* if there is some $L \in R^{>0}$ such that

$$\|D^\alpha f(x)\| \leq \frac{L}{d(x, \partial\Omega)^{|\alpha|-1}} \quad \text{for all } x \in \Omega \text{ and } \alpha \in \mathbb{N}^d \text{ with } 1 \leq |\alpha| \leq m.$$

We also define every map $R^0 \to R^n$ to be Λ^m-regular.

NOTATION. Let $\Omega \subseteq R^d$ be definable and open. Set

$$\Lambda^m(\Omega) := \{f\colon \Omega \to R : f \text{ is definable and } \Lambda^m\text{-regular}\},$$
$$\Lambda^m_\infty(\Omega) := \Lambda^m(\Omega) \cup \{-\infty, +\infty\},$$

where $+\infty$ and $-\infty$ are considered as constant functions on Ω.

DEFINITION 1.3. *Standard open Λ^m-regular cells in R^n* are defined inductively on n as follows:

(1) $n = 0$: R^0 is the standard open Λ^m-regular cell in R^0;
(2) $n \geq 1$: a set of the form (f, g) where $f, g \in \Lambda^m_\infty(D)$ with $f < g$, and D is a standard open Λ^m-regular cell in R^{n-1}.

We say that a subset of R^n is a *standard Λ^m-regular cell in R^n* if it is either a standard open Λ^m-regular cell in R^n or one of the following:

(1) a singleton; or
(2) the graph of a definable Λ^m-regular map $D \to R^{n-d}$, where D is a standard open Λ^m-regular cell in R^d with $1 \leq d < n$.

A subset $E \subseteq R^n$ is called a *Λ^m-regular cell in R^n* if there is a linear orthogonal transformation $\phi\colon R^n \to R^n$ such that $\phi(E)$ is a standard Λ^m-regular cell in R^n.

REMARK. Every Λ^m-regular map on an open Λ^m-regular cell is Lipschitz.

DEFINITION 1.4. By a *Λ^m-regular stratification of R^n* we mean a finite partition $\mathscr{D}$ of R^n into Λ^m-regular cells such that each ∂D ($D \in \mathscr{D}$) is a union of sets from $\mathscr{D}$. Given $E_1, \ldots, E_N \subseteq R^n$, the Λ^m-regular stratification $\mathscr{D}$ of R^n is said to be *compatible with* $E_1, \ldots, E_N$ if each E_i is a union of sets from $\mathscr{D}$.

THEOREM 1.5 (Kurdyka & Pawłucki [7], Fischer [3]). *Let $E_1, \ldots, E_N$ be definable subset of R^n. There exists a Λ^m-regular stratification of R^n compatible with $E_1, \ldots, E_N$.*

By the same idea as in [3, Proposition 2.1], we obtain the following modification of the above theorem. For the sake of brevity, we leave the proof to the reader.

LEMMA 1.6. *Let $f_1, \ldots, f_k\colon U \to R$ be definable continuous functions where U is a definable open subset of R^d. There is a Λ^m-regular stratification $\mathscr{D}$ of R^d compatible with U and some $L \in R$ with the following property: for each $D \in \mathscr{D}$ which is contained in U, each $f_i{\restriction}D$ is C^m and*

$$|D^\alpha f_i(u)| \leq \frac{L}{d(u, \partial D)^{|\alpha|}} \sup\{|f_i(v)| : v \in D,\ \|u - v\| < d(u, \partial D)\}$$
$$\text{for } |\alpha| \leq m \text{ and } u \in D.$$

1.2. Separation. The following important definition goes back to Malgrange's regularly situated condition (see [9]). Let X and Y be closed subsets of R^n. Define $\delta\colon \mathscr{E}^m(X\cup Y) \to \mathscr{E}^m(X)\oplus\mathscr{E}^m(Y)$ and $\pi\colon \mathscr{E}^m(X)\oplus\mathscr{E}^m(Y) \to \mathscr{E}^m(X\cap Y)$ by

$$\begin{aligned} \delta(F) &:= (F{\restriction}X, F{\restriction}Y), \\ \pi(G,H) &:= G{\restriction}X\cap Y - H{\restriction}X\cap Y \end{aligned}$$

for $F \in \mathscr{E}^m(X \cup Y)$ and $G, H \in \mathscr{E}^m(X \cap Y)$. We say that X and Y are *regularly situated* if the sequence

$$0 \to \mathscr{E}^m(X \cup Y) \xrightarrow{\delta} \mathscr{E}^m(X) \oplus \mathscr{E}^m(Y) \xrightarrow{\pi} \mathscr{E}^m(X \cap Y) \to 0$$

is exact. In other words, a C^m-Whitney field on X and another C^m-Whitney field on Y can be glued whenever they agree on $X \cap Y$.

DEFINITION 1.7. Let $X, Y, Z \subseteq R^n$. We say that X and Y are *Z-separated* if there exists $C \in R^{>0}$ such that

$$d(x, Y) \geq C d(x, Z) \quad \text{for every } x \in X.$$

Equivalently, there is a $C' > 0$ such that

$$d(x, X) + d(x, Y) \geq C' d(x, Z) \quad \text{for every } x \in R^n.$$

In [10], Pawłucki gave a special stratification of $\mathbb{R}^n$ providing separability between each pair of sets in the partition. The proof also works in o-minimal exansions of real closed fields, and therefore is omitted here.

DEFINITION 1.8. We say that a subset E of R^n of dimension d is a *Λ^m-pancake* if E is a finite disjoint union of graphs of Lipschitz, Λ^m-regular maps $\Omega \to R^{n-d}$ on a common domain Ω, which is an open Λ^m-regular cell in R^d.

THEOREM 1.9 (Pawłucki [10]). *Let E be a definable closed subset of R^n of dimension d. There is a finite partition $E = M_1 \cup \cdots \cup M_s \cup A$ such that*

(1) *each M_i is a Λ^m-pancake of dimension d in a suitable coordinate system;*
(2) *A is a small, closed, definable subset of E;*
(3) *for all $i \neq j$, $\mathrm{cl}(M_i)$ and $\mathrm{cl}(M_j)$ are ∂M_i-separated;*
(4) *for each i, $\mathrm{cl}(M_i)$ and A are ∂M_i-separated.*

1.3. Hestenes' Lemma. The classical incarnation of the following theorem is one of the keys to the study of Whitney fields. Here, we give an o-minimal version of Hestenes' Lemma. (See [5, Lemma 1] for the classical result.)

THEOREM 1.10 (Definable Hestenes' Lemma). *Let Ω be a definable open subset of R^n. Let $F = (F^\alpha)_{|\alpha|\le m}$ be a jet of order m on Ω. Let E be a closed definable subset of Ω such that $F\restriction E \in \mathscr{E}^m_{\mathrm{def}}(E)$ and $F\restriction(\Omega\setminus E) \in \mathscr{E}^m_{\mathrm{def}}(\Omega\setminus E)$. Then $f := F^0$ is C^m on Ω and $D^\alpha f = F^\alpha$ on Ω. In particular, $F \in \mathscr{E}^m_{\mathrm{def}}(\Omega)$.*

Proof. Let $e_1,\dots,e_n \in \mathbb{N}^n$ be the standard basis of R^n. It is sufficient to show that f is of class C^1 on Ω and, for every $a\in\Omega$ and $i\in\{1,\dots,n\}$, $\frac{\partial f}{\partial x_i}(a) = F^{e_i}(a)$, i.e., for every $\epsilon>0$, there is $\delta>0$ such that

$$|f(a+te_i) - (f(a)+F^{e_i}(a)t)| \le \epsilon|t| \quad \text{for } 0<|t|<\delta. \tag{1.1}$$

Let $a\in\Omega$ and $i\in\{1,\dots,n\}$. Since $\frac{\partial f}{\partial x_i} = F^{e_i}$ on $\Omega\setminus E$, we may assume that $a\in E$. Let $\epsilon>0$ be given. For $x,y\in R^n$ set

$$(x,y) := \{x+t(y-x) : t\in(0,1)\}.$$

By the Cell Decomposition Theorem, there exists $\delta_0>0$ such that either $(a,a+\delta_0 e_i)$ is contained in E, or in $\Omega\setminus E$. If $(a,a+\delta_0 e_i)\subseteq E$, then, since $a\in E$ and $F\restriction E\in\mathscr{E}^m_{\mathrm{def}}(E)$, there is $0<\delta_1<\delta_0$ such that

$$|f(a+te_i) - (f(a)+F^{e_i}(a)t)| \le \epsilon t \quad \text{for } 0<t<\delta_1,$$

so (1.1) holds with $\delta=\delta_1$. Now suppose $(a,a+\delta_0 e_i)\subseteq \Omega\setminus E$. By continuity of F^{e_i}, we may assume that

$$|F^{e_i}(x) - F^{e_i}(a)| < \epsilon \quad \text{for every } x\in(a,a+\delta_0 e_i).$$

Let $t\in(0,\delta_0)$. Since f is C^1 on $\Omega\setminus E$ with $\frac{\partial f}{\partial x_i} = F^{e_i}$ on $\Omega\setminus E$, by the Mean Value Theorem we have

$$\begin{aligned}|f(a+te_i)-&(f(a)+F^{e_i}(a)t)|\\ &\le |(F^{e_i}(\xi)-F^{e_i}(a))t| \quad \text{for some } \xi\in(a,a+te_i)\\ &< \epsilon t.\end{aligned}$$

Therefore, there is $\delta_1>0$ such that

$$|f(a+te_i) - (f(a)+F^{e_i}(a)t)| < \epsilon t \quad \text{for } 0<t<\delta_1.$$

By the same argument, we can also find $\delta_2>0$ such that

$$|f(a-te_i) - (f(a)+F^{e_i}(a)(-t))| < \epsilon t \quad \text{for } 0<t<\delta_2.$$

Then (1.1) holds with $\delta=\min\{\delta_1,\delta_2\}$. ∎

1.4. Pullbacks. Let $E\subseteq R^n$, $E'\subseteq R^{n'}$ be definable and $\varphi=(\varphi_1,\dots,\varphi_n)$ be a definable C^m-map from U' to U, where $U\subseteq R^n$, $U'\subseteq R^{n'}$ are open definable neighborhoods of E, E', respectively, such that $\varphi(E')\subseteq E$. Then φ induces an R-linear map $F\mapsto\varphi^*F\colon \mathscr{E}^m_{\mathrm{def}}(E)\to\mathscr{E}^m_{\mathrm{def}}(E')$ as follows: Suppose $a'\in E'$, $a=\varphi(a')\in E$, and view

$$T^m_a F = \sum_{|\alpha|\le m} F^\alpha(a)\frac{(x-a)^\alpha}{\alpha!}$$

as an element of the polynomial ring $R[x_1 - a_1, \dots, x_n - a_n]$. Then φ^*F is the jet of order m on E' such that for each $a' \in E'$, the Taylor polynomial $T^m_{a'}\varphi^*F$ can be obtained by substituting $T^m_{a'}\varphi_i \in R[x'_1 - a'_1, \dots, x_{n'} - a'_{n'}]$ for x_i in the polynomial $T^m_a F$ and dropping the terms of degree $> m$ in $x' - a'$. It is easy to verify that φ^*F is a (definable) C^m-Whitney field on E' (the *pullback* of F under φ).

If $f\colon U \to R$ is a definable C^m-function, then $\varphi^*(J^m(f)) = J^m(f \circ \varphi)$. Moreover, if $E_1 \subseteq E$ and $E'_1 \subseteq E'$ are definable such that $\varphi(E'_1) \subseteq E_1$, then

$$(\varphi^*F)\restriction E'_1 = \varphi^*(F\restriction E_1) \quad \text{for all } F \in \mathscr{E}^m_{\mathrm{def}}(E).$$

If $\varphi'\colon U'' \to U'$ is another definable C^m-map and $E'' \subseteq U''$ definable with $\varphi(E'') \subseteq E'$, then $(\varphi \circ \varphi')^* = (\varphi')^* \circ \varphi^*$.

Given a pair $E' \subseteq E$ of definable subsets of R^n, we say that a jet F of order m on E is *flat on* E' if $F\restriction E' = 0$, and we let $\mathscr{E}^m_{\mathrm{def}}(E, E')$ be the subspace of $\mathscr{E}^m_{\mathrm{def}}(E)$ consisting of the definable C^m-Whitney fields on E which are flat on E'.

Proposition 1.11 (Kurdyka & Pawłucki [7, Proposition 3], [8, Proposition 3]). *Let Ω be a definable open Λ^m-regular cell in R^n, and E a definable closed subset of Ω such that* $\mathrm{cl}(E)$ *and* $\partial\Omega$ *are* $(\mathrm{cl}(E) \cap \partial\Omega)$*-separated. Let $\varphi\colon \Omega \to R^n$ be a definable Λ^m-regular map with continuous extension $\overline{\varphi}\colon \mathrm{cl}(\Omega) \to R^n$ to $\mathrm{cl}(\Omega)$. Let E' be a definable closed subset of R^n containing $\varphi(E)$ and $F = (F^\alpha)_{|\alpha| \le m}$ be a jet of order m on E' such that, for every $x'_0 \in \overline{\varphi}(\partial E')$ and $|\alpha| \le m$,*

$$F^\alpha(x) = o(d(x, \partial E')^{m-|\alpha|}) \quad \text{as } E' \ni x \to x'_0.$$

Then, for any $x_0 \in \partial E$ and $|\alpha| \le m$,

$$(\varphi^*F)^\alpha(x) = o(d(x, \partial E)^{m-|\alpha|}) \quad \text{as } E' \ni x \to x_0.$$

The following is an immediate consequence of the above proposition. For the sake of brevity, the proof is omitted.

Corollary 1.12. *Let Ω be an open Λ^m-regular cell in R^d and $E := \Omega \times \{0\} \subseteq R^{d+l}$. Suppose that $\varphi\colon \Omega \times R^l \to R^{d+l}$ is a definable Λ^m-regular map and $\overline{\varphi}\colon \mathrm{cl}(\Omega) \times R^l \to R^{d+l}$ is the continuous extension of φ. Assume further that $\overline{\varphi}(\partial E) = \partial(\varphi(E))$. Let $F \in \mathscr{E}^m_{\mathrm{def}}(\mathrm{cl}(\varphi(E)), \partial(\varphi(E)))$. For each $|\alpha| \le m$, define $\overline{F}^\alpha\colon \mathrm{cl}(E) \to R$ by*

$$\overline{F}^\alpha(x) := \begin{cases} (\varphi^*F)^\alpha(x) & \text{if } x \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\overline{\varphi}^*F := (\overline{F}^\alpha)_{|\alpha| \le m}$. Then $\overline{\varphi}^*F \in \mathscr{E}^m_{\mathrm{def}}(\mathrm{cl}(E), \partial E)$.

From now on, if all conditions in Corollary 1.12 hold, we denote $\overline{\varphi}^*F$ just by φ^*F for notational simplicity.

1.5. The sets $\Delta_\epsilon(E)$. For $\epsilon > 0$ and definable $E, E' \subseteq R^n$ with $E' \subseteq \mathrm{cl}(E)$, we let

$$\Delta_\epsilon(E, E') := \{x \in R^n : d(x, E) < \epsilon d(x, E')\},$$

and we set $\Delta_\epsilon(E) := \Delta_\epsilon(E, \partial E)$. The following propositions and lemma are devoted to useful properties of the sets $\Delta_\epsilon(E)$.

PROPOSITION 1.13. *Let Ω be an open cell in R^d. Then, for each $\epsilon > 0$ and each l,*

$$\Delta_\epsilon(\Omega \times \{0\}^l) = \left\{(x, y) \in \Omega \times R^l : \|y\| \leq \frac{\epsilon}{\sqrt{1-\epsilon^2}} d(x, \partial\Omega)\right\}.$$

We leave the proof of this proposition to the reader.

PROPOSITION 1.14. *Let $E = \Gamma(\varphi)$ where $\varphi \colon \Omega \to R^l$ is definable and Lipschitz and Ω is an open cell in R^d. Then there is $\epsilon_0 > 0$ with $\Delta_\epsilon(E) \subseteq \Omega \times R^l$ for all $0 < \epsilon < \epsilon_0$.*

Proof. For any Lipschitz constant L of φ, we set $\epsilon_0 = \frac{1}{1+\sqrt{1+L^2}}$, and the proof is straightforward. ∎

LEMMA 1.15. *Let $\Omega \subseteq R^n$ be open and $E = \bigcup_{i=1}^N \Gamma(\varphi_i)$ where each $\varphi_i \colon \Omega \to R^l$ is definable and Lipschitz. Set*

$$\varphi_{i+}(x, y) := (x, y + \varphi_i(x)) \qquad \text{for } (x, y) \in \Omega \times R^l \text{ and } i = 1, \dots, N.$$

Then

$$\varphi_{i+}(\Delta_\epsilon(\Omega \times \{0\}^l)) \subseteq \Delta_{2\epsilon}(E) \qquad \text{for all } 0 < \epsilon < 1/\sqrt{2} \text{ and } i \in \{1, \dots, N\}.$$

Proof. This follows from Proposition 1.13. ∎

Next, Proposition 6.2 in [11], which is a main step in Pawłucki's version of Whitney's Extension Theorem, can be o-minimalized and the idea of the proof is straightforward.

PROPOSITION 1.16 (Pawłucki [11, Proposition 6.2]). *Assume $m \leq q$. Let $E_i \supseteq E_i'$ $(i = 1, \dots, s)$ be definable closed subsets of R^n and $C > 0$ be a constant such that for any $i, j \in \{1, \dots, s\}$, $i \neq j$,*

$$d(x, E_i) + d(x, E_j) \geq C d(x, E_i') \qquad \text{for all } x \in R^n.$$

Set $E = E_1 \cup \dots \cup E_N$, $E' = E_1' \cup \dots \cup E_N'$, and let $F \in \mathscr{E}^m(E, E')$ and $\epsilon \in (0, C/2)$. Suppose $F \restriction E_i$ has a definable C^m-extension f_i which is m-flat outside $\Delta_\epsilon(E_i, E_i')$ and C^q outside E_i, for each $i = 1, \dots, s$. Then $f = \sum_{i=1}^s f_i$ is a definable C^m-extension of F which is C^q outside E.

1.6. The functions associated with a standard open Λ^m-regular cell. Let $\Omega \subseteq R^n$ be a standard open Λ^m-regular cell. Kurdyka and Pawłucki introduced functions $\rho_j \colon \mathrm{cl}(\Omega) \to R$ $(j = 1, \dots, 2n)$ corresponding to such a cell, which we call the **functions associated with** Ω, and used them in the proof of their main theorems (see [7, 11]). These functions also become useful in our construction of definable C^m-extensions. We define the ρ_j by induction on n:

(1) For $n = 1$ and $\Omega = (a, b)$,

$$\rho_1(x) = \begin{cases} x - a & \text{if } a \in R, \\ 0 & \text{if } a = -\infty, \end{cases} \qquad \rho_2(x) = \begin{cases} b - x & \text{if } b \in R, \\ 0 & \text{if } b = +\infty. \end{cases}$$

(2) Suppose Ω' is a standard open Λ^m-regular cell in R^n and $f, g\colon \Omega' \to R_{\pm\infty}$ are definable Λ^m-regular functions with

$$\Omega = \{(x, x_{n+1}) \in \Omega' \times R : f(x) < x_{n+1} < g(x)\}.$$

Let σ_j ($j = 1, \ldots, 2n$) be the functions associated with Ω'. Let $(x, x_{n+1}) \in \mathrm{cl}(\Omega)$. Set $\rho_j(x, x_{n+1}) = \sigma_j(x)$ for $j = 1, \ldots, 2n$ and

$$\rho_{2n+1}(x, x_{n+1}) = \begin{cases} x_{n+1} - f(x) & \text{if } f(\Omega') \subseteq R, \\ 0 & \text{if } f \equiv -\infty, \end{cases}$$

$$\rho_{2n+2}(x, x_{n+1}) = \begin{cases} g(x) - x_{n+1} & \text{if } g(\Omega') \subseteq R, \\ 0 & \text{if } g \equiv +\infty. \end{cases}$$

The proofs of the following facts from [7] (Lemmas 3 and 4) go through in our setting:

LEMMA 1.17. *Let Ω be a standard open Λ^m-regular cell in R^n. As above, let $\rho_1, \ldots, \rho_{2n}$ be the functions associated with Ω.*

(1) *There is a constant $C > 0$ such that*

$$\min_j \rho_j(x) \le d(x, \partial\Omega) \le C \min_j \rho_j(x) \quad \textit{for every } x \in \Omega.$$

(2) *The ρ_j are Λ^m-regular.*

Pawłucki's proof of Whitney's Extension Theorem in [11] heavily relies on integration of definable functions with respect to parameters, which generally takes us outside our given o-minimal structure $\boldsymbol{R}$, so we cannot immediately follow his proof in our context. In order to overcome this problem, we need to find other definable tools which work in o-minimal expansions of real closed ordered fields.

LEMMA 1.18 (Kurdyka & Pawłucki [8, Lemma 5]). *Let Ω be a definable open subset of R^d and $\rho\colon \Omega \to R$ be a definable Λ^m-regular function which does not vanish on Ω. Then, for $|\alpha| \le m$,*

$$D^\alpha(1/\rho)(x) = O\big((\min\{\rho(x), d(x, \partial\Omega)\})^{-|\alpha|-1}\big) \quad \textit{as } d(x, \partial\Omega) \to 0 \textit{ and } x \in \Omega.$$

Corollary 1.19. *Let $\Omega \subseteq R^d$ be an open Λ^m-regular cell, and let A be an orthogonal isomorphism of R^d such that $A(\Omega)$ is a standard open Λ^m-regular cell. Let $\rho_1, \dots, \rho_{2d}\colon A(\Omega) \to R$ be the functions associated with $A(\Omega)$. Then, for $|\alpha| \le m$ and $j = 1, \dots, 2d$,*

$$D^\alpha(1/\rho_j)(x) = O\big(d(x, \partial A(\Omega))^{-|\alpha|-1}\big) \quad \text{as } d(x, \partial A(\Omega)) \to 0 \text{ and } x \in A(\Omega).$$

Thus if we let $\nu_j = \rho_j \circ A$, then

$$D^\alpha(1/\nu_j)(x) = O\big(d(x, \partial\Omega)^{-|\alpha|-1}\big) \quad \text{as } d(x, \partial\Omega) \to 0 \text{ and } x \in \Omega.$$

Proof. Since each ρ_j is Λ^m-regular and $d(x, \partial\Omega) \le C\rho_j(x)$ for some $C > 0$, by the above lemma we are done. ∎

Lemma 1.20. *Let Ω be an open subset of R^d, let $f\colon \Omega \times R^l \to R$ and $\rho\colon \Omega \to R$ be definable C^m functions, and let $t\colon \Omega \to R^{>0}$ be definable. Suppose there is $C > 0$ such that*

$$t(x) \le d(x, \partial\Omega) \le C\rho(x) \quad \text{for every } x \in \Omega.$$

Let $\epsilon > 0$. Assume, for every $x_0 \in \partial\Omega$ and $\alpha \in \mathbb{N}^d$ with $|\alpha| \le m$,

$$D^\alpha(1/\rho) = O(t(x)^{-|\alpha|-1}) \quad \text{as } x \to x_0,$$

and for $x_0 \in \partial\Omega$ and $\kappa \in \mathbb{N}^{d+l}$, $|\kappa| \le m$,

$$D^\kappa f(x, y) = o(t(x)^{m-|\kappa|}) \quad \text{as } \Delta_\epsilon(\Omega \times \{0\}^l) \ni (x, y) \to (x_0, 0).$$

Fix $i \in \{1, \dots, l\}$. For every definable C^n-function $\xi\colon R \to R$, where $n \le m$, set

$$g_\xi(x, y) := \xi\left(\frac{y_i}{\rho(x)}\right) f(x, y) \quad \text{for } (x, y) \in \Omega \times R^l.$$

Then for every such ξ and n we have, for $|\kappa| \le n$ and $x_0 \in \partial\Omega$,

$$D^\kappa g_\xi(x, y) = o(t(x)^{n-|\kappa|}) \quad \text{as } \Delta_\epsilon(\Omega \times \{0\}^l) \ni (x, y) \to (x_0, 0).$$

Proof. Write $h_0(x, y) = y_i/\rho(x)$ and $h_\xi = \xi \circ h_0$. By the Leibniz formula, it is enough to check that

$$D^\lambda h_\xi(x, y) = O(t(x)^{-|\lambda|}) \quad \text{as } \Delta_\epsilon(\Omega \times \{0\}^l) \ni (x, y) \to (x_0, 0).$$

We proceed by induction on $|\lambda|$. Suppose $|\lambda| = 0$. For $(x, y) \in \Delta_\epsilon(\Omega \times \{0\}^l)$,

$$|y_i| \le d((x, y), \Omega \times \{0\}^l) < \epsilon d(x, \partial\Omega) \le \epsilon C\rho(x);$$

so $|h_0(x, y)| \le \epsilon C$. Thus $\xi([-\epsilon C, \epsilon C])$ contains $h_\xi(\Delta_\epsilon(\Omega \times \{0\}^l))$. Since ξ is continuous, the former set is bounded, and hence so is the latter. Therefore $h_\xi(x, y) = O(1)$ as $\Delta_\epsilon(\Omega \times \{0\}^l) \ni (x, y) \to (x_0, 0)$.

Assume the claim holds true for some value of $|\lambda| \le n - 1$, where $n \ge 1$.

By induction hypothesis,

$$\begin{aligned} D^{\lambda+e_j}h_\xi(x,y) &= \left[D^\lambda\left(\frac{\partial h_\xi}{\partial x_j}\right)\right](x,y) \\ &= \sum_{\mu\le\lambda}\binom{\lambda}{\mu}[D^\mu(\xi'\circ h_0)](x,y)\left[D^{\lambda-\mu}\left(\frac{\partial h_0}{\partial x_j}\right)\right](x,y) \\ &= \sum_{\mu\le\lambda}\binom{\lambda}{\mu}[D^\mu h_{\xi'}](x,y)\left[D^{\lambda-\mu}\left(\frac{\partial h_0}{\partial x_j}\right)\right](x,y) \\ &= \sum_{\mu\le\lambda} O(t(x)^{-|\mu|})O(t(x)^{-|\lambda|+|\mu|}), \end{aligned}$$

and so $D^{\lambda+e_j}h_\xi(x,y) = O(t(x)^{-|\lambda|})$ as $\Delta_\epsilon(\Omega\times\{0\}^l)\ni(x,y)\to(x_0,0)$. ∎

In the rest of this section, we let $0<\epsilon<1/\sqrt{2}$ and $m\le q$, and we let Ω be a standard open Λ^q-regular cell in R^d, with associated functions $\rho_1,\dots,\rho_{2d}$. We also let $F\in\mathscr{E}^m_{\mathrm{def}}(\mathrm{cl}(\Omega)\times\{0\}^l,\partial\Omega\times\{0\}^l)$.

Definition 1.21. Let $\xi\colon R\to R$ be a semialgebraic C^q-function which is 1 in a neighborhood of 0, and 0 outside $(-1,1)$. Define $r_\epsilon\colon R^{d+l}\to R$ by

$$r_\epsilon(x,y) = \prod_{i=1}^{l}\prod_{j=1}^{2d}\xi\left(Q_\epsilon\frac{y_i}{\rho_j(x)}\right)$$

where Q_ϵ is a constant (depending on Ω, ϵ, d, and l) large enough so that r_ϵ is m-flat outside $\Delta_\epsilon(\Omega\times\{0\}^l)$.

Lemma 1.22. *Let $h\colon\Omega\times R^l\to R$ be definable and C^q. Suppose that, for $\kappa\in\mathbb{N}^{d+l}$ with $|\kappa|\le m$ and $x_0\in\partial\Omega$,*

$$D^\kappa h(x,0) = F^\kappa(x,0)\quad \textit{for all } x\in\Omega$$

and

$$D^\kappa h(x,y) = o(d(x,\partial\Omega)^{m-|\kappa|})\qquad \textit{as } \Delta_\epsilon(\Omega\times\{0\}^l)\ni(x,y)\to(x_0,0).$$

Define $f_\epsilon\colon R^{d+l}\to R$ by

$$f_\epsilon(x,y) = \begin{cases} r_\epsilon(x,y)h(x,y) & \text{if } x\in\Omega,\\ 0 & \text{otherwise.}\end{cases}$$

Then f_ϵ is a definable C^m-extension of F which is m-flat outside $\Delta_\epsilon(\Omega\times\{0\}^l)$ and C^q outside $\mathrm{cl}(\Omega)\times\{0\}^l$.

Proof. Obviously, $f_\epsilon\restriction(\Omega\times R^l)$ is m-flat outside $\Delta_\epsilon(\Omega\times\{0\}^l)$ and f_ϵ is C^q outside $\partial\Omega\times\{0\}^l$. First, we will show that f_ϵ extends F. Let $x\in\Omega$. Then

$$f_\epsilon(x,0) = r_\epsilon(x,0)h(x,0) = F^0(x,0).$$

By the Leibniz formula,

$$D^\kappa f_\epsilon(x,y) = D^\kappa(r_\epsilon(x,y)h(x,y)) = \sum_{\sigma\le\kappa}\binom{\kappa}{\sigma}(D^{\kappa-\sigma}r_\epsilon(x,y))(D^\sigma h(x,y)).$$

Since $D^\gamma r_\epsilon(x,0)=0$ if $|\gamma|>0$ and $r_\epsilon(x,0)=1$, we obtain

$$D^\kappa f_\epsilon(x,0) = D^\kappa h(x,0) = F^\kappa(x,0).$$

It remains to show that f_ϵ is actually C^m on R^{d+l}. Let $y\neq 0\in R^l$. It is enough to find $\delta>0$ such that $(x,y)\notin \Delta_\epsilon(\Omega\times\{0\}^l)$ for all $x\in\Omega$ with $d(x,\partial\Omega)<\delta$. Since

$$(x,y)\notin \Delta_\epsilon(\Omega\times\{0\}^l) \iff |y| \ge \frac{\epsilon}{\sqrt{1-\epsilon^2}}d(x,\partial\Omega),$$

it suffices to pick $\delta=|y|/2$. Therefore, f_ϵ is C^m on $R^{d+l}\setminus(\partial\Omega\times\{0\}^l)$. Let $x_0\in\partial\Omega$. By Corollary 1.19 and Lemma 1.20, $D^\kappa f_\epsilon(x,y) = o(d(x,\partial\Omega)^{m-|\kappa|})$ as $\Delta_\epsilon(\Omega\times\{0\}^l)\ni(x,y)\to(x_0,0)$. Since f_ϵ is m-flat outside $\Delta_\epsilon(\Omega\times\{0\}^l)$, f_ϵ is C^m at $(x_0,0)$. ∎

Corollary 1.23. *For $\beta\in\mathbb{N}^l$ with $|\beta|\le m$, suppose*

$$h^\beta\colon \Omega\times R^l\to R,\qquad h^\beta(x,y) = F^{(0,\beta)}(x,0)y^\beta,$$

is C^q and, for $\kappa\in\mathbb{N}^{d+l}$ with $|\kappa|\le m$ and $x_0\in\partial\Omega$,

$$D^\kappa h^\beta(x,y) = o(d(x,\partial\Omega)^{m-|\kappa|})\qquad \text{as } \Delta_\epsilon(\Omega\times\{0\}^l)\ni(x,y)\to(x_0,0).$$

Define $f_\epsilon\colon R^{d+l}\to R$ by

$$f_\epsilon(x,y) = \begin{cases} r_\epsilon(x,y)\displaystyle\sum_{|\beta|\le m}\frac{h^\beta(x,y)}{\beta!} & \text{if } x\in\Omega,\\ 0 & \text{otherwise.}\end{cases}$$

Then f_ϵ is a definable C^m-extension of F which is m-flat outside $\Delta_\epsilon(\Omega\times\{0\}^l)$ and C^q outside $\mathrm{cl}(\Omega)\times\{0\}^l$.

Proof. Clearly,

$$D^\kappa\Big(\sum_{|\beta|\le m}\frac{h^\beta(x,0)}{\beta!}\Big) = F^\kappa(x,0).$$

By Lemma 1.22, we are done. ∎

2. The first four steps. In this section, we assume $m\le q$. Pawłucki's construction of an extension operator for C^m-Whitney fields from [11] can be divided into five steps, depending on the nature of the Whitney field F and its domain E:

Step 1: $E = R^d\times\{0\}^{n-d}$;
Step 2: $E = \mathrm{cl}(\Omega)\times\{0\}^{n-d}$ where Ω is an open Λ^q-regular cell and F is flat on $\partial\Omega\times\{0\}^{n-d}$;

STEP 3: $E = \mathrm{cl}(E_0)$ where E_0 is the graph of a Lipschitz Λ^q-regular map on an open Λ^q-regular cell and F is flat on ∂E_0;
STEP 4: $E = \mathrm{cl}(E_0)$ where E_0 is a Λ^q-regular pancake and F is flat on ∂E_0;
STEP 5: E is any closed definable set.

In this section, we work on the first four steps under the following assumption:

(∗) *For every closed definable set $E \subseteq R^n$ with $\dim(E) < d$, every F in $\mathscr{E}^m_{\mathrm{def}}(E)$ has a definable C^m-extension which is C^q on $R^n \setminus E$.*

Thus, in the rest of this section we assume that condition (∗) holds.

2.1. Step 1

LEMMA 2.1. *Let $F \in \mathscr{E}^m_{\mathrm{def}}(R^d \times \{0\}^{n-d})$. Then F has a definable C^m-extension which is C^q outside $R^d \times \{0\}^{n-d}$.*

Proof. For $\beta \in \mathbb{N}^{n-d}$, define $F_\beta := (\widetilde{F}^{(\sigma,\delta)})_{|(\sigma,\delta)|\le m}$ where

$$\widetilde{F}^{(\sigma,\delta)} := \begin{cases} F^{(\sigma,\beta)} & \text{if } \beta = \delta, \\ 0 & \text{otherwise.} \end{cases}$$

By the definition of C^m-Whitney fields, we can easily see that $F_\beta \in \mathscr{E}^m_{\mathrm{def}}(R^d \times \{0\}^{n-d})$ for every $|\beta| \le m$. Obviously, $F = \sum_{|\beta|\le m} F_\beta$. Hence, we may assume that $F = F_\beta$. By Smooth Cell Decomposition, there is a cell decomposition $\mathscr{C}$ of R^d such that, for each $C \in \mathscr{C}$ and $|(\alpha,\beta)| \le m$, the function $F^{(\alpha,\beta)}\restriction(C \times \{0\}^{n-d})$ is C^q. By (∗), we may assume the F is flat on $\bigcup_{C\in\mathscr{C}\setminus\mathscr{C}^o} C \times \{0\}^{n-d}$. Note that for each C_1 and C_2 in C^o, $C_1 \times \{0\}^{n-d}$ and $C_2 \times \{0\}^{n-d}$ are $(\partial C_i \times \{0\}^{n-d})$-separated for $i = 1, 2$.

Let $C \in \mathscr{C}^o$. By Proposition 1.16, it is sufficient to find a definable C^m-extension f_C of $F\restriction(\mathrm{cl}(C) \times \{0\}^{n-d})$ which is m-flat outside $\Delta_\epsilon(C \times \{0\}^{n-d})$, for some $\epsilon > 0$ small enough, and C^q outside $\mathrm{cl}(C) \times \{0\}^{n-d}$. Therefore, we may assume that F is flat on $(R^d \setminus C) \times \{0\}^{n-d}$ and $F^{(\alpha,\beta)}$ is C^q for every $|(\alpha,\beta)| \le m$. By Lemma 1.6, we may write $\mathrm{cl}(C) = D_1 \cup \cdots \cup D_s \cup B$ where the D_i's are open Λ^q-regular cells and $B = \partial D_1 \cup \cdots \cup \partial D_s$, such that, defining, for $|\alpha| \le m$,

$$g^\alpha \colon R^d \to R, \qquad g^\alpha(x) = F^\alpha(x, 0),$$

there is $L > 0$ such that for $\kappa \in \mathbb{N}^d$ with $|\kappa| \le q$ and $u \in D_i$, each $g^\alpha\restriction D_i$ is C^q and

$$|D^\kappa g^\alpha(u)| \le \frac{L}{d(u,\partial D_i)^{|\kappa|}} \sup\{|g^\alpha(v)| : v \in D_i,\ \|u-v\| < d(u,\partial D_i)\} \quad \text{for } u \in D_i. \tag{2.1}$$

By (∗), let $f_0 \colon R^n \to R$ be a definable C^m-extension of $F\restriction(B \times \{0\}^{n-d})$

which is C^q outside $B \times \{0\}^{n-d}$, and set

$$\widetilde{F} := F - J^m(f_0)\lceil(R^d \times \{0\}^{n-d}) \in \mathscr{E}^m_{\mathrm{def}}(R^d \times \{0\}^{n-d}).$$

Clearly,

$$F_i := \widetilde{F}\lceil(\mathrm{cl}(D_i) \times \{0\}^{n-d}) \in \mathscr{E}^m_{\mathrm{def}}(\mathrm{cl}(D_i) \times \{0\}^{n-d}, \partial D_i \times \{0\}^{n-d}).$$

By Proposition 1.16, it is sufficient to find a definable C^m-extension f_i for each F_i which is m-flat outside $\Delta_\epsilon(D_i \times \{0\}^{n-d})$, for some $\epsilon > 0$ small enough, and C^q outside $\mathrm{cl}(D_i) \times \{0\}^{n-d}$. Fix some $i \in \{1, \dots, s\}$, and let

$$h_i(x, y) := \frac{1}{\beta!} F^{(0,\beta)}(x, 0)y^\beta - f_0(x, y).$$

Obviously, $D^\kappa h_i(x, 0) = \widetilde{F}^\kappa(x, 0)$ for all $x \in D_i$ and $|\kappa| \le m$. Therefore, by Lemma 1.22, it is enough to show the following claim:

Claim. *For $\kappa = (\sigma, \tau) \in \mathbb{N}^d \times \mathbb{N}^{n-d}$ with $|\kappa| \le m$, and $x_0 \in \partial D_i$,*

$$D^\kappa h_i(x, y) = o(d(x, \partial D_i)^{m-|\kappa|}) \qquad \textit{as } \Delta_\epsilon(D_i \times \{0\}^{n-d}) \ni (x, y) \to (x_0, 0).$$

If $x_0 \in C$, by Taylor's formula we are done. Assume $x_0 \in \partial C$. We use induction on $m - |\kappa|$. First assume $|\kappa| = m$. Clearly,

$$|D^\kappa h_i(x, y)| \le \left| D^\kappa\left(\frac{1}{\beta!} F^{(0,\beta)}(x, 0)y^\beta\right)\right| + |D^\kappa f_0(x, y)|.$$

Since f_0 is m-flat at $(x_0, 0)$, we have $D^\kappa f_0(x, y) \to 0$ as $(x, y) \to (x_0, 0)$. Suppose $\tau \le \beta$ (otherwise, $D^\kappa(\frac{1}{\beta!} f_0^{(0,\beta)}(x, 0)y^\beta) = 0$). Then

$$D^\kappa\left(\frac{1}{\beta!} f_0^{(0,\beta)}(x, 0)y^\beta\right) = \frac{1}{(\beta - \tau)!} D^\gamma(f_0^{(\alpha,\beta)}(x, 0)y^{\beta-\tau})$$

where $\sigma = \alpha + \gamma$ and $|\alpha| + |\beta| = m$. We have

$$|\beta| - |\tau| - |\gamma| = |\beta| - |\tau| - |\sigma| + |\alpha| = m - |\tau| - |\sigma| = m - |\kappa| = 0.$$

Since $F^{(\alpha,\beta)}(x_0, 0) = 0$,

$$s(z) := \sup\{|F^{(\alpha,\beta)}(x, 0)| : x \in D_i,\ |x - z| < d(z, \partial D_i)\} \to 0 \quad \text{as } D_i \ni z \to x_0.$$

By (2.1),

$$\begin{aligned} \left| D^\kappa\left(\frac{1}{\beta!} f_0^{(0,\beta)}(x, 0)y^\beta\right)\right| &\le \frac{L}{d(x, \partial D_i)^{|\gamma|}} s(z)\left(\frac{\epsilon}{\sqrt{1-\epsilon^2}} d(x, \partial D_i)\right)^{|\beta|-|\tau|} \\ &= L\left(\frac{\epsilon}{\sqrt{1-\epsilon^2}}\right)^{|\beta|-|\tau|} s(z) \\ &\to 0 \quad \text{as } \Delta_\epsilon(D_i \times \{0\}^{n-d}) \ni (x, y) \to (x_0, 0). \end{aligned}$$

Next, assume that $|\kappa| < m$ and for every $|\lambda| > |\kappa|$,

$$D^\lambda h_i(x,y) = o(d(x,\partial D_i)^{m-|\lambda|}) \quad \text{as } \Delta_\epsilon(D_i \times \{0\}^{n-d}) \ni (x,y) \to (x_0,0).$$

Let $(x,y) \in \Delta_\epsilon(D_i \times \{0\}^{n-d})$. Let $z \in \partial D_i$ with $|x-z| = d(x,\partial D_i)$ and S be the line segment connecting (x,y) and $(z,0)$. By Proposition 1.13, we see that $S \subseteq \Delta_\epsilon(D_i \times \{0\}^{n-d})$ and $d((x,y),(z,0)) \le \big(1+\frac{\epsilon}{\sqrt{1-\epsilon^2}}\big)d(x,\partial D_i)$. Let $C := \sup\{|D^{\kappa+\lambda}h_i(u,w)| : |\lambda| = 1,\ (u,w) \in S\}$ and

$$t(x) := \sup\{|D^{\kappa+\lambda}h_i(u,w)| : |\lambda| = 1,\ (u,w) \in \Delta_\epsilon(D_i \times \{0\}^{n-d}),\ d(u,\partial D_i) < 2d(x,\partial D_i)\}.$$

Observe that $C \le t(x)$. By the Mean Value Theorem, we have

$$\begin{aligned} |D^\kappa h(x,y)| &\le \sqrt{n}\, C\sqrt{|x-z|^2+|y|^2} \\ &\le \sqrt{n}\, t(x)\left(1+\frac{\epsilon}{\sqrt{1-\epsilon^2}}\right)d(x,\partial D_i) \end{aligned}$$

Inductively, we have $t(x) = o(d(x,\partial D_i)^{m-|\kappa|-1})$ as $\Delta_\epsilon(D_i \times \{0\}^{n-d}) \ni (x,y) \to (x_0,0)$. Therefore,

$$\begin{aligned} D^\kappa h_i(x,y) &= o(d(x,\partial D_i)^{m-|\kappa|-1})d(x,\partial D_i) \\ &= o(d(x,\partial D_i)^{m-|\kappa|}) \quad \text{as } \Delta_\epsilon(D_i \times \{0\}^{n-d}) \ni (x,y) \to (x_0,0). \ \blacksquare \end{aligned}$$

2.2. Step 2

LEMMA 2.2. *Let Ω be an open Λ^q-regular cell in R^d, and $F \in \mathscr{E}^m_{\mathrm{def}}(\mathrm{cl}(\Omega) \times \{0\}^{n-d}, \partial\Omega \times \{0\}^{n-d})$. Then, for every $\epsilon > 0$, F has a definable C^m-extension which is m-flat outside $\Delta_\epsilon(\Omega \times \{0\}^{n-d})$ and C^q outside $\mathrm{cl}(\Omega) \times \{0\}^{n-d}$.*

Proof. First, we extend F to $\widetilde{F} \in \mathscr{E}^m_{\mathrm{def}}(R^d \times \{0\}^{n-d})$ as follows:

$$\widetilde{F}^\alpha(x,0) = \begin{cases} F^\alpha(x,0) & \text{if } x \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

By the above lemma, we can find a definable C^m-extension $\widetilde{f}$ of $\widetilde{F}$. However, $\widetilde{f}$ is possibly not m-flat outside $\Delta_\epsilon(\Omega \times \{0\}^{n-d})$. In order to guarantee this, we have to slightly modify $\widetilde{f}$. Define

$$f_\epsilon(x,y) = \begin{cases} r_\epsilon(x,y)\widetilde{f}(x,y) & \text{if } x \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

Here, r_ϵ is as introduced in Definition 1.21. Clearly, f_ϵ is m-flat outside $\Delta_\epsilon(\Omega \times \{0\}^{n-d})$. Moreover, since $\widetilde{f}$ is C^q outside $R^d \times \{0\}^{n-d}$ and r_ϵ is C^q on $\Omega \times R^{n-d}$, f_ϵ is C^q outside $\mathrm{cl}(\Omega) \times \{0\}^{n-d}$. Since $\widetilde{f}$ is C^m on R^{d+l}, by Corollaries 1.19 and 1.20, f_ϵ is C^m on R^{d+l}. ■

2.3. Step 3. Let $\varphi\colon \Omega \to R^{n-d}$ be a definable Lipschitz Λ^q-regular map and Ω be an open Λ^q-regular cell in R^d. Let $\overline{\varphi}\colon \operatorname{cl}(\Omega) \to R^{n-d}$ be the continuous extension of φ, and

$$\begin{aligned}\varphi_+\colon \operatorname{cl}(\Omega)\times R^{n-d} \to R^n, &\quad \varphi_+(x,y) := (x, y+\overline{\varphi}(x)),\\ \varphi_-\colon \operatorname{cl}(\Omega)\times R^{n-d} \to R^n, &\quad \varphi_-(x,y) := (x, y-\overline{\varphi}(x)).\end{aligned}$$

To apply Step 2 to $E = \operatorname{cl}(\Gamma(\varphi))$, we first show that for each C^m-Whitney field on E, there is a corresponding C^m-Whitney field on $\operatorname{cl}(\Omega)\times\{0\}^{n-d}$.

Let $E_0 := \Gamma(\varphi)$, $E := \operatorname{cl}(E_0) = \Gamma(\overline{\varphi})$, and $F \in \mathscr{E}^m_{\mathrm{def}}(E, \partial E_0)$. Obviously,

$$\varphi_+(\operatorname{cl}(\Omega)\times\{0\}^{n-d}) = E, \quad \varphi_+(\partial\Omega\times\{0\}^{n-d}) = \partial E_0.$$

By Corollary 1.12,

$$\varphi_+^* F \in \mathscr{E}^m_{\mathrm{def}}(\operatorname{cl}(\Omega)\times\{0\}^{n-d}, \partial\Omega\times\{0\}^{n-d}).$$

Now we show:

Lemma 2.3. *Let $E_0 := \Gamma(\varphi)$, $E := \operatorname{cl}(E_0) = \Gamma(\overline{\varphi})$, and $F \in \mathscr{E}^m_{\mathrm{def}}(E, \partial E_0)$. Then, for every $\epsilon > 0$, F has a definable C^m-extension which is m-flat outside $\varphi_+(\Delta_\epsilon(\Omega\times\{0\}^{n-d}))$ and C^q outside E.*

Proof. By Proposition 1.14, there is $\epsilon_0 > 0$ such that $\Delta_\delta(E) \subseteq \Omega\times R^{n-d}$ for all $0 < \delta < \epsilon_0$. Let $\epsilon > 0$. We may assume $\epsilon < \epsilon_0$. By Lemma 2.2, take a definable C^m-extension $f_{-\varphi}$ of $\varphi_+^* F$ which is m-flat outside $\Delta_{\epsilon/2}(\Omega\times\{0\}^{n-d})$ and C^q outside $\operatorname{cl}(\Omega)\times\{0\}^{n-d}$. Define $f\colon R^n \to R$ by

$$f(x,y) := \begin{cases} f_{-\varphi}(\varphi_-(x,y)) & \text{if } x \in \Omega,\\ 0 & \text{otherwise.}\end{cases}$$

Since $J^m(f)\restriction E = \varphi_-^*(\varphi_+^* F) = (\varphi_+ \circ \varphi_-)^* F$ and $\varphi_+\circ\varphi_- = \mathrm{id}_{\operatorname{cl}(\Omega)\times R^{n-d}}$ we have $J^m(f)\restriction E = F$. Therefore, f is a C^m-extension of F which is m-flat outside $\varphi_+(\Delta_{\epsilon/2}(\Omega\times\{0\}^{n-d}))$ and C^q outside E. ∎

2.4. Step 4

Lemma 2.4. *Let E_0 be a Λ^q-pancake of dimension d with common domain $\Omega \subseteq R^d$, let $E = \operatorname{cl}(E_0)$, and $F \in \mathscr{E}^m_{\mathrm{def}}(E, \partial E_0)$. Then, for every $\epsilon > 0$, F has a definable C^m-extension which is m-flat outside $\Delta_\epsilon(E_0)$ and C^q outside E.*

Proof. Suppose $E = \operatorname{cl}(E_1 \cup \cdots \cup E_s)$ where $E_i = \Gamma(\varphi_i)$ with $\varphi_i\colon \Omega \to R^{n-d}$ a definable Λ^q-regular Lipschitz map. For each $i \in \{1,\ldots,s\}$, let $\overline{\varphi_i}\colon \operatorname{cl}(\Omega) \to R^{n-d}$ be the continuous extension of φ, and

$$\begin{aligned}\varphi_{i+}\colon \operatorname{cl}(\Omega)\times R^{n-d} \to R^n, &\quad \varphi_{i+}(x,y) := (x, y+\overline{\varphi_i}(x)),\\ \varphi_{i-}\colon \operatorname{cl}(\Omega)\times R^{n-d} \to R^n, &\quad \varphi_{i-}(x,y) := (x, y-\overline{\varphi_i}(x)).\end{aligned}$$

By Lemma 1.15, it is enough to prove that, *for* $0 < \epsilon < 1/\sqrt{2}$, *there exists a definable* C^m*-extension of* F *which is* m*-flat outside* $\bigcup_{i=1}^{s} \varphi_{i+}(\Delta_\epsilon(\Omega \times \{0\}^{n-d}))$ *and* C^q *outside* $\bigcup_{i=1}^{s} \mathrm{cl}(E_i)$. We proceed by induction on s. The case $s = 1$ follows immediately from Lemmas 1.15 and 2.3. Suppose $s > 1$, and the statement is true for $s - 1$ in place of s. Let $0 < \epsilon < 1/\sqrt{2}$. Then we can find a definable C^m-extension $\widetilde{f_\epsilon}$ of $F\lceil\bigcup_{i=1}^{s-1} \mathrm{cl}(E_i)$ which is m-flat outside $\bigcup_{i=1}^{s-1} \varphi_{i+}(\Delta_\epsilon(\Omega \times \{0\}^{n-d}))$ and C^q outside $\bigcup_{i=1}^{s-1} \mathrm{cl}(E_i)$. Note that $\bigcup_{i=1}^{s-1} \varphi_{i+}(\Delta_\epsilon(\Omega \times \{0\}^{n-d}))$ and $\partial\Omega \times R^{n-d}$ are disjoint. After replacing F by $F - J^m(\widetilde{f_\epsilon})\lceil E$, we may assume that

$$F \in \mathscr{E}^m_{\mathrm{def}}\Big(\bigcup_{i=1}^{s} \mathrm{cl}(E_i), \bigcup_{i=1}^{s-1} \mathrm{cl}(E_i) \cup \partial E_s\Big).$$

Next, consider $\varphi^*_{s+}(F\lceil\mathrm{cl}(E_s)) \in \mathscr{E}^m_{\mathrm{def}}(\mathrm{cl}(\Omega) \times \{0\}^{n-d}, \partial\Omega \times \{0\}^{n-d})$ (by Corollary 1.12.) By Lemma 2.2, let f be a C^m-extension of $\varphi^*_{s+}(F\lceil\mathrm{cl}(E_s))$ which is m-flat outside $\Delta_\epsilon(\Omega \times \{0\}^{n-d})$ and C^q outside $\mathrm{cl}(\Omega) \times \{0\}^{n-d}$. For $i = 1, \ldots, s-1$ and $x \in \Omega$, we define $r_i(x) := \|\varphi_i(x) - \varphi_s(x)\|$. Each function $r_i\colon \Omega \to R^{>0}$ is Λ^m-regular. Let $\xi\colon R \to R$ be any semialgebraic C^q-function which is 1 in a neighborhood of 0 and 0 outside $(-1, 1)$. Then, define

$$g(x, y) = \begin{cases} \displaystyle\prod_{i=1}^{s-1} \prod_{j=1}^{n-d} \xi\Big(\sqrt{l}\frac{y_j}{r_i(x)}\Big) f(x, y) & \text{if } x \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

Since f is C^m, by Lemma 1.18 and 1.20, g is a C^m-extension of $\varphi^*_{s+}(F\lceil\mathrm{cl}(E_s))$ which is m-flat outside $\Delta_\epsilon(\Omega \times \{0\}^{n-d})$. Moreover, by the choice of r_i and ξ, we also see that g is m-flat on $\varphi_{s-}(E_i)$ for all $i = 1, \ldots, s - 1$. Define $f_\epsilon\colon R^n \to R$ by

$$f_\epsilon(x, y) := \begin{cases} g(\varphi_{s-}(x)) & \text{if } x \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, $\mathrm{cl}(E_i) = \varphi_{s+}(\varphi_{s-}(\mathrm{cl}(E_i)))$ for all $i \in \{1, \ldots, s\}$. Thus, f_ϵ is a C^m-extension of $F\lceil\mathrm{cl}(E_s)$ which is m-flat on $\mathrm{cl}(E_i)$ and outside $\varphi_{s+}(\Delta_\epsilon(\Omega \times \{0\}^{n-d}))$. Therefore, f_ϵ is a C^m-extension of F which is m-flat outside $\bigcup_{i=1}^{s} \varphi_{i+}(\Delta_\epsilon(\Omega \times \{0\}^{n-d}))$. In addition, f_ϵ is C^q outside $\bigcup_{i=1}^{s} \mathrm{cl}(E_i)$. ∎

3. Proof of Theorem A. Suppose $m \le q$. We prove by induction on d that *every* $F \in \mathscr{E}^m_{\mathrm{def}}(E)$, *where* E *is a definable closed subset of* R^n *of dimension* d, *has a definable* C^m*-extension which is* C^q *on* $R^n \setminus E$. When $d = 0$, E is just a finite subset of R^n, and this case is easy. Suppose $d > 0$, and the statement is true for all smaller values of d; that is, condition $(*)$ from the previous section holds. Let E be a definable closed subset of R^n

of dimension d and $F \in \mathscr{E}^m_{\mathrm{def}}(E)$. By the Λ^m-regular Separation Theorem, decompose $E = M_1 \cup \cdots \cup M_s \cup A$ where

(1) each M_i is a Λ^q-pancake of dimension d in a suitable coordinate system;
(2) A is a small, closed, definable subset of E;
(3) for all $i \neq j$, $\mathrm{cl}(M_i), \mathrm{cl}(M_j)$ are ∂M_i-separated; and
(4) for each i, $\mathrm{cl}(M_i), A$ are ∂M_i-separated.

By $(*)$, take a definable C^m-extension f_A of $F \restriction A$. By replacing F by $F - J^m(f_A)\restriction E$, we may assume that F is flat on $\bigcup_{i=1}^s \partial M_i$. Now, by separability, Proposition 1.16, and Lemma 2.4, we obtain a C^m-extension of F which is C^q outside E. ∎

As usual in the o-minimal context, there is a certain uniformity inherent in the above constructions; this can be exhibited by redoing these constructions "uniformly in parameters," or perhaps more elegantly, by using the Compactness Theorem of first-order logic:

Theorem 3.1. *Assume $\boldsymbol{R}$ is o-minimal. Let $(F_a)_{a \in A}$, where $A \subseteq R^N$, be a definable family of definable C^m-Whitney fields F_a on a closed definable set $E_a \subseteq R^n$. Then there is a definable family $(f_a)_{a \in A}$ of definable C^m-functions $f_a \colon R^n \to R$ such that f_a is an extension of F_a for each $a \in A$.*

Proof. Let $\mathscr{L}$ be the language of $\boldsymbol{R}$, assumed to include a name for each element of R, so that every definable set in R is definable by an $\mathscr{L}$-formula. For each $\alpha \in \mathbb{N}^n$ with $|\alpha| \le m$, let $\phi^\alpha(x, y, z)$ be a formula in $\mathscr{L}$ where the lengths of x, y, and z are n, 1, and k, respectively, such that for each $a \in A$, $\phi^\alpha(x, y, a)$ defines the graph of $(F_a)^\alpha$. For each formula $\psi(x, y, z)$, let $\chi_\psi(z)$ be a formula such that, for each $a \in R^N$, $\chi_\psi(a)$ holds in $\boldsymbol{R}$ precisely when $\psi(x, y, a)$ defines the graph of a C^m-extension of F_a. Next, add N fresh constants $c_1, \dots, c_N$ to $\mathscr{L}$ and call the resulting language $\mathscr{L}'$. For notational convenience, we write $c = (c_1, \dots, c_N)$. By our main theorem, the $\mathscr{L}'$-theory

$$\mathrm{Th}(\boldsymbol{R}) \cup \{\neg\chi_\psi(c) : \psi = \psi(x, y, z) \text{ is an } \mathscr{L}\text{-formula}\}$$

is inconsistent. Therefore, by the Compactness Theorem, there are formulas

$$\psi_1(x, y, z), \dots, \psi_M(x, y, z)$$

such that, for each $a \in A$, one of $\psi_i(x, y, a)$ defines the graph of a C^m-extension of F_a in $\boldsymbol{R}$. We can now easily construct a single formula $\psi(x, y, z)$ which works for every $a \in A$, i.e., for each $a \in A$, $\psi(x, y, a)$ defines the graph of a C^m-extension of F_a. ∎

Acknowledgements. This paper is a continuation of the author's Ph.D. dissertation at University of California, Los Angeles. The author thanks

M. Aschenbrenner for his guidance when the author started studying this topic and writing this paper.

References

[1] L. van den Dries, *Tame Topology and O-minimal Structures*, London Math. Soc. Lecture Note Ser. 248, Cambridge Univ. Press, Cambridge, 1998.

[2] L. van den Dries and C. Miller, *Geometric categories and o-minimal structures*, Duke Math. J. 84 (1996), 497–540.

[3] A. Fischer, *O-minimal Λ^m-regular stratification*, Ann. Pure Appl. Logic 147 (2007), 101–112.

[4] M. Gromov, *Entropy, homology and semialgebraic geometry*, in: Séminaire Bourbaki, Vol. 1985/86, Astérisque 145-146 (1987), 225–240.

[5] M. R. Hestenes, *Extension of the range of a differentiable function*, Duke Math. J. 8, (1941), 183–192.

[6] K. Kurdyka and A. Parusiński, *Quasi-convex decomposition in o-minimal structures. Application to the gradient conjecture*, in: Singularity Theory and its Applications, Adv. Stud. Pure Math. 43, Math. Soc. Japan, Tokyo, 2006, 137–177.

[7] K. Kurdyka and W. Pawłucki, *Subanalytic version of Whitney's extension theorem*, Studia Math. 124 (1997), 269–280.

[8] K. Kurdyka and W. Pawłucki, *o-minimal version of Whitney's extension theorem*, Studia Math. 224 (2014), 81–96.

[9] B. Malgrange, *Ideals of Differentiable Functions*, Oxford Univ. Press, 1966.

[10] W. Pawłucki, *A decomposition of a set definable in an o-minimal structure into perfectly situated sets*, Ann. Polon. Math. 79 (2002), 171–184.

[11] W. Pawłucki, *A linear extension operator for Whitney fields on closed o-minimal sets*, Ann. Inst. Fourier (Grenoble) 58 (2008), 383–404.

[12] H. Whitney, *Analytic extension of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. 36 (1934), 63–89.

[13] H. Whitney, *Differentiable functions defined in closed sets. I*, Trans. Amer. Math. Soc. 36 (1934), 369–389.

[14] H. Whitney, *Functions differentiable on the boundaries of regions*, Ann. of Math. 35 (1934), 482–485.

Athipat Thamrongthanyalak
Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University
Bangkok 10400, Thailand
Current address:
Department of Mathematics
The Ohio State University
231 West 18th Avenue
Columbus, OH 43212, U.S.A.
E-mail: thamrongthanyalak.1@osu.edu