COLLOQUIUM MATHEMATICUM

VOL. 148 2017 NO. 2

A CLASSICAL APPROACH TO SMOOTH SUPERMANIFOLDS

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Abstract. A differential-geometric approach to supergeometry is considered, in the sense that our objects of study are superalgebra bundles over smooth manifolds. Our definition is not to be confused with Batchelor's Theorem, for which we provide a direct proof. Rather, our objects are abstract superalgebra bundles, a special case of which are the so-called split supermanifolds constructed from the exterior algebra functor applied to a given vector bundle.

The highlights of this work are the results proving equivalence between our approach and the usual "algebro-geometric" one using ringed spaces, and a supergeometric version of the Flowbox Theorem.

Introduction. The theory of supermanifolds was developed in order to supply theoretical physics with a mathematical basis for the concept of supersymmetry. Developed essentially in the 1970s, this theory is in the middle of the most important discussions surrounding one of the most feasible candidates for a unification theory.

In this work we develop the theory from first principles to give it a differential-geometric basis. We undertake the task of proving the equivalence between our theory and the usual approach to supergeometry via ringed spaces for the sake of completeness. This is done not only as a theoretical exercise of "translation" between the sheaf-theoretic language and the differential-geometric one; rather, this work is written with two basic purposes in mind: to serve as an introduction to the subject that deals with objects closer to geometric intuition, and to offer a concise summary of elementary differential supergeometry in a complete and essentially self-contained manner. We leave for a later work the development of supergeometry *per se*, that is, the theory of differential superforms, superconnections and other objects. In this work we are content with giving the first few steps towards a very practical approach to supergeometry.

2010 *Mathematics Subject Classification*: Primary 53-02; Secondary 53C05, 53C80.
Key words and phrases: supergeometry, superalgebra bundles, supermanifolds.
Received 23 September 2015; revised 6 July 2016.
Published online 23 March 2017.

DOI: 10.4064/cm6772-7-2016

We now give a summary of the work. Section 1 deals with the basics of our approach; we give a completely differential sense to smooth supermaps (Proposition 1.19) and establish a useful characterisation of them in geometric and algebraic terms (Theorem 1.22). Section 2 gives a complete characterisation of superderivations as sections of a vector bundle over the smooth manifold embedded in a supermanifold (Theorem 2.4). Section 3 is devoted to the proof of our main result, namely that the sheaf-theoretic approach and ours are completely equivalent. Section 4 deals with a geometrical construction analogous to the Flowbox Theorem for smooth manifolds (Theorem 4.2) and some consequences of it. We include two appendices. In Appendix A we gather all the facts on linear differential operators we need to make this work as self-contained as possible. Appendix B is devoted to the proof of an algebraic result (which we call "the Cartan–Poincaré Lemma") used in the proof of Theorem 4.2.

1. Smooth supermanifolds as superalgebra bundles. In this first section we establish the first steps in the theory of smooth supermanifolds. The results here are written in an invariant way, without unnecessary coordinates. In this section we completely characterise our category of study. First of all let us specify our objects:

DEFINITION 1.1. Let M be a smooth m-dimensional manifold. A *smooth supermanifold of superdimension* $(m|k)$ over M is a pair $(M|\mathcal{R}M)$ such that $\mathcal{R}M$ is a supercommutative-algebra bundle of rank k; that is, for every point $p \in M$ the fibre $\mathcal{R}_pM$ is isomorphic to the exterior algebra $\Lambda\mathbf{S}^*$ of some vector space $\mathbf{S}$ of dimension k. The sections $\eta \in \Gamma(\mathcal{R}M)$ are the *smooth superfunctions* of $(M|\mathcal{R}M)$.

REMARK 1.2. The reason for requiring the fibres to be isomorphic to $\Lambda\mathbf{S}^*$ is technical: one wants to think of a superfunction as a function of odd-coordinates, which locally are trivializations using a basis for $\mathbf{S}$; it is then natural to think of a smooth superfuncion as an anticommutative polynomial in these coordinates.

Now we specify the appropriate morphisms in our category:

DEFINITION 1.3. Let $(M|\mathcal{R}M)$ and $(N|\mathcal{R}N)$ be smooth supermanifolds. A *supersmooth map* is a pair $(\phi|\Phi)\colon (M|\mathcal{R}M) \to (N|\mathcal{R}N)$ such that

- $\phi\colon M \to N$ is a smooth map;
- $\Phi\colon \Gamma(\mathcal{R}N) \to \Gamma(\mathcal{R}M)$ is a unital homomorphism of superalgebras that is *local*, i.e. it is a sheaf morphism.

What the second condition means is the following: if $U \subset N$ is an open set then $\Phi\Gamma(U) = \Gamma(\phi^{-1}(U))$.

Let us now recall the usual definition of a supermanifold; it can be found in [L] and [K].

Definition 1.4. A *supermanifold in the sense of Kostant–Leites* (abbreviated to KL-supermanifold) of dimension $(m|n)$ is a pair $(M, \mathcal{O})$ where M is a smooth manifold of dimension m and $\mathcal{O}$ is a sheaf of unital supercommutative algebras such that there is an open cover $\{U_\alpha\}_\alpha$ of M with the following property: for each α the superalgebra $\mathcal{O}_\alpha := \mathcal{O}(U_\alpha)$ is isomorphic to $\mathcal{C}^\infty_{U_\alpha} \otimes \Lambda\mathbb{R}^n$; the integer n is the *odd dimension* of the supermanifold, whereas m is the *even dimension*.

We have written $\mathcal{C}^\infty_U$ for the sheaf of smooth functions on an open set $U \subseteq M$. Let us now state our main result:

Theorem 1.5. *Let $(M, \mathcal{O})$ be a KL-supermanifold of dimension $(m|n)$ and let $\mathbf{S}$ be a vector space of dimension n. There exists a bundle of superalgebras $\mathcal{R}M$ such that every fibre $\mathcal{R}_pM$ is isomorphic to $\Lambda\mathbf{S}^*$, and $\Gamma(\mathcal{R}M) \cong \mathcal{O}$ as sheaves.*

In order to prove Theorem 1.5 let us first analyse the problem. In her paper, Batchelor reduced the classification problem to a cohomological one. To wit, Theorems 2.3 and 2.5 of [B] are used to reduce the structure of the sheaf $\mathcal{O}$ to a sheaf of $\mathcal{C}^\infty_M$-modules. Those theorems are equivalent to the statement that the sequences

$$0 \to \mathcal{N}^k \hookrightarrow \mathcal{O} \to \mathcal{O}/\mathcal{N}^k \to 0$$

split for each $k \geq 0$, which for $k = 1$ reduces to the fact that there is a unital superalgebra sheaf surjection $\varepsilon\colon \mathcal{O} \to \mathcal{C}^\infty_M$ (the augmentation map). Then by defining

$$\mathcal{O}_k = \mathcal{O}/\mathcal{N}^k \quad \text{and} \quad \widetilde{\mathcal{O}} = \bigoplus_{k\geq 0} \mathcal{O}_k$$

the result follows.

Corollary 1.6 (Batchelor's Theorem, [B]). *The structure sheaf of any supermanifold $(M|\mathcal{R}M)$ can be realised as the sheaf of sections of a bundle of exterior algebras of finite rank.*

The missing ingredient is a corresponding result for morphisms of supermanifolds. We take care of this in Theorem 1.24. We postpone the proof of Theorem 1.5 to Section 3. Now we turn to the development of our approach.

1.1. Derivations of exterior algebras. In this subsection we state without proof relevant facts about the space of superderivations of an exterior algebra. In any case the proofs are all straightforward computations.

Definition 1.7. Let $\mathcal{A}$ be a superalgebra. A homogeneous endomorphism D of $\mathcal{A}$ is a *superderivation* if for all homogeneous $a, b \in \mathcal{A}$ the

$\mathbb{Z}_2$-graded Leibniz identity

$$D(ab) = D(a)b + (-1)^{\lceil a \rfloor \lceil D \rfloor} aD(b)$$

holds; here $\lceil \cdot \rfloor$ denotes the parity of a homogenous element in the appropriate space. We denote the space of even (resp. odd) superderivations on $\mathcal{A}$ by $\mathrm{sder}_+ \mathcal{A}$ (resp. $\mathrm{sder}_- \mathcal{A}$).

We use the term *derivation* for an even superderivation. Note that even derivations preserve the $\mathbb{Z}_2$-grading of any superalgebra. The proof of the following is an easy computation on a decomposable form:

LEMMA 1.8. *Let D and $\widetilde{D}$ be derivations of the exterior algebra ΛV^*. If $D|_{V^*} = \widetilde{D}|_{V^*}$ then $D = \widetilde{D}$.*

Let us recall that, for $v \in V$, the operator $v \lrcorner$ is the map

$$\Lambda^k V^* \to \Lambda^{k-1} V^*, \qquad \omega \mapsto \omega(v, \cdot),$$

i.e. $v \lrcorner \omega$ is the $(k-1)$-form obtained from ω by inserting v as its first argument. A well known fact about this operator is that it is a *derivation of degree* -1, which means

$$v \lrcorner (\alpha \wedge \beta) = (v \lrcorner \alpha) \wedge \beta + (-1)^{\lceil \alpha \rfloor} \alpha \wedge (v \lrcorner \beta) \tag{1.1}$$

for all forms α and β. Identity (1.1) is a consequence of the fact that $v \lrcorner$ is the dual operator of the exterior multiplication by a vector: if α is an exterior form then for all multivectors X,

$$\langle v \lrcorner \alpha, X \rangle = \langle \alpha, v \wedge X \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the evaluation pairing between ΛV^* and ΛV. This operator allows us to construct many derivations of the exterior algebra:

LEMMA 1.9. *Let $F : V^* \to \Lambda_- V^*$ be a linear map and let $\{v_1, \ldots, v_n\}$ and $\{dv_1, \ldots, dv_n\}$ be dual bases for V and V^* respectively. The map*

$$D_F := \sum_{\mu=1}^{n} F(dv_\mu) \wedge \circ (v_\mu \lrcorner)$$

is a derivation of ΛV^*.

This is an easy consequence of Lemma 1.8.

We can now characterise the space of derivations of ΛV^*.

THEOREM 1.10. *Let V be a vector space of dimension n. We have a natural isomorphism*

$$\mathrm{der}(\Lambda V^*) \cong V \otimes \Lambda_- V^* \oplus \Lambda_- V^* / (\Lambda_- V^* \cap \Lambda^n V^*). \tag{1.2}$$

We can now characterise the space of all superderivations of ΛV^*.

THEOREM 1.11. *The superspace* $\operatorname{sder}(\Lambda V^*)$ *of all superderivations is isomorphic to* $V \otimes \Lambda V^*$*, the* $\mathbb{Z}_2$*-grading being given by*

$$\operatorname{sder}_\bullet(\Lambda V^*) = V \otimes \Lambda_{-\bullet} V^* \tag{1.3}$$

where $-\bullet$ *means a change of parity.*

The following result is proved in Appendix B. It relies on an algebraic fact we call the Cartan–Poincaré Lemma.

LEMMA 1.12. *Let* $\mathcal{A}$ *be a free supercommutative finite-dimensional superalgebra and denote by* $\mathbf{S}^*$ *its space of generators. Let* $D : V \to \operatorname{sder}_- \mathcal{A}$ *be a linear map such that the composition*

$$f : V \xrightarrow{D} \operatorname{sder}_- \mathcal{A} \xrightarrow{\mathrm{pr}} \mathbf{S}$$

is injective and such that if $v, \widetilde{v}$ *are in* V *then* $[D_v, D_{\widetilde{v}}] = 0$*. Then there exists an isomorphism* $G : \Lambda\mathbf{S}^* \to \mathcal{A}$ *of* $\mathbb{Z}_2$*-graded algebras with unit such that* G *induces the identity*

$$\overline{G} : \mathbf{S}^* \to \mathcal{A}^{\geq 1}/\mathcal{A}^{\geq 2} =: \mathbf{S}^*,$$

and for all $v \in V$ *and all* $\sigma \in \Lambda\mathbf{S}^*$*,*

$$D_v(G\sigma) = G(f(v) \lrcorner\, \sigma).$$

Furthermore, up to the ideal generated by $\Lambda^3 \ker(f^*)$ *in* $\Lambda^3\mathbf{S}^*$ *the isomorphism* G *is unique in the sense that if* G' *is any other such isomorphism then*

$$G^{-1} \circ G' : \Lambda\mathbf{S}^* \to \Lambda\mathbf{S}^*, \qquad \sigma \mapsto \sigma + \langle \Lambda^3 \ker(f^*) \rangle.$$

CARTAN–POINCARÉ LEMMA. *Let* $F\colon V \to W$ *and* $G\colon W \to V$ *be linear maps and define the bigraded algebra* $A^{\bullet,\circ} = \operatorname{Sym}^\bullet V \otimes \Lambda^\circ W$*. The maps*

$$d_F := \sum_{\mu=1}^{n} dv_\mu \lrcorner \otimes F(v_\mu) \wedge : A^{\bullet,\circ} \to A^{\bullet-1,\circ+1}, \tag{1.4a}$$

$$d_G^* := \sum_{\mu=1}^{m} G(w_\mu) \cdot \otimes dw_\mu \lrcorner : A^{\bullet,\circ} \to A^{\bullet+1,\circ-1}, \tag{1.4b}$$

which we call the Cartan–Poincaré operators*, are bigraded boundary maps of the algebra* $A^{\bullet,\circ}$*; their cohomology satisfies*

$$H^{\bullet,\circ}(d_F) \cong \operatorname{Sym}^\bullet(\ker F) \otimes \Lambda^\circ(\operatorname{coker} F),$$
$$H_{\bullet,\circ}(d_G^*) \cong \operatorname{Sym}^\bullet(\operatorname{coker} G) \otimes \Lambda^\circ(\ker G).$$

Sketch of proof. We shall require the so-called commutation and anti-commutation relations; if V is a finite-dimensional vector space then for any vectors $v, \tilde{v}$ and any $\alpha, \tilde{\alpha} \in V^*$ we have the following identities:

$$\{v\lrcorner, \tilde{v}\lrcorner\} = \{\alpha\wedge, \tilde{\alpha}\wedge\} = 0,$$
$$\{v\lrcorner, \alpha\wedge\} = \alpha(v) \cdot \mathrm{id} \qquad \text{in } \Lambda V^*,$$

where $\{\cdot,\cdot\}$ denotes the anticommutator, and

$$[v\lrcorner, \tilde{v}\lrcorner] = [\alpha\,\cdot, \tilde{\alpha}\,\cdot] = 0,$$
$$[v\lrcorner, \alpha\,\cdot] = \alpha(v)\cdot \mathrm{id} \quad \text{in } \operatorname{Sym} V^*.$$

With the aid of the above identities it is possible to prove that for linear maps as in the hypothesis of the lemma the operators

$$D_{FG} := \sum_{\mu=1}^{\dim W} FG(w_\mu)\cdot \circ dw_\mu\lrcorner,$$
$$D_{GF} := \sum_{\nu=1}^{\dim V} GF(v_\nu)\wedge \circ dv_\nu\lrcorner$$

are derivations of the algebras $\operatorname{Sym} W$ and ΛV respectively (here, $\alpha\lrcorner v$ just means $\alpha(v)$, since the pairing is "symmetrical" between a vector space and its dual). Then one shows that the operator $\Delta_{F,G} := \{d_F, d_G^*\}$ is diagonalizable on the subspaces $U^{\bullet,\circ}_{a,b}$ defined as follows: let C be a subspace of V such that $V = \ker F \oplus C$ and let Z be a subspace of W such that $W = \operatorname{im} F \oplus Z$; these subspaces are complete systems of representatives for $V/\ker F$ and $\operatorname{coker} F$ respectively. Now form the subspaces

$$U^{\bullet,\circ}_{a,b} = \big(\operatorname{Sym}^a(\ker F)\otimes \operatorname{Sym}^{\bullet-a} C\big)\otimes\big(\Lambda^b(\operatorname{im} F)\otimes \Lambda^{\circ-b} Z\big).$$

One finally uses this decomposition of $A^{\bullet,\circ}$ to show that any other choice for representatives of $V/\ker F$ and $\operatorname{coker} F$ gives a (co)homologous class; the case of G is handled in the same manner, mutatis mutandis. ∎

1.2. Bundles associated to a smooth supermanifold. Since we are working in the category of vector bundles (more precisely of algebra bundles), we can exploit all the tools available in this setting.

Let $(M|\mathcal{R}M)$ be of superdimension $(m|n)$. For each point p in M the fibre $\mathcal{R}_pM$ is a free supercommutative algebra of rank n; as such it has a unique maximal ideal, denoted

$$\mathcal{R}_p^{\geq 1}M := \{\eta\in\mathcal{R}_pM \mid \varepsilon_p(\eta)=0\} \tag{1.5}$$

where $\varepsilon_p : \mathcal{R}_pM \to \mathbb{R}$ is the augmentation map of the algebra $\mathcal{R}_pM$. By defining $\mathcal{R}_p^{\geq k}M = (\mathcal{R}_p^{\geq 1}M)^k$ this algebra is filtered,

$$\mathcal{R}_pM = \bigcup_{k=0}^{n} \mathcal{R}_p^{\geq k}M. \tag{1.6}$$

Definition 1.13. Let $(M|\mathcal{R}M)$ be a supermanifold of dimension $(m|n)$ and let $p\in M$. The vector space

$$\mathbf{S}_p^*M := \mathcal{R}_p^{\geq 1}M/\mathcal{R}_p^{\geq 2}M$$

is the *space of odd codirections at the point* p. Its dual S_pM is the space of *odd directions* at p.

Now we proceed as differential geometers and define the bundles $\mathcal{R}^{\geq k}M$, with $k \geq 0$ an integer, the *nilpotent bundle* of $(M|\mathcal{R}M)$ being the case $k = 1$. The sections of each of these vector bundles are the nilpotent superfunctions on $(M|\mathcal{R}M)$. We then get

PROPOSITION 1.14. *Let* $(M|\mathcal{R}M)$ *be a supermanifold. The algebra* $\Gamma(\mathcal{R}M)$ *of smooth superfunctions is a filtered algebra:*

$$\Gamma(\mathcal{R}^{\geq 0}M) \supset \Gamma(\mathcal{R}^{\geq 1}M) \supset \cdots \supset \Gamma(\mathcal{R}^{\geq n-1}M) \supset \Gamma(\mathcal{R}^{\geq n}M) \supset \{0\}. \tag{1.7}$$

DEFINITION 1.15. The vector bundles

$$\mathbf{S}M = \bigsqcup_{p \in M} \mathbf{S}_pM \quad \text{and} \quad \mathbf{S}^*M = \bigsqcup_{p \in M} \mathbf{S}_p^*M$$

are called the bundles of *odd directions* and *odd codirections* respectively.

REMARK 1.16. An important distinction arises when considering the $\mathbb{Z}_2$-grading of the algebra $\mathcal{R}_pM$: the even elements, which we denote by $\mathcal{R}_{+,p}M$, and the odd elements, denoted by $\mathcal{R}_{-,p}M$. Accordingly we get the bundles $\mathcal{R}_+M$ and $\mathcal{R}_-M$, and their sections are called, respectively, the *even* and *odd* superfunctions of $(M|\mathcal{R}M)$.

1.3. Morphisms. Let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ be ringed spaces. Recall that a morphism between them is defined to be a pair $(\phi, \phi^\#)$ of maps such that $\phi : M \to N$ is continuous and for each open set $U \subseteq N$ the *localized map* $\phi^\# : \mathcal{O}_U \to \mathcal{O}_{\phi^{-1}(U)}$ is a morphism of rings (cf. [U, Section 2.3(b)]). In the case of supermanifolds, the morphism $\phi^\#$ is required to be a unital morphism of supercommutative algebras, i.e. $\phi^\#(1) = 1$. This forces $\phi^\#$ to be even.

To make a differential-geometric sense out of this definition, let us recall a well-known fact about the algebra $\mathcal{C}^\infty(M)$ of smooth functions of a smooth manifold M, whose proof is straightforward:

LEMMA 1.17. *Let* $\phi : M \to N$ *be a smooth map. Then the map*

$$\phi^* : \mathcal{C}^\infty(N) \to \mathcal{C}^\infty(M), \qquad f \mapsto f \circ \phi,$$

is a unital homomorphism of associative algebras.

Our definition of a supersmooth map takes the above morphism into account. We first consider a special case: let $\mathcal{R}M = M \times \Lambda V^*$ be a trivial bundle and consider the corresponding supermanifold $(M|M \times \Lambda V^*)$. The sheaf of sections is nothing other than $\mathcal{C}^\infty(M, \Lambda V^*)$. A smooth superfunction on the supermanifold in question is expressed as

$$f = f_0 + \text{nilpotent part}$$

where f_0 is a smooth function on M, and so we get the inclusion $\iota : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M, \Lambda V^*)$. If $(N|N \times \Lambda W^*)$ is another such supermanifold then, associated to a smooth map $\phi : M \to N$, we must obtain an even morphism of supercommutative algebras $\Phi : \mathcal{C}^\infty(N, \Lambda W^*) \to \mathcal{C}^\infty(M, \Lambda V^*)$ such that the diagram

$$\begin{array}{ccc} \mathcal{C}^\infty(N, \Lambda W^*) & \underset{\varepsilon_N}{\overset{\iota}{\rightleftarrows}} & \mathcal{C}^\infty(N) \\ \downarrow \Phi & & \downarrow \phi^* \\ \mathcal{C}^\infty(M, \Lambda V^*) & \underset{\iota}{\overset{\varepsilon_M}{\rightleftarrows}} & \mathcal{C}^\infty(M) \end{array} \tag{1.8}$$

commutes, where ε_M and ε_N are the augmentation maps, and ι_M and ι_N the corresponding inclusions. Now let f and g be smooth functions on N and let $\eta \in \mathcal{C}^\infty(N, \Lambda W^*)$; considering the twisted commutator

$$[\Phi_\phi f](\eta) := \Phi(f\eta) - (f \circ \phi)\Phi(\eta) \tag{1.9}$$

we get

$$[\Phi_\phi f](\eta) = (\Phi(f) - f \circ \phi)\Phi(\eta) \tag{1.10}$$

because Φ is an algebra morphism; moreover, since Φ is even it preserves the $\mathbb{Z}_2$-grading, and therefore $\Phi(f) - f \circ \phi$ is an even superfunction on N. Considering diagram (1.8) we see that $\Phi \circ \varepsilon_M = \varepsilon_N \circ \phi^*$, and therefore the term $\Phi(f) - f \circ \phi = \Phi(f) - \phi^*(f)$ is nilpotent. Hence, iterating this commutator we get zero after finitely many steps. If q is the dimension of W then putting $k = \lfloor q/2 \rfloor$ (integer part of $q/2$), it is manifest that

$$[\ldots, [[\Phi_\phi f_0]_\phi f_1]_\phi \cdots_\phi f_k] \equiv 0 \tag{1.11}$$

for any $k+1$ smooth functions on N. The above property defines the class of morphisms we are interested in:

Definition 1.18. Let $E \to N$ and $F \to M$ be smooth vector bundles and $\phi \colon M \to N$ a smooth map. An $\mathbb{R}$-linear map $\Phi \colon \Gamma(E) \to \Gamma(F)$ is a *linear differential operator along* ϕ if condition (1.11) above is satisfied for every choice of $k+1$ smooth functions on N.

Now, given two smooth supermanifolds $(M|\mathcal{R}M)$ and $(N|\mathcal{R}N)$ of superdimensions $(m|p)$ and $(n|q)$ respectively, the above constructions are valid on trivializing neighbourhoods of the corresponding bundles $\mathcal{R}M$ and $\mathcal{R}N$. Using a partition of unity we arrive at the same results: we have a commutative diagram

(1.12)
$$\begin{array}{ccc} \Gamma(\mathcal{R}N) & \xrightarrow[\varepsilon_N]{} & \mathcal{C}^\infty(N) \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \phi^*} \\ \Gamma(\mathcal{R}M) & \xrightarrow{\varepsilon_M} & \mathcal{C}^\infty(M) \end{array}$$

Note that the inclusions are not considered in this diagram. Summarising the considerations above we have:

PROPOSITION 1.19. *Let* $(M|\mathcal{R}M)$ *and* $(N|\mathcal{R}N)$ *be smooth supermanifolds and* $(\phi|\Phi) : (M|\mathcal{R}M) \to (N|\mathcal{R}N)$ *a supersmooth map. Then the map* $\Phi : \Gamma(\mathcal{R}N) \to \Gamma(\mathcal{R}M)$ *is a* linear differential operator along the smooth map $\phi : M \to N$ *of order at most* $k = \lfloor (\operatorname{rank} \mathbf{S}N)/2 \rfloor$.

Another important property of supersmooth maps is the following:

PROPOSITION 1.20. *If* $(\phi|\Phi) : (M|\mathcal{R}M) \to (N|\mathcal{R}N)$ *is a supersmooth map then*

$$\Phi(\Gamma(\mathcal{R}^{\geq k}N)) \subseteq \Gamma(\mathcal{R}^{\geq k}M)$$

for all non-negative integers k*. That is, supermanifold morphisms preserve the filtration* (1.7)*.*

Proof. Since Φ is an even superalgebra morphism, it preserves the nilpotency of superfunctions, so nilpotent sections go to nilpotent sections; also, algebra morphisms preserve powers, and the result follows. ∎

COROLLARY 1.21. *The morphism* $\Phi : \Gamma(\mathcal{R}N) \to \Gamma(\mathcal{R}M)$ *defines bundle morphisms*

$$\Phi^k : \mathcal{R}^{\geq k}N/\mathcal{R}^{\geq k+1}N \to \mathcal{R}^{\geq k}M/\mathcal{R}^{\geq k+1}M$$

by $\Phi^k(\eta + \mathcal{R}^{\geq k+1}N) = \Phi(\eta) \bmod \mathcal{R}^{\geq k+1}M$ *for each* $k \geq 0$.

Proof. Let f be a smooth function on N and η a section of $\mathcal{R}^{\geq k}N$. Because of the identity

$$[\Phi_\phi f](\eta) = (\Phi(f) - (f \circ \phi))\Phi(\eta)$$

we know that $\Phi(f) - (f \circ \phi)$ is a section of $\mathcal{R}^{\geq 2}M$ and because of the inclusions of filtration (1.7) we know that $(\Phi(f) - (f \circ \phi))\Phi(\eta)$ is a section of $\mathcal{R}^{\geq k+2}M$, which is contained in $\mathcal{R}^{\geq k+1}M$; therefore $[\Phi_\phi f](\eta) \in \Gamma(\mathcal{R}^{\geq k+1})$, and thus Φ^k is a bundle morphism over the smooth map ϕ. ∎

1.3.1. *Structure results for supersmooth maps.* It is immediate from the definition that if $f : A \to B$ is an algebra homomorphism and A is generated by a set $\{a_\alpha\}$ then f is completely determined by the images $\{f(a_\alpha)\}$. Since we are working in a finite-dimensional setting, we can, at least locally, use a basis for the superalgebras considered. The first result of this subsection is concerned with the structure of the space of supersmooth maps.

To wit, let $(V|\mathbf{S})$ be a supervector space; the linear supermanifold $(V|V \times \Lambda\mathbf{S}^*)$ has as structure superalgebra the space of smooth functions

$\mathcal{C}^\infty(V, \Lambda\mathbf{S}^*)$. We claim that a smooth supermap $(\phi|\Phi)\colon (M|\mathcal{R}M) \to (V|\mathbf{S})$ can be completely characterised in terms of the morphism $\Phi\colon \mathcal{C}^\infty(V, \Lambda\mathbf{S}^*) \to \Gamma(\mathcal{R}M)$.

Since Φ is a unital superalgebra homomorphism it is completely characterised by the images of a set of generators of $\mathcal{C}^\infty(V, \Lambda\mathbf{S}^*)$. If we choose bases $\{v_1, \dots, v_m\}$ and $\{s_1, \dots, s_n\}$ of V and $\mathbf{S}$ respectively, the dual bases $\{dv_1, \dots, dv_m\}$ and $\{ds_1, \dots, ds_n\}$ furnish generators of $\operatorname{Sym} V^* \otimes \Lambda\mathbf{S}^*$ which is seen to be dense in $\mathcal{C}^\infty(V, \Lambda\mathbf{S}^*)$ by the Stone–Weierstrass Theorem. The morphism Φ restricts to a unital superalgebra morphism

$$\Phi_0\colon \operatorname{Sym} V^* \otimes \Lambda_\bullet\mathbf{S}^* \to \Gamma(\mathcal{R}_\bullet M).$$

The images $\Phi_0(dv_\mu)$ and $\Phi_0(ds_\nu)$ of the generators of $\mathcal{C}^\infty(V, \Lambda\mathbf{S}^*)$ are, respectively, even and odd superfunctions on $\mathcal{R}M$. Given $\alpha \in V^*$ we can define a smooth map $f\colon M \to V$ by the implicit formula

$$\langle f(x), \alpha\rangle := \varepsilon(\Phi_0(\alpha))(x) \tag{1.13}$$

for all $x \in M$, using the natural pairing $\langle\cdot,\cdot\rangle\colon V \otimes V^* \to \mathbb{R}$. Again by the Stone–Weierstrass Theorem we deduce that $f = \phi$.

It is quite evident that the above considerations can be reversed. That is, given a unital superalgebra homomorphism

$$\Phi_0\colon \operatorname{Sym} V^* \otimes \Lambda_\bullet\mathbf{S}^* \to \Gamma(\mathcal{R}_\bullet M),$$

equation (1.13) defines a smooth map $\phi\colon M \to V$. Likewise, the images under Φ_0 of the bases considered above furnish even and odd superfunctions on $\Gamma(\mathcal{R}M)$. To extend Φ_0 to a morphism $\Phi\colon \mathcal{C}^\infty(V, \Lambda\mathbf{S}^*) \to \Gamma(\mathcal{R}M)$ that is a differential operator along ϕ we consider the following: for every $x \in M$, the map Φ_0 yields a superalgebra homomorphism

$$\Phi_x^-\colon \Lambda_\bullet\mathbf{S}^* \to \mathcal{R}_{\bullet,x}M$$

given by $\Phi_x^-(ds_{\mu_1} \wedge \cdots \wedge ds_{\mu_k}) = \Phi_0\big(ds_{\mu_1}\big)(x) \cdots \Phi_0\big(ds_{\mu_k}\big)(x)$. Also, using the function defined by (1.13) we can define a map from $\operatorname{Sym} V^*$ with values in $\mathcal{R}^{\geq 2}_{+,x}M$ by the relation

$$\Phi_x^+(\alpha) = \Phi_0\big(\alpha\big)(x) - \langle\phi(x), \alpha\rangle$$

defined for $\alpha \in V^*$ and extending it to all of $\operatorname{Sym} V^*$; it is clearly multiplicative. Both maps can be bundled together by forgetting the evaluation at $x \in M$, and we thus get two maps $\Phi^-\colon \Lambda_\bullet\mathbf{S}^* \to \Gamma(\mathcal{R}_\bullet M)$ and $\Phi^+\colon \operatorname{Sym} V^* \to \Gamma(\mathcal{R}^{\geq 2}_+ M)$. Finally we can define, for $h \in \mathcal{C}^\infty(V, \Lambda\mathbf{S}^*)$,

$$\Phi(h) = \sum_{\mu_1 + \cdots + \mu_r = k \geq 0} \frac{1}{k!} \Phi^-\left(\frac{\partial^k (h \circ \phi)}{\partial x^{\mu_1} \cdots \partial x^{\mu_r}}\right) \Phi^+(dv_{\mu_1} \cdots dv_{\mu_r}) \tag{1.14}$$

for $1 \leq \mu_1, \dots, \mu_r \leq m = \dim V$. Note that if $r > \lfloor(\operatorname{rank} \mathbf{S}M)/2\rfloor$ then

$$\Phi^+(dv_{\mu_1}, \dots, dv_{\mu_r}) = 0.$$

This is seen to be a differential operator along ϕ; it is also not difficult to see that it is a unital homomorphism of superalgebras $\mathcal{C}^\infty(V, \Lambda_\bullet \mathbf{S}^*) \to \Gamma(\mathcal{R}_\bullet M)$ by using the Leibniz identity for partial differential operators. Summarising, we get:

THEOREM 1.22. *Let* $\mathcal{C}^\infty((M|\mathcal{R}M), (V|\mathbf{S}))$ *denote the space of supersmooth maps. There are isomorphisms of superalgebras*

$$\begin{aligned}\mathcal{C}^\infty((M|\mathcal{R}M), (V|\mathbf{S})) &\cong \mathrm{Hom}_{\mathrm{alg}}\big(\mathcal{C}^\infty(V, \Lambda_\bullet \mathbf{S}^*), \Gamma(\mathcal{R}_\bullet M)\big)\\ &\cong \mathrm{Hom}_{\mathrm{alg}}\big(\mathrm{Sym}\, V^* \otimes \Lambda_\bullet \mathbf{S}^*, \Gamma(\mathcal{R}_\bullet M)\big)\\ &\cong V \otimes \Gamma(\mathcal{R}_+ M) \oplus \mathbf{S} \otimes \Gamma(\mathcal{R}_- M)\end{aligned}$$

where $\mathrm{Hom}_{\mathrm{alg}}$ *denotes the superalgebra homomorphisms, given by the definitions above of* Φ_0, Φ^+, Φ^- *and formula* (1.14).

We will now establish the result we mentioned at the end of the introduction to this section, namely one concerning the structure of a supersmooth map. First we need the following

LEMMA 1.23. *Let* M *and* N *be smooth manifolds and let* $\mathcal{C}^\infty_M$ *and* $\mathcal{C}^\infty_N$ *denote the sheaves of smooth functions over* M *and* N *respectively. Given a* continuous *map* $\phi : M \to N$, *if* $\Phi : \mathcal{C}^\infty_N \to \phi_* \mathcal{C}^\infty_M$ *is a unital morphism of sheaves of algebras over* ϕ *then* ϕ *is smooth and* Φ *equals the map*

$$\phi^* \colon \mathcal{C}^\infty_N \to \mathcal{C}^\infty_M, \qquad f \mapsto f \circ \phi.$$

Proof. Recall that if $U \subseteq N$ is an open set, the sheaf $\phi_* \mathcal{C}^\infty_M(U)$ is defined as $\mathcal{C}^\infty_M(\phi^{-1}(U))$, the restriction of $\mathcal{C}^\infty_M$ to the open set $\phi^{-1}(U)$. Let $f \in \mathcal{C}^\infty_N(U)$ and $x \in \phi^{-1}(U)$. Then if $V \subset \mathbb{R}$ is any open subset containing $\phi(f(x))$, the set $W := \phi^{-1}(f^{-1}(V))$ is relatively open on $\phi^{-1}(U)$. The restriction morphism $\mathrm{res}^U_{f^{-1}(V)}$ of the sheaf $\mathcal{C}^\infty_N$ induces the morphism

$$\Phi_{f^{-1}(V)} : \mathcal{C}^\infty_N(f^{-1}(V)) \to \mathcal{C}^\infty_M(W) \tag{1.15}$$

of sheaves. Consider the function $h := f - f(\phi(x))\mathbf{1}$, where $\mathbf{1}$ is the function identically 1 on N. The image of h under $\Phi_{f^{-1}(V)}$ is

$$g := \Phi_{f^{-1}(V)}(h) = \Phi_{f^{-1}(V)}(f) - \Phi_{f^{-1}(V)}(f)(x)\mathbf{1} \tag{1.16}$$

(recall that $\Phi_{f^{-1}(V)}(f)$ is by definition a smooth funcion on W). The function g is not invertible on W because $x \in W$, and therefore g has a zero there. Since U is arbitrary, we have $\Phi_U(f)(x) \in V$ for all V containing $f(\phi(x))$. Therefore

$$\Phi_U(f) = \phi^*(f) = f \circ \phi. \tag{1.17}$$

Now let $x_1, \ldots, x_n$ be any local coordinates on an open subset $\widetilde{U}$ of U. Each of these coordinates is a smooth function on $\widetilde{U}$, and therefore $\Phi_{\widetilde{U}}(x_\mu) = x_\mu \circ \phi$ is smooth for all $1 \leq \mu \leq n$. This is precisely the definition of a smooth map between smooth manifolds. Therefore ϕ is smooth. As a consequence of (1.17) we get $\Phi = \phi^*$. ■

The final result of this subsection will extend the above lemma to algebraic homomorphisms of the sheaves of superfunctions.

THEOREM 1.24. *Let $(M|\mathcal{R}M)$ and $(N|\mathcal{R}N)$ be supermanifolds; denote by $\mathcal{O}_M$ the sheaf of sections of $\mathcal{R}M$ and let $\Phi\colon \mathcal{O}_N \to \mathcal{O}_M$ be a sheaf morphism along the continuous map $\phi : M \to N$. Then Φ is a differential operator along ϕ.*

Proof. Consider a splitting $\iota_N\colon \mathcal{C}^\infty_N \to \mathcal{O}_N$ as in Corollary 1.6. It is clear that $\varepsilon_M \circ \Phi \circ \iota_N\colon \mathcal{C}^\infty_N \to \mathcal{C}^\infty_M$ is a sheaf morphism over the continuous map ϕ. The above lemma implies that ϕ is smooth and $\varepsilon_M \circ \Phi \circ \iota_N = \phi^*$; therefore we get the commutative diagram

$$\begin{array}{ccc} \Gamma(\mathcal{R}N) & \underset{\varepsilon_N}{\overset{\iota}{\rightleftarrows}} & \mathcal{C}^\infty(N) \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \phi^*} \\ \Gamma(\mathcal{R}M) & \underset{\iota}{\overset{\varepsilon_M}{\rightleftarrows}} & \mathcal{C}^\infty(M) \end{array}$$

so $\varepsilon_M \circ \Phi = \phi^* \circ \varepsilon_N$; hence defining the commutator $[\Phi_\phi f](\eta) = \Phi(f\eta) - (f \circ \phi)\Phi(\eta)$ for all smooth functions f on M and $\eta \in \Gamma(\mathcal{R}M)$ it is seen that $\Phi(f) - \phi^*(f)$ is nilpotent, so Φ is a differential operator along ϕ. ■

The two theorems above guarantee that at the level of morphisms, the usual approach via sheaf maps and our approach are equivalent. It remains to prove the equivalence at the level of objects.

1.4. $\mathbb{Z}$-graded supermanifolds. Let $(M|\mathcal{R}M)$ be a supermanifold. Recall that for every point p of M we have the bundle $\mathbf{S}_pM$ of odd directions and its dual $\mathbf{S}^*_pM$, and that choosing a connection of algebras on $\mathcal{R}M$ we have an isomorphism $\mathcal{R}M \cong \Lambda\mathbf{S}^*M$. In this section we study supermanifolds arising from the inverse construction.

Let $\xi\colon E \to M$ and $\widetilde{\xi}\colon \widetilde{E} \to N$ be smooth vector bundles, and let $\Phi\colon E \to \widetilde{E}$ be a bundle morphism along a smooth map $\phi\colon M \to N$. Then the lift $\Lambda\Phi\colon \Lambda E \to \Lambda\widetilde{E}$ is a morphism of the associated supermanifolds $(M|\Lambda E)$ and $(N|\Lambda\widetilde{E})$. Associated to Φ we get a map $\Phi^*\colon \Gamma(\widetilde{E}) \to \Gamma(E)$ as in Lemma 1.17. This kind of morphisms are not only $\mathbb{Z}_2$-graded but also $\mathbb{Z}$-graded. As a differential operator along ϕ, the map Φ^* is as simple as it could be:

PROPOSITION 1.25. *In the above setting, the map Φ^* is a differential operator of order* 0.

Proof. It suffices to observe that $\Lambda\Phi$ is $\mathcal{C}^\infty(M)$-linear, that is,

$$\Lambda\Phi(f\omega) = (f \circ \phi)\Lambda\Phi(\omega)$$

for all sections ω of $\Lambda(\widetilde{E})$. Therefore the commutator $[\Phi^*{}_\phi f]$ vanishes for all smooth functions f on M. ■

In this setting we can make the identification $\mathbf{S}M = E$ and the supermanifold structure is thus completely determined by the vector bundle $\mathbf{S}M$.

Definition 1.26. A *$\mathbb{Z}$-graded supermanifold* is a supermanifold of the form $(M|\Lambda(\mathbf{S}M))$ for some vector bundle $\mathbf{S}M$. A *$\mathbb{Z}$-graded smooth supermap* is the lifting to the exterior bundle of a bundle map $F\colon \mathbf{S}M \to \mathbf{S}N$.

These supermanifolds are also referred to as *split supermanifolds*. We will denote such a supermanifold as $(M|\mathbf{S}M)$. Batchelor's Theorem (Corollary 1.6) is tantamount to saying that every smooth supermanifold $(M|\mathcal{R}M)$ is (non-naturally) isomorphic to a split supermanifold.

2. The tangent and cotangent superbundles. In this section we give the construction of the tangent and cotangent superbundles of a smooth supermanifold. We rely on the contents of Appendix A. We begin with a technical albeit rather simple fact.

Lemma 2.1. *Given a supersmooth map $(\phi|\Phi)\colon (M|\mathcal{R}M) \to (N|\mathcal{R}N)$, there are two bundle maps $F\colon \mathbf{S}M \to \mathbf{S}N$ and $F^*\colon \mathbf{S}^*N \to \mathbf{S}^*M$ naturally associated to it.*

Proof. The map F^* is just the bundle map Φ^1 of Corollary 1.21, and F is its dual. ■

Now recall that the tangent bundle of a smooth manifold M has as sections the derivations of the algebra $\mathcal{C}^\infty(M)$. That is, $\mathrm{der}(\mathcal{C}^\infty(M)) = \Gamma(TM)$. This is a Lie algebra under the commutator of vector fields. If $(M|\mathcal{R}M)$ is any supermanifold, then the space of its superderivations should be a Lie superalgebra. Because of the projection $\varepsilon\colon \Gamma(\mathcal{R}M) \to \mathcal{C}^\infty(M)$ it is manifest that the Lie algebra of vector fields on M must be a Lie subalgebra of the even part of this superalgebra. Now each fibre $\mathcal{R}_pM$ is a free supercommutative algebra with only odd generators, the space $\mathbf{S}^*_pM$. The Lie superalgebra $\mathrm{sder}(\mathcal{R}_pM)$ of superderivations of the algebra $\mathcal{R}_pM$ is generated by $\mathbf{S}_pM$. Now, fixing an isomorphism $\phi : \Lambda\mathbf{S}^*_pM \to \mathcal{R}_pM$ we know (by Theorem 1.11) that the space of superderivations of $\mathcal{R}_pM$ is isomorphic to $\mathbf{S}_pM \otimes \Lambda\mathbf{S}^*_pM$. This procedure singles out the following space:

Definition 2.2. For every point p of M define the space

$$\mathrm{sder}_p(\mathcal{R}M) = \mathrm{sder}(\mathcal{R}_pM)$$

of *pointwise derivations* of $\mathcal{R}M$. The vector bundle

$$\mathrm{sder}(\mathcal{R}M) := \bigsqcup_{p\in M} \mathrm{sder}_p(\mathcal{R}M)$$

is the *bundle of pointwise derivations* of $(M|\mathcal{R}M)$.

The elements of the fibres of this bundle are derivations of the corresponding fibre of $\mathcal{R}M$. A section of this bundle, then, acts only fibrewise and it is linear on each fibre and therefore linear over the ring $\mathcal{C}^\infty(M)$. That is, if η is a superfunction and f is smooth on M, then $D(f\eta)_p = f(p)D(\eta)_p$ for every section D of $\mathrm{sder}(\mathcal{R}M)$. Summarising:

PROPOSITION 2.3. *The sections of the bundle* $\mathrm{sder}(\mathcal{R}M)$ *are differential operators of order* 0 *over* $\mathcal{C}^\infty(M)$.

Clearly in the space $\Gamma(\mathrm{sder}(\mathcal{R}M))$ we are missing a lot of superderivations of $\Gamma(\mathcal{R}M)$, as we expect the graded Leibniz rule to apply. That is, if D is a superderivation of $\Gamma(\mathcal{R}M)$, we expect

$$D(\eta\psi) = D(\eta)\psi + (-1)^{\lceil\eta\rfloor\lceil D\rfloor}\eta D(\psi), \tag{2.1}$$

and this identity is clearly not linear for smooth functions on M; however, it must be satisfied for all superderivations of $\Gamma(\mathcal{R}M)$. Thus they are differential operators of positive order.

We now proceed to the construction of the bundle of superderivations. Since smooth vector fields must be even elements of the Lie superalgebra $\mathrm{sder}(\Gamma(\mathcal{R}M))$, we must include $\Gamma(TM)$ in the even subspace of $\mathrm{sder}(\Gamma(\mathcal{R}M)$. On the other hand, the space of generators of $\mathrm{sder}(\Gamma(\mathcal{R}M))$ must also contain $\Gamma(\mathbf{S}M)$ (by Theorem 1.11) and at the same time the pointwise derivations (Definition 2.2). With these considerations in mind, let f be a smooth function on M and D a superderivation of $\Gamma(\mathcal{R}M)$. We know that $D(f)$ is an even superfunction, and because of the requirements imposed by the Leibniz identity (2.1), we know $D(f)$ must be of the form $X(f) + \eta(f)$ with X a vector field on M and η an even nilpotent superfunction (i.e., a section of $\mathcal{R}^{\geq 2}_+M$). Because of Proposition 2.3 we know the bundle must include $\mathrm{sder}(\mathcal{R}M)$, so there seems to be a decomposition

$$\mathrm{sder}(\Gamma(\mathcal{R}M)) \cong \Gamma(\mathcal{R}M \otimes (TM \oplus \mathbf{S}M)),$$

where the $\mathbf{S}M$ factor accounts for the generators of the space of superderivations. We now prove this is the case:

THEOREM 2.4. *The following sequence of vector bundles is exact:*

$$0 \to \mathrm{sder}(\mathcal{R}M) \overset{\iota}{\hookrightarrow} \mathrm{Der}(\mathcal{R}M) \overset{\sigma}{\twoheadrightarrow} \mathcal{R}M \otimes TM \to 0. \tag{2.2}$$

Here σ *denotes the principal symbol* (*Definition* A.5).

Proof. We have seen that the bundle $\Lambda\mathbf{S}^*M \otimes (TM \oplus \mathbf{S}M)$ actually contains operators that act on the superfunctions as derivations. We also know (Proposition 2.3) that the sections of $\operatorname{sder}\mathcal{R}M$ are differential operators of order zero, so their principal symbol is zero. This proves $\operatorname{im}(\iota) \subseteq \ker(\sigma)$. Also, this condition is sufficient for a differential operator to be of order zero, so $\operatorname{im}(\iota) = \ker(\sigma)$.

In order to prove that σ is surjective we choose a connection ∇ on the bundle $\mathcal{R}M$. Let $X \otimes r$ be a section of $\mathcal{R}M \otimes TM$ and define a map $g: \Gamma(\mathcal{R}M \otimes TM) \to \Gamma(\operatorname{Der}(\mathcal{R}M))$ by $g(X \otimes r) = r\nabla_X$, which is evidently a differential operator of order 1, so we compute its principal symbol: let η be a superfunction and f a smooth function; then

$$\begin{aligned}\sigma(r\nabla_X; f)(\eta) &= [r\nabla_X, f](\eta) = r\nabla_X(f\eta) - fr\nabla_X(\eta)\\ &= rX(f)\eta + rf\nabla_X(\eta) - fr\nabla_X(\eta)\\ &= rX(f)\eta \quad \text{(because } fr = rf)\\ &= r \otimes X(f \otimes \eta),\end{aligned} \tag{2.3}$$

which proves that $g \circ \sigma$ is the identity on $TM \otimes \mathcal{R}M$, making σ a surjective map. ∎

Corollary 2.5. *Superderivations of the algebra $\Gamma(\mathcal{R}M)$ are differential operators of order at most* 1 *(cf. Definition* A.1*).*

2.1. Tangential maps. Let $(\phi|\Phi)\colon (M|\mathcal{R}M) \to (N|\mathcal{R}N)$ be a supersmooth map, fixed throughout this subsection. Recall from Lemma 2.1 that there are two associated bundle maps, $F\colon \mathbf{S}M \to \mathbf{S}N$ and $F^*\colon \mathbf{S}^*M \to \mathbf{S}^*N$. Recall that in the classical case, given a smooth map $\phi\colon M \to N$ there exist two naturally associated bundle maps, $\phi_*\colon TM \to TN$ and $\phi^*\colon T^*N \to T^*M$. In the supermanifold case, the tangent superbundle $T(M|\mathcal{R}M) = \operatorname{Der}(\mathcal{R}M)$ is generated by $TM \oplus \mathbf{S}M$. So now we can consider the maps

$$(\phi|\Phi)_* := \phi_* \oplus F\colon TM \oplus \mathbf{S}M \to TN \oplus \mathbf{S}N, \tag{2.4}$$

$$(\phi|\Phi)^* := \phi^* \oplus F^*\colon T^*N \oplus \mathbf{S}^*N \to T^*M \oplus \mathbf{S}^*M, \tag{2.5}$$

where $F = (\Phi^1)^*$ is the map of Corollary 1.21, and call them, respectively, the *differential* and *codifferential* of $(\phi|\Phi)$. It is manifest they satisfy analogous properties to those of the classical differential and codifferential. A noteworthy fact is the following

Proposition 2.6. *Let $\mathcal{I}$ be the functor that associates to any given supermanifold $(M|\mathcal{R}M)$ the corresponding $\mathbb{Z}$-graded supermanifold $(M|\mathbf{S}M)$, and let $F\colon \mathbf{S}M \to \mathbf{S}N$ be the map of Lemma* 2.1*. If $(\phi|\Phi)\colon (M|\mathcal{R}M) \to (N|\mathcal{R}N)$ is a morphism of supermanifolds then $\mathcal{I}(\phi|\Phi) = (\phi|F)$.*

Proof. Recall that a morphism of $\mathbb{Z}$-graded supermanifolds is the exterior lift $\Lambda(G)$ of a morphism $G: \mathbf{S}M \to \mathbf{S}N$ over the smooth map $\phi: M \to N$.

By Corollary 1.21 we know $\Phi^1 = F$ is naturally associated to Φ when taking the quotient $\mathcal{R}^{\geq 1}M/\mathcal{R}^{\geq 2}M$. Since $\Lambda(\Phi^1) = \Lambda(F)$, we get a morphism $(\phi|F)$ of $\mathbb{Z}$-graded supermanifolds naturally associated to $(\phi|\Phi)$ via the quotient map. Thus $\mathcal{I}(\phi|\Phi) = (\phi|F)$. ∎

2.2. The auxiliary differential. When we analysed smooth supermaps, we found that they are differential operators along smooth maps; the proof relied on the operator

$$[\Phi_\phi\cdot]\colon \mathcal{C}^\infty(N) \to \Gamma(\mathcal{R}^{\geq 2}M),$$

which is a derivation along ϕ, that is,

$$[\Phi_\phi fg] = [\Phi_\phi f](g \circ \phi) + (f \circ \phi)[\Phi_\phi g],$$

which can be seen by evaluating the commutator on the section **1** and using the fact that Φ is a superalgebra morphism. By the universal property of jets (Theorem A.11) we get the following diagram:

$$\begin{array}{ccc} \mathcal{C}^\infty(N) & \xrightarrow{[\Phi,\cdot]} & \Gamma(\mathcal{R}^{\geq 2}M) \\ {\scriptstyle \mathrm{jet}^1}\big\downarrow & \nearrow{\scriptstyle \mathrm{jet}^1([\Phi.\cdot])} & \\ \Gamma(\mathrm{Jet}^1(N)) & & \end{array}$$

By Corollary A.15 the above diagram turns into

$$\begin{array}{ccc} \mathcal{C}^\infty(N) & \xrightarrow{[\Phi_\phi\cdot]} & \Gamma(\mathcal{R}^{\geq 2}M) \\ {\scriptstyle d}\big\downarrow & \nearrow{\scriptstyle \mathrm{jet}^1([\Phi_\phi\cdot])} & \\ \Gamma(T^*N) & & \end{array} \tag{2.6}$$

Let us now compute

$$[\Phi_\phi fg] = \Phi(fg) - (fg) \circ \phi = \Phi(f)\Phi(g) - (f \circ \phi)(g \circ \phi),$$

which turns into

$$\Phi(f)\Phi(g) - (f \circ \phi)(g \circ \phi) = [\Phi, f][\Phi, g] + [\Phi, g](f \circ \phi) + ([\Phi, f])(g \circ \phi),$$

Now the term $[\Phi_\phi f][\Phi_\phi g]$ is an even section of $\mathcal{R}^{\geq 4}M$, so it makes sense to consider the class of $[\Phi_\phi fg]$ in the quotient $\Gamma(\mathcal{R}^{\geq 2}M/\mathcal{R}^{\geq 3}M)$. Thus we get the map

$$\begin{aligned} \Phi^!\colon T^*N &\to \Lambda^2\mathbf{S}^*M = \mathcal{R}^{\geq 2}M/\mathcal{R}^{\geq 3}M, \\ df_{\phi(p)} &\mapsto \Phi(f)_p - (f \circ \phi)(p) + \mathcal{R}^{\geq 3}_p M, \end{aligned} \tag{2.7}$$

which is a derivation along ϕ in the sense described above.

Definition 2.7. Let $(\phi|\Phi) : (M|\mathcal{R}M) \to (N|\mathcal{R}N)$ be a supersmooth map. The *auxiliary codifferential* of the supersmooth map $(\phi|\Phi)$ is the map $\Phi^!$ of (2.7). The *auxiliary differential* is the dual map $\Phi_! := (\Phi^!)^*$.

LEMMA 2.8. *If the map $\Phi : \Gamma(\mathcal{R}N) \to \Gamma(\mathcal{R}M)$ is a differential operator of order* 0 *then* $\Phi_! \equiv 0$.

Proof. Since $\Phi^!$ is the natural extension to T^*M of the derivation $[\Phi_\phi f]$, if one is zero so is the other. But $[\Phi_\phi f] = \Phi(f) - f \circ \phi = 0$ means $\Phi(f) = f \circ \phi$, which is a necessary condition for Φ to be a differential operator of order zero. To wit: given an arbitrary superfunction η on $(M|\mathcal{R}M)$ the vanishing of the auxiliary (co)differential implies $0 = [\Phi_\phi f](\eta) = (\Phi(f) - f \circ \phi)\Phi(\eta)$, so we get $\Phi(f\eta) = (f \circ \phi)\Phi(\eta)$. ∎

3. Proof of Theorem 1.5. Let $(M, \mathcal{O})$ be a KL-supermanifold. As we remarked at the end of the introduction to Section 1, the theorem is equivalent to the splitting of the sequence

$$0 \to \mathcal{N} \hookrightarrow \mathcal{O} \overset{\varepsilon}{\to} \mathcal{C}^\infty \to 0$$

so as to give $\mathcal{O}$ the structure of a sheaf of $\mathcal{C}^\infty$-modules. We state this as

THEOREM 3.1. *There exists a unital sheaf homomorphism* $j\colon \mathcal{C}^\infty \to \mathcal{O}$ *such that* $\varepsilon \circ j = \mathrm{id}_{\mathcal{C}^\infty}$.

Let us define an equivalence relation for these maps:

DEFINITION 3.2. Let $\iota, \iota'\colon \mathcal{C}^\infty \to \mathcal{O}$ be unital morphisms and $k \geq 0$. We say they are *equivalent up to degree k*, written $\iota \underset{k}{\sim} \iota'$, if $\mathrm{im}(\iota - \iota') \subseteq \mathcal{N}^{k+1}$.

It is clear that if the odd dimension of $(M, \mathcal{O})$ is n and $k \geq \lfloor n/2 \rfloor$, then always $\iota \underset{k}{\sim} \iota'$. This observation will be the basis of the proof.

Fix an open cover $\{U_\alpha\}_{\alpha \in \mathrm{A}}$ of M such that

(a) there exists an isomorphism $\tau_\alpha\colon \mathcal{O}_\alpha := \mathcal{O}|_{U_\alpha} \to \mathcal{C}^\infty_\alpha \otimes \Lambda\mathbf{S}^*$ of superalgebras with unit, and
(b) there exist $y_\alpha = (y^1_\alpha, \ldots, y^m_\alpha)$ smooth functions on U_α such that (U_α, y_α) is a coordinate chart on M; that is, U_α is diffeomorphic to an open set in $\mathbb{R}^m$.

Let $U_{\alpha\beta} := U_\alpha \cap U_\beta$, and let $\mathcal{O}_{\alpha\beta}$ and $\mathcal{C}^\infty_{\alpha\beta}$ denote the corresponding sheaves. Define $\tau_{\alpha\beta} = \tau_\alpha \circ \tau_\beta^{-1}$.

From now on, until otherwise stated, we work with the supermanifolds $(U_\alpha, \mathcal{O}_\alpha) \cong (U_\alpha, \mathcal{C}^\infty(U_\alpha, \Lambda\mathbf{S}^*))$ We will use the following

LEMMA 3.3. *Let* $\iota, \iota'\colon \mathcal{C}^\infty_\alpha \to \mathcal{C}^\infty_\alpha \otimes \Lambda\mathbf{S}^*$. *Then* $\iota \underset{2k}{\sim} \iota'$ *if and only if* $\mathrm{im}((\iota - \iota')\mathcal{C}^\infty_\alpha) \subseteq \mathcal{C}^\infty_\alpha \otimes \Lambda^{\geq 2k+2}\mathbf{S}^*$; *in this case there exists a vector field X on* U_α *with values in* $\Lambda^{2k+2}\mathbf{S}^*$ (*exact degree*) *such that* $(\iota - \iota')f \equiv X(f) \bmod \mathcal{C}^\infty_\alpha \otimes \Lambda^{\geq 2k+4}\mathbf{S}^*$.

Proof. The proof of the equivalence is trivial from the definition and the fact that both ι and ι' are even maps. Let f and g be smooth functions on M. Using the fact that both ι and ι' are unital algebra homomorphisms, it is straightforward to compute

$$(\iota-\iota')(fg)=\iota'(f)(\iota-\iota')(g)+(\iota-\iota')(f)\iota(g).$$

Therefore if we define $X(f)=(\iota-\iota')(f) \bmod \mathcal{C}^\infty\otimes\Lambda^{\geq 2r+4}\mathbf{S}^*$, we immediately see that X is the required vector field since it satisfies $X(fg)\equiv fX(g)+X(f)g \bmod \mathcal{C}^\infty_\alpha\otimes\Lambda^{\geq 2k+4}\mathbf{S}^*$. ■

We will need the following simplified form of Theorem 1.22:

LEMMA 3.4. *Let U be an open set of $\mathbb{R}^d$ with coordinates $(y_1,\dots,y_d)$ and $\{\mathcal{Y}_1,\dots,\mathcal{Y}_d\}\subseteq\mathcal{C}^\infty\otimes\Lambda_+\mathbf{S}^*$ satisfy $\varepsilon(\mathcal{Y}_\mu)=y_\mu$ for $1\leq\mu\leq d$. Then there exists a unique unital morphism $\iota\colon\mathcal{C}^\infty\to\mathcal{C}^\infty\otimes\Lambda\mathbf{S}^*$ such that $\iota(y_\mu)=\mathcal{Y}_\mu$.*

Proof. It suffices to define $\Phi^+(y_\mu)=\mathcal{Y}_\mu-y_\mu$ in formula (1.14). ■

COROLLARY 3.5. *Let $\iota,\iota'\colon\mathcal{C}^\infty\to\mathcal{C}^\infty\otimes\Lambda\mathbf{S}^*$, and let $\mathcal{Y}_\mu$ and $\mathcal{Y}'_\mu$ denote the corresponding even superfunctions. Then $\iota \underset{2k}{\sim} \iota'$ if and only if $\mathcal{Y}_\mu\equiv\mathcal{Y}'_\mu \bmod \Lambda^{\geq 2k+2}$ for $1\leq\mu\leq d$.*

Proof. Set $X=\sum_{\lambda=1}^d(\mathcal{Y}'_\lambda-\mathcal{Y}_\lambda)\frac{\partial}{\partial y_\lambda}$; by Lemma 3.3 the result follows. ■

LEMMA 3.6. *For all $r\geq 0$ there exist $\{\iota_{\alpha,r}\}_{\alpha\in\mathbf{A}}$ where $\iota_{\alpha,r}\colon\mathcal{C}^\infty_\alpha\to\mathcal{C}^\infty_\alpha\otimes\Lambda\mathbf{S}^*$ is such that*

$$\tau_{\alpha\beta}\circ\iota_{\alpha,r}-\iota_{\beta,r}\equiv 0 \bmod \mathcal{C}^\infty_\alpha\otimes\Lambda_+^{\geq 2r+2}\mathbf{S}^*. \tag{3.1}$$

Proof. The proof is by induction on r. By condition (b) we can use Lemma 3.4 since we are working with the supermanifolds $(U_\alpha|U_\alpha\times\mathbb{R})$ and $(U_\alpha|U_\alpha\times\Lambda\mathbf{S}^*)$. Also, in using Lemma 3.3 we will identify U_α with its image in $\mathbb{R}^m$ and we will freely use this identification to construct the ι's by considering the images of the generators y^α_μ. Recall that $\tau_{\alpha\beta}=\tau_\alpha\circ{\tau_\beta}^{-1}$; this is an algebra automorphism of $\mathcal{C}^\infty_{\alpha\beta}\otimes\Lambda\mathbf{S}^*$.

Set $\iota_{\alpha,0}(f)=f\otimes 1$; this is obviously the simplest inclusion of $\mathcal{C}^\infty_\alpha$ into $\mathcal{O}_\alpha$.

Let us state the inductive hypothesis from $r-1$ to r: for $r\geq 1$ define

$$X^{(r)}_{\alpha\beta}=\tau_{\alpha\beta}\circ\iota_{\beta,r-1}-\iota_{\alpha,r-1} \bmod \mathcal{C}^\infty_{\alpha\beta}\otimes\Lambda^{\geq 2r+2}\mathbf{S}^*. \tag{3.2}$$

By Lemma 3.3 we can find $\iota_{\alpha,r+1}$ such that $\iota_{\alpha,r}-\iota_{\alpha,r+1}\equiv X^{(r)}_{\alpha\beta} \bmod \Lambda^{\geq 2r+4}\mathbf{S}^*$. Then for all α,β,γ we have

$$\begin{aligned}\tau_{\alpha\gamma}\circ X^{(r)}_{\alpha\beta}&\equiv X^{(r)}_{\gamma\beta}-X^{(r)}_{\gamma\alpha} \bmod \Lambda_+^{\geq 2r+4}\\ &=\tilde\tau_{\alpha\beta}(X^{(r)}_{\alpha\beta})\quad\text{(equality, not congruence)}\end{aligned}$$

where $\tilde\tau$ is the odd differential (2.5), and therefore

$$X^{(r)}_{\alpha\beta}=-X^{(r)}_{\beta\alpha}. \tag{3.3}$$

The important fact here is that $\tilde\tau_{\alpha\beta}$ is actually $\mathcal{C}^\infty_{\alpha\beta}$-linear. Choose a partition of unity $\{\psi_\alpha\}_{\alpha\in\mathbf{A}}$ whose supports form a locally finite subcover of $\{U_\alpha\}_\alpha$.

Now we set

$$X_\alpha^{(r)} = \sum_{\gamma\in\mathbf{A}} \psi_\gamma X_{\alpha\gamma}^{(r)}$$

and note it can be extended by 0 to the rest of U_α. Define

$$\iota_{\alpha,r} = \iota_{\alpha,r-1} + X_\alpha^{(r-1)}. \tag{3.4}$$

Now we compute

$$\begin{aligned}\tau_{\alpha\beta}\circ\iota_{\beta,r} - \iota_{\alpha,r} &\equiv \tau_{\alpha\beta}(\iota_{\beta,r-1} + X_\beta^{(r-1)}) - (\iota_{\alpha,r-1} + X_\alpha^{(r-1)}) \bmod \Lambda^{\geq 2r+2}\mathbf{S}^*\\ &\equiv X_{\alpha\beta}^{(r-1)} + \tilde\tau_{\alpha\beta}(X_\beta^{(r-1)}) - X_\alpha^{(r-1)} \bmod \Lambda^{\geq 2r+2}\mathbf{S}^*\\ &= X_{\alpha\beta}^{(r-1)} + \sum_{\gamma\in\mathbf{A}}\psi_\gamma(\tilde\tau_{\alpha\beta}(X_{\beta\gamma}^{(r-1)}) - X_{\alpha\gamma}^{(r-1)})\\ &= X_{\alpha\beta}^{(r-1)} + \sum_{\gamma\in\mathbf{A}}\psi_\gamma(X_{\alpha\gamma}^{(r-1)} - X_{\alpha\beta}^{(r-1)} - X_{\alpha\gamma}^{(r-1)})\end{aligned}$$

by the remarks immediately before identity (3.3). Now observe that the term in the last parentheses is simply $-X_{\alpha\beta}^{(r-1)}$, and thus the result is 0; so under definition (3.4) condition (3.1) is immediately satisfied for all $r \geq 0$, remembering $\iota_{\alpha,0}(f) = f\otimes 1$ and using Lemma 3.4 to recursively construct even superfunctions that project to the coordinates. ∎

To finish the proof of Theorem 3.1 we construct for each $r \geq 1$ the inclusions $\iota_{\alpha,r}$; then, by Theorem 1.22, we can extend the algebraic homomorphisms $\iota_{\alpha,r}$ to smooth supermaps $(\mathrm{id}|\iota_{\alpha,r})\colon (U_\alpha, \mathcal{C}_\alpha^\infty) \to (U_\alpha, \mathcal{C}_\alpha^\infty\otimes\Lambda\mathbf{S}^*)$. As observed before, if $R \geq \lfloor n/2\rfloor$ we get the identity

$$\tau_{\alpha\beta}\circ\iota_{\beta,R} = \iota_{\alpha,R}. \tag{3.5}$$

Then we can define $j_\alpha\colon \mathcal{C}_\alpha^\infty \to \mathcal{O}_\alpha$ by $j_\alpha(f) = \iota_{\alpha,R}(f)$ for a minimum such R; this map yields a well defined section of the sheaf $\mathcal{O}$ since by (3.5) the local sections $j_\alpha(f)$ can be consistently glued to a global one, by which we define $j(f)$. Since Theorem 3.1 is equivalent to Theorem 1.5, the latter is completely proved. ∎

Now our claim that our approach yields the same theory as the KL approach is completely established by considering Theorem 1.24.

4. Straightenings. A reasonable question to ask in view of Definition 2.7 and Lemma 2.8 is whether or not there are any "higher order" auxiliary differentials associated to the supermanifold morphism $(\phi|\Phi)$. Since Φ preserves the filtration (1.7), it seems possible that there is a map

$$\Phi^{\mathrm{aux}} : \Gamma(\mathbf{S}^*N) \to \Gamma(\mathcal{R}^{\geq 3}M/\mathcal{R}^{\geq 4}M) \cong \Gamma(\Lambda^3\mathbf{S}^*M). \tag{4.1}$$

Let us analyse this possibility. First note that (2.7) is well defined because the commutator $[\Phi_\phi f]$ has a class modulo $\Gamma(\mathcal{R}^{\geq 3}M)$ independent of any isomorphism between the vector bundles $\mathcal{R}^{\geq 2}M/\mathcal{R}^{\geq 3}M$ and $\Lambda^2\mathbf{S}^*M$; this in turn is due to the fact that the map ε gives a natural "truncation" of $\Phi(f)$ to its non-nilpotent part. When trying to construct a map Φ^{aux} as above we see that if $\sigma \in \Gamma(\mathbf{S}^*N)$ then $\Phi(\sigma)$ is a section of $\mathcal{R}^{\geq 1}M$. So in order to get a section of $\Lambda^3\mathbf{S}^*M$, an isomorphism is needed that chooses an appropriate class for $\Phi(\sigma)$. This isomorphism in turn depends on some other choices:

PROPOSITION 4.1. *If $\Psi_k : \Lambda^k\mathbf{S}^*M \to \mathcal{R}^{\geq k}M/\mathcal{R}^{\geq k+1}M$ is a vector bundle isomorphism for all non-negative integers k then there exists an algebra bundle isomorphism $\Psi : \Lambda\mathbf{S}^*M \to \mathcal{R}M$ such that $\Psi_k = \Psi|_{\Lambda^k\mathbf{S}^*M}$.*

Proof. Choose a connection on $\mathbf{S}M$; this gives a local basis $\{s_1, \dots, s_n\}$ for this bundle; then using the dual basis $\{ds_1, \dots, ds_n\}$ extend to an algebra isomorphism via $ds_{\mu_1} \wedge \cdots \wedge ds_{\mu_k} \mapsto ds_{\mu_1} \cdots ds_{\mu_k} + \Gamma(\mathcal{R}^{\geq k+1}M)$. ∎

Hence, in order to consistently define a class for $\Phi(\sigma)$ we need a bundle isomorphism $\Psi\colon \Lambda\mathbf{S}^* \to \mathcal{R}M$ and a connection ∇ on $\mathbf{S}M$. These pairs are in correspondence with a very special class of maps:

THEOREM 4.2 (Flowbox coordinates for supermanifolds). *Let $(M|\mathcal{R}M)$ be a supermanifold; let*

- $\mathrm{Conn}(\mathbf{S}^*M)$ *denote the set of connections in the vector bundle of odd directions;*
- $\mathrm{Inc}(M|\mathcal{R}M)$ *denote the set $\{\psi \mid (TM \oplus \mathbf{S}M)_\bullet \hookrightarrow \mathrm{Der}_\bullet(\mathcal{R}M)\}$ of $\mathbb{Z}_2$-graded inclusions of the bundle of generators $TM \oplus \mathbf{S}M$ into the bundle of superderivations, and*
- $\mathcal{S}(M|\mathcal{R}M) = \{\Psi \mid \Lambda\mathbf{S}^*M \overset{\cong}{\to} \mathcal{R}M\}$, *the set of unital superalgebra bundle isomorphisms.*

There is a bijection of sets $\mathrm{Inc}(M|\mathcal{R}M) \times \mathrm{Conn}\,\mathbf{S}M \leftrightarrow \mathcal{S}(M|\mathcal{R}M)$ satisfying the following properties:

(1) *if $s, \widetilde{s}$ are sections of $\mathbf{S}M$ then $[\![\psi(s), \psi(\widetilde{s})]\!] = 0$;*
(2) *for any $s \in \Gamma(\mathbf{S}M)$ and any vector field X on M we have $[\![\psi(X), \psi(s)]\!] = \psi(\nabla_X s)$;*
(3) *if s is a section of $\mathbf{S}M$ and σ a section of $\Lambda\mathbf{S}^*M$ then $\Psi(s \lrcorner \sigma) = \psi(s)(\Psi(\sigma))$.*

Before proving the above theorem let us analyse the geometric meaning of the bijection. As we saw in Proposition 1.25, morphisms in the category of split supermanifolds are rather simple: they are just exterior bundle maps associated to vector bundle morphisms; this means that they are differential operators of order 0, or—what is the same–module morphisms $\Phi\colon \Gamma(\Lambda\mathbf{S}^*M) \to \Gamma(\Lambda\mathbf{S}^*N)$ covering a smooth map $\phi\colon N \to M$, which in turn provides a unital algebra homomorphism $\phi^*\colon \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(N)$. On the other hand, morphisms of supermanifolds are in general quite more complicated. So what the Flowbox Coordinates Theorem allows us to do is "straighten up" an arbitrary supermanifold $(M|\mathcal{R}M)$ into a split one. We thus call the mappings of the set $\mathcal{S}(M|\mathcal{R}M)$ *straightenings* of $(M|\mathcal{R}M)$.

Proof of Theorem 4.2. Recall that the bundle of derivations is locally isomorphic to the space $\mathcal{C}^\infty(U, \operatorname{sder} \Lambda\mathbf{S}^*)$ once a trivialisation is chosen on $\mathbf{S}U$. We observe that the linear map $\psi\colon (TU \oplus \mathbf{S}U)_\bullet \to \mathcal{C}^\infty(U, \operatorname{sder}_\bullet \Lambda\mathbf{S}^*)$ satisfies the hypotheses of Lemma 1.12: it is injective and the image of the odd part $\mathbf{S}U$ consists of supercommuting odd derivations. Thus this lemma gives us an isomorphism $\Psi\colon \Lambda\mathbf{S}^*U \to \mathcal{R}U$ with properties (1) and (3) of the statement of the Theorem. For property (2) we observe that $[\![\psi(X), \psi(s)]\!]$ is an odd supervector field and the result of the bracket operation is uniquely determined by X and s.

The uniqueness of ψ for a given (Ψ, ∇) is also guaranteed by Lemma 1.12. The theorem follows from a partition of unity argument. ■

Let a straightening ψ of $(M|\mathcal{R}M)$ be given. If $(\phi|\Phi)\colon (M|\mathcal{R}M) \to (N|\mathcal{R}N)$ then we have a morphism $(\phi|\widetilde{\Phi})\colon (M|\mathbf{S}M) \to (N|\mathcal{R}N)$ given by the following diagram:

$$\begin{array}{ccc} \Gamma(\mathcal{R}N) & \xrightarrow{\widetilde{\Phi}} & \Gamma(\Lambda\mathbf{S}^*M) \\ {\scriptstyle\Phi}\downarrow & \nearrow_{\Gamma(\Psi^{-1})} & \\ \Gamma(\mathcal{R}M) & & \end{array} \tag{4.2}$$

When dealing with the bundle $\Lambda\mathbf{S}^*M$ it is now possible to talk about the truncation of one of its sections: consider the map $\operatorname{tr}^k\colon \Lambda\mathbf{S}^*M \to \Lambda^{\leq k}\mathbf{S}^*M$ that truncates a given form to its part of degree at most k.

Now let $\sigma \in \Gamma(\mathbf{S}^*N)$ and consider the section $\widetilde{\Phi}(\sigma) \in \Gamma(\Lambda_-\mathbf{S}^*M)$, i.e. $\widetilde{\Phi}(\sigma)$ is of positive degree in the bundle $\Lambda\mathbf{S}^*M$. Taking the class of $\widetilde{\Phi}(\sigma) - \operatorname{tr}^1(\widetilde{\Phi}(\sigma))$ modulo $\Gamma(\Lambda^{\geq 5}\mathbf{S}^*M)$ we get a well defined map $\check{\Phi}^3\colon \mathbf{S}^*N \to \Lambda^3\mathbf{S}^*M$. If we continue in this manner we get bundle maps

$$\check{\Phi}^\bullet\colon (T^*N|\mathbf{S}^*N)_\circ \to \Lambda^\bullet_\circ\mathbf{S}^*M$$

where $\circ$ denotes the $\mathbb{Z}_2$-grading and $\bullet$ the $\mathbb{Z}$-grading. We then have

PROPOSITION 4.3. *Let $(\phi|\Phi)\colon (M|\mathcal{R}M) \to (N|\mathcal{R}N)$. The unital superalgebra morphism $\Phi\colon \Gamma(\mathcal{R}M) \to \Gamma(\mathcal{R}N)$ is a differential operator of order zero along the smooth map ϕ if and only if each $\check{\Phi}^k$ is zero for $k \geq 2$.*

Proof. The "if" part is rather trivial: observe that if $\Phi(f\eta) = (f \circ \phi)\eta$ then for $\sigma \in \Gamma(\mathbf{S}^*M)$ we know $\Phi(f\sigma) = (f \circ \phi)\sigma$ is a section of $\mathbf{S}^*M$; this is equivalent to $\check{\Phi}^3 \equiv 0$. Likewise, Lemma 2.8 tells us that $\Phi_! \equiv 0$ forces Φ to be a differential operator of order 0. Since, under the choice of a straightening, all sections of $\mathcal{R}N$ can be written as linear combinations of products of sections of $\mathbf{S}^*N$ and $\mathcal{C}^\infty(N)$, we get one implication by the above observations.

For the converse, let $\Phi\colon \Gamma(\mathcal{R}M) \to \Gamma(\mathcal{R}N)$ and recall that given a straightening of $\mathcal{R}M$ we get a map $\widetilde{\Phi}$ associated to Φ given by diagram (4.2). Suppose $\check{\Phi}^k \equiv 0$ for all $k \geq 2$. Choosing a straightening for $\mathcal{R}M$ we know $\check{\Phi}^k \equiv 0$ means that for any $\eta \in \Gamma(\mathcal{R}^{\geq k}N)$ the section $\check{\Phi}(\eta) = \widetilde{\Phi}(\eta) - \mathrm{tr}^{k-1}(\Phi(\eta)) = 0$ modulo $\Gamma(\Lambda^{k+1}\mathbf{S}^*M)$ for all k, so it is actually a section of $\Lambda^k\mathbf{S}^*M$. In particular $\widetilde{\Phi}$ maps sections of $\mathbf{S}^*N$ to sections of $\mathbf{S}^*M$. This means $[\Phi_\phi f]$ preserves the grading of $\Lambda\mathbf{S}^*N$ (choosing a straightening for $\mathcal{R}N$ if necessary), and is thus identically zero for all f; that is, Φ is a differential operator of order zero. ■

Thus we have seen that any "higher-order auxiliary differentials" of

$$(\phi|\Phi)\colon (M|\mathcal{R}M) \to (N|\mathcal{R}N)$$

are codified by the set of straightenings of $(N|\mathcal{R}N)$, albeit the property of being a bundle map (equivalently, a differential operator of order zero) is independent of any straightening.

Appendix A. Linear differential operators

A.1. Basic concepts

Definition A.1. Let E and F be the total spaces of two smooth vector bundles over a smooth manifold M. A *linear differential operator of order* $n \geq 0$ is an $\mathbb{R}$-linear map

$$D : \Gamma(E) \to \Gamma(F)$$

such that the commutator

$$[D, f](\psi) := D(f\psi) - fD(\psi) \tag{A.1}$$

with $f \in \mathcal{C}^\infty(M)$ and $\psi \in \Gamma(E)$ is zero when iterated with $f_0, \dots, f_n$ smooth functions on M.

Observe that a differential operator of order 0 is $\mathcal{C}^\infty(M)$-linear and is therefore a bundle morphism.

Lemma A.2. *Let $\{f_1, \dots, f_k\}$ be smooth functions on M, $k > 1$, and set $K := \{1, \dots, k\}$. The iterated commutator of D and the above k smooth functions is given by*

$$[\dots[[D, f_1], f_2], \dots, f_k](\eta) = \sum_{A \subseteq K} (-1)^{\#(A)} f_A D(f_{K-A}\eta), \tag{A.2}$$

where η is any section of E, $\#(A)$ is the cardinality of A and we set

$$f_A = \begin{cases} \prod_{a \in A} f_a, & A \neq \emptyset, \\ 1, & A = \emptyset. \end{cases}$$

Proof. The proof is by induction on k. If $k = 2$ then we compute

$$\begin{aligned}[[D, f_1], f_2](\eta) &= [D, f_1](f_2\eta) - f_2[D, f_1](\eta)\\ &= D(f_1 f_2\eta) - f_1 D(f_2\eta) - f_2 D(f_1\eta) + f_1 f_2 D(\eta)\\ &= \sum_{A\subseteq\{1,2\}} (-1)^{\#(A)} f_A D(f_{\{1,2\}-A}\eta).\end{aligned}$$

Now suppose that (A.2) holds for all integers $l < k$. Let $L := \{1, \ldots, k-1\}$. We compute

$$\begin{aligned}[[\cdots[[D, f_1], f_2], \ldots, f_{k-1}], f_k](\eta) = &\sum_{A\subseteq L} (-1)^{\#(A)} f_A D(f_{L-A} f_k\eta)\\ &- \sum_{A\subseteq L} (-1)^{\#(A)} f_A f_k D(f_{L-A}\eta).\end{aligned}$$

If $A \subseteq L$ is non-empty and we set $A = \{\mu_1, \ldots, \mu_r\}$ and $L - A = \{\nu_1, \ldots, \nu_s\}$, then the above formula is equivalent to

$$\begin{aligned}&[[\cdots[[D, f_1], f_2], \ldots, f_{k-1}], f_k](\eta)\\ &\quad= \sum_{r+s=k-1} (-1)^r f_{\mu_1}\cdots f_{\mu_r} D(f_{\nu_1}\cdots f_{\nu_s} f_k\eta)\\ &\qquad - \sum_{r+s=k-1} (-1)^r f_{\mu_1}\cdots f_{\mu_r} f_k D(f_{\nu_1}\cdots f_{\nu_s}\eta)\\ &\qquad - f_k D(f_1\cdots f_{k-1}\eta) + D(f_1\cdots f_k\eta)\\ &\quad= \sum_{A\subseteq K} (-1)^{\#(A)} f_A D(f_{K-A}\eta),\end{aligned} \tag{A.3}$$

where $K := \{1, \ldots, k\}$, since $-(-1)^{k-1} D(f_1\cdots f_k\eta) = (-1)^k D(f_1\cdots f_k\eta)$. ■

We keep the notation $K = \{1, \ldots, k\}$ in the proof below.

Proposition A.3. *If $f_1, \ldots, f_k$ are smooth functions on M then*

$$[\cdots[[D, f_1], f_2], \ldots, f_k] = [\ldots[[D, f_{\sigma(1)}], f_{\sigma(2)}], \ldots, f_{\sigma(k)}]$$

for any permutation $\sigma \in S_k$.

Proof. The action of S_k on $\mathcal{P}(K)$ (the power set of K) is transitive on subsets of a given cardinality, therefore formula (A.2) is invariant under permutations of $\{f_1, \ldots, f_k\}$. ■

Remark A.4. The above proposition allows us to write $[D; f_1, \ldots, f_k]$ for the iterated commutator.

Definition A.5. Let $D : \mathit{\Gamma}(E) \to \mathit{\Gamma}(F)$ be a linear differential operator of order $k \geq 0$ and let $\{f_1, \ldots, f_k\}$ be a set of smooth functions on M. The *principal symbol* of D evaluated on $f_1\cdots f_k$ is

$$\sigma_D(f_1\cdots f_k) := [D; f_1, \ldots, f_k] \tag{A.4}$$

Since the commutation bracket is symmetric and linear in the smooth functions we get the following:

PROPOSITION A.6. *The principal symbol of a linear differential operator D of order $k \geq 0$ is a section of* $\operatorname{Sym}^k T^*M \otimes E^* \otimes F$.

A.2. Jets of sections. We now discuss the concept of jets. These generalise, in local coordinates at least, the Taylor polynomials of smooth transformations; their importance, nevertheless, lies in the fact that the bundle of jets is a universal construction for factoring smooth differential operators. Our setting is a smooth vector bundle E over a smooth manifold M.

DEFINITION A.7. Let k be a non-negative integer and p a point of M. Two sections η and $\bar{\eta}$ are in *contact up to order k at p* if given a bundle chart (U, x) around p we have

$$\partial_p^\mu(\eta) = \partial_p^\mu(\bar{\eta}), \quad |\mu| \leq k, \quad \partial_p^\mu := \left(\frac{\partial^{|\mu|}}{\partial x^\mu}\right)\bigg|_p .$$

We denote this relation by $\eta \sim_{k,p} \bar{\eta}$.

It is clear that this relation is independent of the bundle coordinates and that it is an equivalence relation. The equivalence class of a section η is called the *k-jet at p* of η and it is denoted by $\operatorname{jet}_p^k(\eta)$. If η and $\bar{\eta}$ have the same k-jet at p then their difference, when written down in local coordinates, is a homogeneous polynomial of degree $k+1$ in those coordinates. So the expression of $\operatorname{jet}_p^k(\eta)$ is the Taylor polynomial of η in coordinates around p. It is in this sense that jets are generalisations of Taylor polynomials.

We now define the *space of k-jets at p* as the space

$$\operatorname{Jet}_p^k(E) = \Gamma(E)/{\sim_{k,p}}, \tag{A.5}$$

and of course the *bundle of k-jets of E* as the vector bundle

$$\operatorname{Jet}^k(E) = \bigsqcup_{p \in M} \operatorname{Jet}_p^k(E). \tag{A.6}$$

Each of these bundles carries a natural map $\operatorname{ev}\colon \operatorname{Jet}^k(E) \to E$ given by $\operatorname{ev}(\operatorname{jet}_p^k \eta) = \eta(p)$. We use a special notation for jets of smooth functions:

$$\operatorname{Jet}^k(M) := \operatorname{Jet}^k(M \times \mathbb{R}). \tag{A.7}$$

We now give an invariant characterisation of jet spaces:

PROPOSITION A.8. *Let I_p be the ideal of $\mathcal{C}^\infty(M)$ consisting of functions that vanish at p. Then* $\operatorname{Jet}_p^k(E) \cong \Gamma(E)\big/\big(I_p^{k+1} \cdot \Gamma(E)\big)$.

This is a consequence of the following result, which can be found in standard books on vector calculus:

Lemma A.9 (Taylor formula with remainder). *Let $h : \mathbb{R}^m \to \mathbb{R}^n$ be smooth. The Taylor polynomial of degree k of h around 0 is given by*

$$h(x^1, \dots, x^m) = \sum_{r=0}^{k} \frac{1}{r!} \sum_{\mu_1,\dots,\mu_r=1}^{m} x^{\mu_1} \dots x^{\mu_r} \frac{\partial^r h}{\partial x^{\mu_1} \dots \partial x^{\mu_r}}(0, \dots, 0)$$
$$+ \frac{1}{k!} \sum_{\mu_0,\dots,\mu_k=1}^{m} x^{\mu_0} \dots x^{\mu_k} H_{\mu_0,\dots,\mu_k}$$

where $H_{\mu_0,\dots,\mu_k} = \int_0^1 \frac{\partial^{k+1} h}{\partial x^{\mu_0} \dots \partial x^{\mu_k}}(tx^1, \dots, tx^m)(1-t)^k \, dt$.

Proof of Proposition A.8. Let $f_1 \cdots f_{k+1}\eta$ be an element of $I_p^{k+1} \cdot \Gamma(E)$ and choose a trivialisation of E around p with coordinates $(x_1, \dots, x_m)$. If $l \leq k+1$ then the generalised Leibniz identity for the product $f_1 \cdots f_{k+1}$ implies

$$\frac{\partial^L}{\partial x^L}(f_1 \cdots f_{k+1})\eta = 0$$

for all multi-indices L of length l, because at least one of the functions $f_1, \dots, f_{k+1}$ will appear in the expansion for the derivatives evaluated at p, where they all vanish. So we get $f_1 \cdots f_{k+1}\eta \sim_{k+1,p} 0$, and therefore $\mathrm{jet}_p^k(f_1 \cdots f_{k+1}\eta) = 0$. Let $\psi \colon \Gamma(E)/(I_p^{k+1} \cdot \Gamma(E)) \to \mathrm{Jet}_p^k(E)$ be defined as $\psi(\eta + I_p^{k+1} \cdot \Gamma(E)) = \mathrm{jet}_p^k(\eta)$. Then ψ is well defined by the coordinate-independence of $\mathrm{jet}_p^k(\eta)$, it is evidently $\mathbb{R}$-linear, and it is surjective because of the definition of $\mathrm{Jet}_p^k(E)$. To see that it is injective, note that any section η, when written in local coordinates, is a smooth function between open sets of vector spaces. By the Taylor formula above we know $\psi(\eta + I_p^{k+1} \cdot \Gamma(E)) = 0$ if and only if the Taylor polynomial vanishes up to order k at p, and therefore $\mathrm{jet}_p^k(\eta + I_p^{k+1} \cdot \Gamma(E)) = 0$ if and only if $\eta \equiv 0 \bmod I_p^{k+1}$; so ψ is injective. ∎

Corollary A.10. *The map $\mathrm{jet}^k : \Gamma(E) \to \Gamma(\mathrm{Jet}^k E)$ assigning to every section η its k-jet $\mathrm{jet}^k(\eta)$ pointwise, is a differential operator of order k.*

Proof. We must show that for given smooth functions $f_0, \dots, f_k$ on M,

$$[\mathrm{jet}^k; f_0, \dots, f_k] \equiv 0.$$

Formula (A.2) in this case turns into

$$[\mathrm{jet}^k; f_0, \dots, f_k](\eta) = \sum_{A \subseteq K} (-1)^{\#(A)} f_A \, \mathrm{jet}^k(f_{K-A}\eta)$$

where K stands for $\{0, \dots, k+1\}$. By the previous proposition we know $\mathrm{jet}^k(f_{K-A}\eta) = f_{K-A}\eta + I_p^{k+1} \cdot \Gamma(E)$, so the summands in the formula above are equal to permutations of $f_0 \cdots f_k\eta$ times a sign; thus the formula is equivalent to

$$[\mathrm{jet}^k; f_0, \ldots, f_k] = \sum_{\sigma \in S_{k+1}} \operatorname{sgn}\sigma \cdot f_{\sigma(0)} \cdots f_{\sigma(k)}\eta + I_p^{k+1} \cdot \Gamma(E).$$

Since all the f's commute and each product appears with a plus sign the same number of times it appears with a minus sign, the above quantity vanishes identically. Thus jet^k is a differential operator of order k. ∎

The advantage of the bundle of jets is that it is a space that universally factorizes differential operators.

THEOREM A.11. *Let $D : \Gamma(E) \to \Gamma(\widetilde{E})$ be a differential operator of order k. Then there is a unique bundle morphism $\widehat{D} : \mathrm{Jet}^k E \to \widetilde{E}$ such that*

$$\begin{array}{ccc} \Gamma(E) & \xrightarrow{\;D\;} & \Gamma(\widetilde{E}) \\ {\scriptstyle \mathrm{jet}^k}\downarrow & \nearrow {\scriptstyle \Gamma(\widehat{D})} & \\ \Gamma(\mathrm{Jet}^k E) & & \end{array}$$

commutes, where $\Gamma(\widehat{D})$ is the associated morphism.

Proof. Since jet^k is a differential operator of order k, we have to prove that $\mathrm{jet}_p^k \eta \mapsto D(\eta)_p = \mathrm{ev}(\mathrm{jet}_p^k(D\eta))$ is a bundle morphism; that is, if f is a smooth function on M we should get $\widehat{D}(\mathrm{jet}^k(f\eta)) = fD(\eta)$. Indeed, with the latter definition of $\widehat{D}$,

$$\begin{aligned}\widehat{D}(\mathrm{jet}^k(f\eta))_p &= D(f\eta)_p + I_p^{k+1}\Gamma(\tilde{E}) = f(p)D(\eta)_p + I_p^{k+1}\Gamma(\tilde{E}) \\ &= \mathrm{jet}_p^k(f(p)D(\eta)) = f(p)\,\mathrm{jet}_p^k(D(\eta)),\end{aligned}$$

and the last section satisfies $\mathrm{ev}(f(p)\,\mathrm{jet}_p^k(D\eta)) = f(p)D(\eta)_p$, which is exactly the definition of $\widehat{D}$. ∎

DEFINITION A.12. Let D be a differential operator of order k. The map $\widehat{D}$ of Theorem A.11 is called the *total symbol* of D. It is denoted by $\sigma^{\mathrm{total}}(D)$.

A.3. Jets and Taylor polynomials. Since jets are generalisations of Taylor polynomials it is natural to look for an expression of jets that reflects this fact. To do so let ∇ be a connection on the vector bundle E, and D a torsion-free connection on the tangent bundle of M.

DEFINITION A.13. The iterated covariant derivatives of a section η of E are given recursively by $\nabla^0\eta = \eta$ and

$$\nabla^{k+1}_{X_0,\ldots,X_k}\eta := \nabla_{X_0}(\nabla^k_{X_1,\ldots,X_k}\eta) - \sum_{\mu=1}^{k} \nabla^k_{X_1,\ldots,D_{X_0}X_\mu,\ldots,X_k}\eta. \tag{A.8}$$

Using the iterated covariant derivatives we define an operator

$$J^{\nabla,D,l}_{X_1\cdots X_l}(\eta) = \frac{1}{l!}\sum_{\sigma\in S_l} \nabla^l_{X_{\sigma(1)},\dots,X_{\sigma(l)}}(\eta), \tag{A.9}$$

which is symmetric in the vector field arguments. We therefore have $J^{\nabla,D,l} \in \Gamma(\mathrm{Sym}^{\le l} T^*M \otimes E)$. Here $\mathrm{Sym}^{\le l} T^*M$ denotes the space

$$\mathrm{Sym}^0 T^*M \oplus \mathrm{Sym}^1 T^*M \oplus \cdots \oplus \mathrm{Sym}^l T^*M.$$

It is clear that this is a "polynomial" in the vector fields, since it is a symmetric form on them.

Proposition A.14. *The map* $\mathrm{jet}^k \eta \mapsto J^{\nabla,D,0}\eta + J^{\nabla,D,1}\eta + \cdots + J^{\nabla,D,k}\eta$ *is a linear isomorphism of bundles* $\Psi : \mathrm{Jet}^k E \to \mathrm{Sym}^{\le k}(T^*M) \otimes E$.

Proof. This is just the Taylor formula for the connections ∇ and D, since any connection gives a trivialisation when properly restricted to an open subset of the base manifold. Since Taylor polynomials are uniquely determined by both their jets and their expression in local coordinates, the result follows. ■

Recall from (A.7) that $\mathrm{Jet}^k(M)$ denotes the jet bundle of smooth functions on M.

Corollary A.15. *There is a natural isomorphism of bundles* $\mathrm{Jet}^1(M) \cong \mathbb{R} \times T^*M$.

Proof. The isomorphism is clear. The naturality comes from the fact that $M \times \mathbb{R}$ has a natural connection, namely the exterior derivative. ■

In order to avoid confusion we henceforth switch to denoting a differential operator by L instead of D, letting the latter denote a torsion-free connection on the tangent bundle of the base manifold.

Definition A.16. The *polynomial total symbol* of a differential operator $L : \Gamma(E) \to \Gamma(\widetilde{E})$ of order k is the polynomial

$$\sigma^{\mathrm{total}}(L,\cdot) \in \Gamma(\mathrm{Sym}^{\le k} T^*M \otimes E^* \otimes \widetilde{E})$$

associated to D via the isomorphism of Proposition A.14.

That is, the total symbol is a section of $\widetilde{E}$ of the form

$$\sigma^{\mathrm{total}}(L, X_1\cdots X_k)(\eta) = \sum_{\mu=0}^{k}\sum_{\tau\in S_\mu} J^{\nabla,D,\mu}_{X_{\tau(1)}\cdots X_{\tau(\mu)}}(\eta)$$

Appendix B. The composition algebra. In this appendix we use the Cartan–Poincaré Lemma to prove Lemma 1.12. We do so by associating to an injective linear map $f\colon V \to \mathbf{S}$ an algebra very similar to the ones in which the Cartan–Poincaré operators act.

Definition B.1. Let $\mathbf{S}$ be a finite-dimensional vector space. The *composition algebra* of $\mathbf{S}$ is

$$\Lambda \mathbf{S}^* \otimes \mathbf{S} \tag{B.1}$$

with product defined by $(\omega \otimes s) \cdot (\widetilde{\omega} \otimes \widetilde{s}) = \omega \wedge (s \lrcorner \widetilde{\omega}) \otimes \widetilde{s}$.

It is quite evident that this product is not associative. Nevertheless it has a very interesting property, as a consequence of Theorem 1.11:

Lemma B.2. *For elements $\sigma \otimes s$ and $\hat{\sigma} \otimes \hat{s}$ of the composition algebra define*

$$[\sigma \otimes s, \hat{\sigma} \otimes \hat{s}] = (\sigma \otimes s) \cdot (\hat{\sigma} \otimes \hat{s}) + (-1)^{(\lceil\sigma\rfloor+1)(\lceil\hat{\sigma}\rfloor+1)} (\hat{\sigma} \otimes \hat{s}) \cdot (\sigma \otimes s). \tag{B.2}$$

Then, under the isomorphism $\Psi\colon \Lambda_\bullet \mathbf{S}^* \otimes \mathbf{S} \to \mathrm{sder}_{-\bullet} \Lambda \mathbf{S}^*$ given by $\Psi(\sigma \otimes s)(\omega) = \sigma \wedge (s \lrcorner \omega)$, the Lie superbracket on $\mathrm{sder}\, \Lambda \mathbf{S}^*$ corresponds to the bracket operation above.

Proof. Let $\omega \in \Lambda \mathbf{S}^*$. We compute

$$\begin{aligned}
[\sigma \otimes s, \hat{\sigma} \otimes \hat{s}]\omega &= \sigma \wedge \big(s \lrcorner (\hat{\sigma} \wedge \hat{s} \lrcorner \omega)\big) + (-1)^{(\lceil\sigma\rfloor+1)(\lceil\hat{\sigma}\rfloor+1)} \hat{\sigma} \wedge (\hat{s} \lrcorner \sigma \wedge s \lrcorner \omega) \\
&= \sigma \wedge (s \lrcorner \hat{\sigma}) \wedge (\hat{s} \lrcorner \omega) + (-1)^{\lceil\hat{\sigma}\rfloor} \sigma \wedge \hat{\sigma} \wedge s \lrcorner \hat{s} \lrcorner \omega \\
&\quad - (-1)^{(\lceil\hat{\sigma}\rfloor+1)(\lceil\sigma\rfloor+1)} \hat{\sigma} \wedge (\hat{s} \lrcorner \sigma) \wedge s \lrcorner \omega \\
&\quad + (-1)^{\lceil\sigma\rfloor\lceil\hat{\sigma}\rfloor+\lceil\hat{\sigma}\rfloor} \hat{\sigma} \wedge \sigma \wedge \hat{s} \lrcorner s \lrcorner \omega \\
&= \big(\sigma \wedge (s \lrcorner \hat{\sigma}) \otimes \hat{s} - (-1)^{(\lceil\sigma\rfloor+1)(\lceil\hat{\sigma}\rfloor+1)} \hat{\sigma} \wedge (\hat{s} \lrcorner \sigma) \otimes s\big)\omega,
\end{aligned}$$

which is exactly the formula for the action of elements $\sigma \otimes s$ as derivations of $\Lambda \mathbf{S}^*$. ■

Note that we can write $(\sigma \otimes s) \cdot (\hat{\sigma} \otimes \hat{s}) = ((\sigma \otimes s) \star \hat{\sigma}) \otimes \hat{s}$, where $\star$ denotes the action of $\Lambda \mathbf{S}^* \otimes \mathbf{S}$ as derivations of the exterior algebra.

Let now $G\colon \mathbf{S}^* \to \Lambda \mathbf{S}^*$ be a linear map; it can obviously be interpreted as an element of the composition algebra. Then, given a derivation D of the exterior algebra, we can form the product

$$D \cdot G \in \Lambda \mathbf{S}^* \otimes \mathbf{S}$$

in the following way: if $\{s_1, \ldots, s_n\}$ is a basis of $\mathbf{S}$ and $\{ds_1, \ldots, ds_n\}$ is the dual basis, then

$$D \cdot G = \sum D(G_{ds_k}) \otimes s_k = D \cdot \Big(\sum G_{ds_k} \otimes s_k\Big) \tag{B.3}$$

where $\sum G_{ds_k} \otimes s_k$ is the element in the composition algebra corresponding to G.

Now let $D\colon V \to \mathrm{sder}_{-} \Lambda \mathbf{S}^*$ be a linear map. In order to use the multiplication of the composition algebra for such a map, we consider it as an element of the algebra $\mathrm{Sym}\, V^* \otimes \Lambda \mathbf{S}^* \otimes \mathbf{S}$ of polynomials in V with values in the composition algebra. The multiplication of the latter algebra is given by

$$(p \otimes \sigma \otimes s) \cdot (q \otimes \hat{\sigma} \otimes \hat{s}) = pq \otimes \sigma \wedge (s \lrcorner \hat{\sigma}) \otimes \hat{s},$$

so given a map D as above, the product $D \cdot D$ can be thought of as a polynomial with values in the even derivations of $\Lambda\mathbf{S}^*$.

LEMMA B.3. *Suppose $D : V \to \operatorname{sder}_- \Lambda\mathbf{S}^*$ is a linear map. Then $[D_v, D_{\widetilde{v}}] = 0$ if and only if $D_v \cdot D_{\widetilde{v}} = 0$ in the algebra $\operatorname{Sym} V^* \otimes \Lambda\mathbf{S}^* \otimes \mathbf{S}$ of polynomials in V with values in the composition algebra.*

Proof. This is just a consequence of polarization:

$$\begin{aligned}(D \cdot D)_{v+\widetilde{v}} + (D \cdot D)_{v-\widetilde{v}} &= \tfrac{1}{4}([D_v, D_v] + [D_{\widetilde{v}}, D_v] + [D_v, D_{\widetilde{v}}] + [D_{\widetilde{v}}, D_{\widetilde{v}}] \\ &\qquad - [D_v, D_v] + [D_{\widetilde{v}}, D_v] + [D_v, D_{\widetilde{v}}] + [D_{\widetilde{v}}, D_{\widetilde{v}}]) \\ &= [D_v, D_{\widetilde{v}}]. \ \blacksquare\end{aligned}$$

Finally, we recall that $\operatorname{sder} \Lambda\mathbf{S}^*$ is a $\Lambda\mathbf{S}^*$-module freely generated by $\mathbf{S}$. So to the projection map $\operatorname{pr}\colon \Lambda\mathbf{S}^* \to \mathbf{S}^*$ corresponds a map

(B.4) $$\operatorname{pr}\colon \operatorname{sder} \Lambda\mathbf{S}^* \to \mathbf{S}$$

from the supermodule of superderivations to its space of generators.

We now restate and prove

LEMMA 1.12. *Let $\mathcal{A}$ be a free supercommutative finite-dimensional superalgebra and denote by $\mathbf{S}^*$ its space of generators. Let $D : V \to \operatorname{der}_- \mathcal{A}$ be a linear map such that the composition*

$$f : V \xrightarrow{D} \operatorname{sder}_- \mathcal{A} \xrightarrow{\operatorname{pr}} \mathbf{S}$$

is injective and such that if $v, \widetilde{v}$ are in V then $[D_v, D_{\widetilde{v}}] = 0$. Then there exists an isomorphism $G : \Lambda\mathbf{S}^* \to \mathcal{A}$ of $\mathbb{Z}_2$-graded algebras with unit such that G induces the identity

$$\overline{G} : \mathbf{S}^* \to \mathcal{A}^{\geq 1}/\mathcal{A}^{\geq 2} =: \mathbf{S}^*$$

and such that, for all $v \in V$ and all $\sigma \in \Lambda\mathbf{S}^*$,

(B.5) $$D_v(G\sigma) = G(f(v) \lrcorner \sigma).$$

Furthermore, up to the ideal generated by $\Lambda^3 \ker(f^*)$ in $\Lambda^3\mathbf{S}^*$ the isomorphism G is unique in the sense that if G' is any other such isomorphism then

(B.6) $$G^{-1} \circ G' : \Lambda\mathbf{S}^* \to \Lambda\mathbf{S}^* : \sigma \mapsto \sigma + \langle \Lambda^3 \ker(f^*) \rangle.$$

Proof. For simplicity we suppose that $\mathcal{A}$ is already an exterior algebra. Recall (Theorem 1.11) that $\operatorname{sder}_- \Lambda\mathbf{S}^* \cong \Lambda_+\mathbf{S}^* \otimes \mathbf{S}$, so let $D\colon V \to \Lambda_+\mathbf{S}^* \otimes \mathbf{S}$ be a linear map of the kind considered in the statement of the lemma. Then $D \cdot D = 0$ in the $\mathbb{Z}$-bigraded algebra $\mathcal{B}^{\bullet,\circ} := \operatorname{Sym}^\bullet V^* \otimes \Lambda^\circ\mathbf{S}^* \otimes \mathbf{S}$. We shall prove that there exists a map

(B.7) $$G\colon \mathbf{S}^* \to \Lambda_-\mathbf{S}^*$$

such that $\mathrm{pr}^1 \circ G = \mathrm{id}_{\mathbf{S}^*}$ and

$$D \cdot G = \mathrm{pr} \circ D, \tag{B.8}$$

where $\mathrm{pr}_1 : \Lambda_-\mathbf{S}^* \to \mathbf{S}^*$ and $\mathrm{pr}\colon \Lambda_-\mathbf{S}^* \otimes \mathbf{S} \to \mathbf{S}$ are the natural projections.

First, observe that left multiplication by an element X of the algebra $\mathcal{B}$ is a boundary operator:

$$X\cdot\colon \mathcal{B}^{\bullet,\circ} \to \mathcal{B}^{\bullet+1,\circ-1},$$

because for all X we get $X^3 = 0$. Now as a linear map $f\colon V \to \Lambda\mathbf{S}^* \otimes \mathbf{S}$ it has the form $f = \alpha \otimes 1 \otimes s$, where $\alpha \in V^*$ and $s \in \mathbf{S}$. So multiplying by f in the algebra $\mathcal{B}$ is tantamount to multiplying by $\alpha \otimes 1 \otimes s$.

In order to prove the existence of $G\colon \mathbf{S}^* \to \Lambda\mathbf{S}^*$ with the desired properties we make an ansatz for the maps D and G as

$$\begin{aligned} D &= D_0 + D_1 + D_2 + \cdots, \\ G &= G_0 + G_1 + G_3 + \cdots, \end{aligned} \tag{B.9}$$

where $G_0 = \mathrm{id}_{\mathbf{S}^*}$ and $D_0 = f$; also, each G_μ is an element of $\Lambda^{2\mu+1}\mathbf{S}^* \otimes \mathbf{S}$ and each D_μ of $V^* \otimes \Lambda^{2\mu}\mathbf{S}^* \otimes \mathbf{S}$. Since this will guarantee that $D_0 \cdot G_0 = f$, we need to show that G can be chosen to satisfy

$$D_0 \cdot G_\mu + D_1 \cdot G_{\mu-1} + \cdots + D_\mu \cdot G_0 = 0. \tag{B.10}$$

For this we will make use of the Cartan–Poincaré Lemma.

Claim. *The operator $D_0\cdot$ equals $d^*_{f^*} \otimes \mathrm{id}_{\mathbf{S}}$.*

With the proposed decomposition we get $d^*_{f^*} = \sum f^*(ds_\mu)\otimes s_\mu \lrcorner \otimes s_\mu$ for a basis $\{s_1, \dots, s_n\}$ of $\mathbf{S}$. Also, since $D_0 = f$, we know $D_0 = \sum f^*(ds_\mu)\otimes 1 \otimes s_\mu$. Now

$$\begin{aligned} D_0 \cdot (p \otimes \alpha \otimes s) &= \sum f^*(ds_\mu) p \otimes s_\mu \lrcorner \alpha \otimes s \\ &= \Big(\sum f^*(ds_\mu) \otimes 1 \otimes s_\mu\Big) \cdot p \otimes \alpha \otimes s \\ &= d^*_{f^*} \otimes \mathrm{id}_{\mathbf{S}}(p \otimes \alpha \otimes s), \end{aligned}$$

so the claim follows. ∎

The Cartan–Poincaré Lemma now implies

$$H_{\bullet,\circ}(d^*_{f^*}) = \begin{cases} 0, & \bullet > 0, \\ \Lambda^\circ \ker f^* \otimes \mathbf{S}, & \bullet = 0, \end{cases}$$

because $\ker f = 0$. Since $D_0\cdot = f\cdot$, the fact that all homology groups vanish for polynomials of positive degree is equivalent to

$$D_0 \cdot X = Y \quad \text{implies} \quad D_0 \cdot Y = 0$$

Our ansatz now requires

$$D_0 \cdot G_\mu = X \;\Leftrightarrow\; D_0 \cdot X = 0,$$

and furthermore the solution X is unique up to the kernel of the Cartan–Poincaré operator $D_0\cdot\colon \Lambda^{2\mu}\mathbf{S}^*\otimes\mathbf{S}\to V^*\otimes\Lambda^{2\mu-1}\mathbf{S}^*\otimes\mathbf{S}$, which is $\Lambda^{2\mu+1}(\ker f^*)\otimes\mathbf{S}$ because of the Cartan–Poincaré Lemma. This yields the second claim of the lemma.

Let $\mu\geq 1$. The lemma will be proved if we can show that, for chosen $G_1,\dots,G_{\mu-1}$ satisfying

$$\begin{aligned} D_0\cdot G_1 + D_1\cdot \mathrm{id}_{\mathbf{S}} &= 0,\\ D_0\cdot G_2 + D_1\cdot G_1 + D_2\cdot G_0 &= 0,\\ &\vdots\\ D_0\cdot G_{\mu-1} + D_1\cdot G_{\mu-2}+\cdots+D_{\mu-1}\cdot G_0 &= 0,\end{aligned} \tag{B.11}$$

one can choose G_μ such that $D_0\cdot G_\mu = 0$. Given the decomposition (B.9) for G we have

$$D_0\cdot G_\mu = -(D_\mu\cdot G_0 + D_{\mu-1}\cdot G_1+\cdots+D_1\cdot G_{\mu-1}) = X$$

by (B.10), so we must have $D_0\cdot X = 0$. To prove this last equation we observe that the sum

$$\sum_{1\leq\alpha+\beta\leq\mu} D_\alpha\cdot(D_\beta\cdot G_{\mu-\alpha-\beta})$$

contains all terms present in (B.11) which we know to be zero, and also contains the sum development for $-X$. Now

$$\begin{aligned}\sum_{1\leq\alpha+\beta\leq\mu} D_\alpha\cdot(D_\beta\cdot G_{\mu-\alpha-\beta}) &= \frac{1}{2}\sum_{1\leq\alpha+\beta\leq\mu} D_\beta\cdot(D_\alpha\cdot G_{\mu-\alpha-\beta})\\ &\quad + D_\alpha\cdot(D_\beta\cdot G_{\mu-\alpha-\beta})\\ &= \frac{1}{2}\sum_{1\leq\alpha+\beta\leq\mu}[D_\alpha,D_\beta]\cdot G_{\mu-\alpha-\beta}.\end{aligned} \tag{B.12}$$

We now see that

$$\sum_{1\leq\alpha+\beta\leq\mu}[D_\alpha,D_\beta] = \sum(D_\alpha\cdot D_\beta + D_\beta\cdot D_\alpha) = 0$$

because all D_λ are in $V^*\otimes\Lambda^{2\lambda}\mathbf{S}^*\otimes\mathbf{S}$; also $D_\mu\cdot(D_0\cdot\mathrm{id}_{\mathbf{S}}) = 0$ is trivially true because $f\in V^*\otimes\mathbb{R}\otimes\mathbf{S}$ gives zero when the operator $D\cdot$ is applied to it.

We now use the general formula

$$A\cdot(B\cdot X) + (-1)^{\lceil A\rfloor\lceil B\rfloor} B\cdot(A\cdot X) = ([A,B]\otimes\mathrm{id}_{\mathbf{S}})X$$

to see that the expansion in (B.12) is zero. So now we have proved that our ansatz for G yields the result. ∎

Acknowledgements. The author wishes to express his gratitude to Prof. Gregor Weingart of the Cuernavaca Branch of the Institute of Mathematics, UNAM, México, for his patience and valuable advice in developing all the ideas contained here. This is part of the author's work under his supervision.

This work was supported by Ph.D. stipend grant No. 339491 of the National Council of Science and Technology, CONACYT, México.

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