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A Mahler-type estimate of weighted Fekete sums on the Berkovich projective line

by

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1. Introduction. Let K be an algebraically closed field of possibly positive characteristic that is complete with respect to a non-trivial and possibly non-archimedean absolute value $|\cdot|$. Recall that K is said to be *non-archimedean* if the *strong triangle inequality*

$$|z+w| \le \max\{|z|,|w|\}$$

holds for every $z,w \in K$; otherwise K is *archimedean*. It is known that $K \cong \mathbb{C}$ if and only if K is archimedean. The *Berkovich* projective line $\mathsf{P}^1 = \mathsf{P}^1(K)$ is a compact augmentation of the (classical) projective line $\mathbb{P}^1 = \mathbb{P}^1(K)$. It is known that $\mathsf{P}^1 \cong \mathbb{P}^1$ if and only if K is archimedean.

For archimedean $K \cong \mathbb{C}$, the (log of the) classical *Mahler's estimate* of the Fekete *product* $\prod_{i=1}^N \prod_{j:\, j\neq i} |z_i - z_j|$ for any N distinct points $z_1,\dots,z_N$ in K is

$$(1.1)\qquad \sum_{i=1}^N \sum_{j:\, j\neq i} (\log|z_i - z_j| - \log\max\{1,|z_i|\} - \log\max\{1,|z_j|\}) \le N\log N$$

([15, Theorem 1]). Aiming to study the field of definition for periodic points of a rational function over a number field, Benedetto [4, Lemma 4.1] (in the case of $f \in K[z]$) and Baker [1, Theorem 1.1] (in the case of $f \in K(z)$) generalized (1.1) as follows; for every $f \in K(z)$ of degree > 1,

$$(1.2)\qquad \sum_{i=1}^N \sum_{j:\, j\neq i} \big(\log[z_i,z_j] - g_f(z_i) - g_f(z_j)\big) \le C\cdot N\log N$$

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for any N distinct points $z_1, \dots, z_N$ in $\mathbb{P}^1$, where $C \geq 0$ is an effective constant independent of $z_1, \dots, z_N$. Here, $[z, w]$ is the normalized chordal distance on $\mathbb{P}^1$ (see Notation 1.1 for the definition) and g_f is the dynamical Green function of f on P^1 (see Fact 1.2 for the definition of g_f).

The proofs of (1.1) and (1.2) were based on Hadamard's inequality, taking into account the geometry of the (homogeneous) filled Julia set of (the non-degenerate homogeneous lift of) f. One of our aims in this article is to give a simple proof of (1.2) with, in general, the asymptotically *best possible* lower estimate of the constant $C > 0$. Our proof is based on a simple formula for *weighted Fekete sums* and some upper and lower estimates of *regularized Fekete sums* (see §2.5 and §2.6, respectively) from potential theory on P^1. For the foundation of potential theory on P^1 for non-archimedean K, see Baker–Rumely [2], Favre–Rivera-Letelier [9], Thuillier [17], and also Jonsson [13]. In the following, we adopt the notation of [16].

Notation 1.1 (Potential theory on P^1). Let $\pi : K^2 \setminus \{(0,0)\} \to \mathbb{P}^1 = \mathbb{P}^1(K)$ be the canonical projection so that $\pi(p_0, p_1) = p_1/p_0 \in K$ if $p_0 \neq 0$ and $\pi(0,1) = \infty$. On K^2, let $\|(p_0, p_1)\|$ be the maximal norm $\max\{|p_0|, |p_1|\}$ (for non-archimedean K) or the Euclidean norm $\sqrt{|p_0|^2 + |p_1|^2}$ (for archimedean K). With the wedge product $(z_0, z_1) \wedge (w_0, w_1) := z_0 w_1 - z_1 w_0$ on K^2, the *normalized chordal metric* $[z, w]$ on $\mathbb{P}^1$ is the function

$$(z, w) \mapsto [z, w] := |p \wedge q| / (\|p\| \cdot \|q\|)$$

on $\mathbb{P}^1 \times \mathbb{P}^1$, where $p \in \pi^{-1}(z)$ and $q \in \pi^{-1}(w)$. For non-archimedean K, the *generalized Hsia kernel* $[\mathcal{S}, \mathcal{S}']_{\mathrm{can}}$ on P^1 *with respect to* $\mathcal{S}_{\mathrm{can}}$ is the unique (jointly) upper semicontinuous and separately continuous extension to $\mathsf{P}^1 \times \mathsf{P}^1$ of the chordal distance function $(z, w) \mapsto [z, w]$ on $\mathbb{P}^1 \times \mathbb{P}^1$ (see §2.2 for the definition of $[\mathcal{S}, \mathcal{S}']_{\mathrm{can}}$). For archimedean K, by convention, the kernel function $[z, w]_{\mathrm{can}}$ on $\mathsf{P}^1 \cong \mathbb{P}^1$ is defined to be $[z, w]$ itself.

Let $\delta_{\mathcal{S}}$ be the Dirac measure on P^1 at a point $\mathcal{S} \in \mathsf{P}^1$. The probability Radon measure Ω_{can} on P^1 is defined as

$$\Omega_{\mathrm{can}} := \begin{cases} \delta_{\mathcal{S}_{\mathrm{can}}} & \text{if } K \text{ is non-archimedean}, \\ \omega & \text{if } K \text{ is archimedean}, \end{cases}$$

where $\mathcal{S}_{\mathrm{can}}$ is the *canonical* (or *Gauss*) point in P^1 for non-archimedean K (see §2.1 for the definition), and ω is the Fubini–Study area element on $\mathbb{P}^1$ normalized as $\omega(\mathbb{P}^1) = 1$ for archimedean K. The Laplacian Δ on P^1 is normalized so that for each $\mathcal{S}' \in \mathsf{P}^1$,

$$\Delta \log[\cdot, \mathcal{S}']_{\mathrm{can}} = \delta_{\mathcal{S}'} - \Omega_{\mathrm{can}}$$

on P^1 (for non-archimedean K, see [2, §5.4], [9, §2.4]; in [2] the opposite sign convention on Δ is adopted).

A *continuous weight* on P^1 is a continuous function g on P^1 such that $\mu^g := \Delta g + \Omega_{\mathrm{can}}$ is a probability Radon measure on P^1. For a continuous weight g on P^1, the *g-potential kernel* on P^1 (or the negative of an Arakelov Green kernel function on P^1 relative to μ^g [2, §8.10]) is a (jointly) upper semicontinuous function

$$\Phi_g(\mathcal{S},\mathcal{S}') := \log[\mathcal{S},\mathcal{S}']_{\mathrm{can}} - g(\mathcal{S}) - g(\mathcal{S}') \tag{1.3}$$

on $\mathsf{P}^1 \times \mathsf{P}^1$, and the *$g$-equilibrium energy* V_g of P^1 (in fact $V_g \in \mathbb{R}$) is the supremum of the g-energy functional $\nu \mapsto \int_{\mathsf{P}^1\times\mathsf{P}^1} \Phi_g \, d(\nu\times\nu) \in [-\infty,+\infty)$ over all probability Radon measures ν on P^1. A probability Radon measure ν on P^1 at which the above g-energy functional attains the supremum V_g is called a *g-equilibrium mass distribution* on P^1; in fact, μ^g is the unique g-equilibrium mass distribution on P^1 (for non-archimedean K, see [2, Theorem 8.67, Proposition 8.70]).

A *normalized weight* on P^1 is a continuous weight g on P^1 satisfying $V_g = 0$ (for every continuous weight g on P^1, $\overline{g} := g + V_g/2$ is the unique normalized weight on P^1 such that $\mu^{\overline{g}} = \mu^g$).

One of our principal results is the following Mahler-type estimate.

Theorem 1. *Let K be an algebraically closed field of possibly positive characteristic that is complete with respect to a non-trivial and possibly non-archimedean absolute value. Let g be a normalized weight on P^1, and suppose that the restriction $g|\mathbb{P}^1$ is a $1/\kappa$-Hölder continuous function on $(\mathbb{P}^1,[z,w])$ for some $\kappa \ge 1$. Then setting $C := \sup_{z,w\in\mathbb{P}^1,\, z\neq w} |g(z)-g(w)|/[z,w]^{1/\kappa} \in \mathbb{R}_{\ge 0}$, for every non-empty finite subset F in $\mathbb{P}^1$, we have*

$$\sum_{z\in F}\sum_{w\in F\setminus\{z\}} \Phi_g(z,w) \le \kappa\cdot(\#F)\log(\#F) + 2(\#F)\Big(C' + \epsilon_K\cdot(\#F)^{1-\kappa} + \sup_{\mathsf{P}^1}|g|\Big), \tag{3.1$'$}$$

where we also set $C' := C\cdot 2^{1/\kappa}$ if K is non-archimedean, and $C' := C$ otherwise, and $\epsilon_K := 1$ if K is archimedean, and $\epsilon_K := 0$ otherwise.

In particular,

$$\limsup_{N\to\infty}\left(\sup_{F\subset\mathbb{P}^1:\, 0<\#F\le N} \frac{\sum_{z\in F}\sum_{w\in F\setminus\{z\}}\Phi_g(z,w)}{(\#F)\log(\#F)}\right) \le \kappa. \tag{1.4}$$

Theorem 1 is a consequence of (3.1) in Theorem 2 stated and shown in Section 3, which is a little technical but applies to any normalized weight g on P^1, involving the *restricted* modulus of continuity

$$\eta_{g,F}(\epsilon) := \max_{z\in F}\Big(\sup_{\mathcal{S}\in\mathsf{P}^1:\, \mathsf{d}(z,\mathcal{S})\le\epsilon} |g(z)-g(\mathcal{S})|\Big) \quad \text{on } [0,1] \tag{1.5}$$

of g around a non-empty finite subset F in $\mathbb{P}^1$ with respect to the metric d on P^1 (see §2.2 for the definition of d). The estimate (1.2) is obtained as a special case of Theorem 1 by recalling the following.

FACT 1.2. For every $f \in K(z)$ of degree $d > 1$, whose action on $\mathbb{P}^1$ canonically extends to that on P^1, the weak limit $\mu_f = \lim_{n\to\infty}(f^n)^*\Omega_{\mathrm{can}}/d^n$ exists on P^1, which is called the *f-equilibrium* (or *canonical*) measure on P^1 (for non-archimedean K, see [2, §10], [6, §2], [9, §3.1]). We call the unique normalized weight g on P^1 such that $\mu^g = \mu_f$ on P^1 the *f-dynamical Green function* on P^1 and denote it by g_f. It is known that $f : (\mathbb{P}^1, [z, w]) \to (\mathbb{P}^1, [z, w])$ is Lipschitz continuous (for non-archimedean K, see [14, Theorem 2]) and that if for every $n \in \mathbb{N}$, $f^n : (\mathbb{P}^1, [z, w]) \to (\mathbb{P}^1, [z, w])$ is M_n-Lipschitz continuous for some $M_n > d^n$, then $g_f|\mathbb{P}^1$ is $1/\kappa$-Hölder continuous on $(\mathbb{P}^1, [z, w])$ for every $\kappa > \limsup_{n\to\infty}(\log(M_n^{1/n}))/\log d$ (see e.g. [8, §6.6]).

Organization of the article. In Section 2, we recall the background from potential theory on P^1 including a few preparatory lemmas and facts, and some details on the regularization of Dirac measures supported in $\mathbb{P}^1$. In Section 3, we state and show Theorem 2, and then deduce Theorem 1 from it. In Section 4, we include a deduction of the lower estimate (2.6) of regularized Fekete sums, which plays a key role in the proof of Theorem 2. In Section 5, we include a few examples for which (1.4) is optimal.

2. Background from potential theory on P^1.

Let K be an algebraically closed field that is complete with respect to a non-trivial absolute value $|\cdot|$.

For more details concerning this section, including references, see [16].

2.1. The Berkovich projective line P^1 for non-archimedean K. Suppose that K is non-archimedean. A subset B in K is a (K-closed) *disk in K* if $B = \{z \in K : |z - a| \le r\}$ for some $a \in K$ and $r \ge 0$. By the strong triangle inequality, *two disks in K are either nested or disjoint.* This alternative extends to any two decreasing infinite sequences of disks in K so that they are either *infinitely nested* or *eventually disjoint*, and induces the so called *cofinal equivalence* relation among them. As a set, the set of all cofinal equivalence classes $\mathcal{S}$ of decreasing infinite sequences (B_n) of disks in K together with $\infty \in \mathbb{P}^1$ is nothing but P^1 [5, p. 17]; if $\mathcal{S} \ne \infty$, then $B_{\mathcal{S}} := \bigcap_n B_n$ is independent of the choice of (B_n) and is itself a disk in K unless $B_{\mathcal{S}} = \emptyset$. For example, the *canonical* (or *Gauss*) *point* $\mathcal{S}_{\mathrm{can}}$ in P^1 is the cofinal equivalence class of the constant sequence (B_n) of disks $B_n \equiv \mathcal{O}_K$ in K, where $\mathcal{O}_K := \{z \in K : |z| \le 1\}$ is the ring of K-integers. Each $z \in \mathbb{P}^1$ is identified with the cofinal equivalence class of the constant sequence (B_n) of disks $B_n \equiv \{z\}$ in K.

The above alternative for decreasing infinite sequences of disks in K also induces a partial ordering $\succeq$ on P^1 so that for any $\mathcal{S},\mathcal{S}'\in\mathsf{P}^1\setminus\{\infty\}$ satisfying $B_{\mathcal{S}},B_{\mathcal{S}'}\neq\emptyset$, $\mathcal{S}\succeq\mathcal{S}'$ if and only if $B_{\mathcal{S}}\supset B_{\mathcal{S}'}$ (the description of $\succeq$ when $B_{\mathcal{S}}$ or $B_{\mathcal{S}'}$ is empty is a little complicated) and that $\infty\succeq\mathcal{S}$ for every $\mathcal{S}\in\mathsf{P}^1$, and endows P^1 with the canonical *tree* structure in the sense of Jonsson [13, §2, Definition 2.2].

The topology of P^1 coincides with the weak topology induced by the tree structure of P^1, and P^1 is uniquely arcwise-connected and contains both $\mathbb{P}^1$ and $\mathsf{H}^1=\mathsf{H}^1(K):=\mathsf{P}^1\setminus\mathbb{P}^1$ as dense subsets.

2.2. The kernels $[\mathcal{S},\mathcal{S}']_{\mathrm{can}}$ and $|\mathcal{S}-\mathcal{S}'|_\infty$ and the distance d on P^1. Suppose first that K is non-archimedean. Let $\operatorname{diam}B$ be the diameter of a disk B in K with respect to $|\cdot|$. For every $\mathcal{S}\in\mathsf{P}^1\setminus\{\infty\}$ represented by a decreasing infinite sequence (B_n) of disks in K, set

$$\operatorname{diam}\mathcal{S}:=\lim_{n\to\infty}\operatorname{diam}B_n\qquad(=\operatorname{diam}B_{\mathcal{S}}\text{ unless }B_{\mathcal{S}}=\emptyset),$$

which is independent of the choice of (B_n), and set $\operatorname{diam}\infty=\infty$. For every $\mathcal{S},\mathcal{S}'\in\mathsf{P}^1$, let $\mathcal{S}\wedge\mathcal{S}'$ be the smallest $\mathcal{S}''\in\mathsf{P}^1$ satisfying $\mathcal{S}''\succeq\mathcal{S}$ and $\mathcal{S}''\succeq\mathcal{S}'$. Under the convention that $\infty/\infty^2=0$, the generalized Hsia kernel $[\mathcal{S},\mathcal{S}']_{\mathrm{can}}$ on P^1 with respect to $\mathcal{S}_{\mathrm{can}}$ is defined by the function

$$[\mathcal{S},\mathcal{S}']_{\mathrm{can}}:=\frac{\operatorname{diam}\mathcal{S}''}{(\operatorname{diam}(\mathcal{S}_{\mathrm{can}}\wedge\mathcal{S}''))^2}\tag{2.1}$$

on $\mathsf{P}^1\times\mathsf{P}^1$, where $\mathcal{S}''$ is the unique point in P^1 lying between $\mathcal{S}$ and $\mathcal{S}'$, between $\mathcal{S}'$ and $\mathcal{S}_{\mathrm{can}}$, and between $\mathcal{S}_{\mathrm{can}}$ and $\mathcal{S}$ with respect to $\succeq$ (see [8, §3.4], [2, §4.4]). Then, as mentioned in Section 1, the kernel function $(\mathcal{S},\mathcal{S}')\mapsto[\mathcal{S},\mathcal{S}']_{\mathrm{can}}$ on $\mathsf{P}^1\times\mathsf{P}^1$ is the unique (jointly) upper semicontinuous and separately continuous extension to $\mathsf{P}^1\times\mathsf{P}^1$ of the chordal distance function $(z,w)\mapsto[z,w]$ on $\mathbb{P}^1\times\mathbb{P}^1$. If K is archimedean, then $[z,w]_{\mathrm{can}}$ is defined by $[z,w]$ itself on $\mathsf{P}^1\cong\mathbb{P}^1$, by convention.

No matter whether K is archimedean or non-archimedean, set

$$\mathsf{d}(\mathcal{S},\mathcal{S}'):=[\mathcal{S},\mathcal{S}']_{\mathrm{can}}-\frac{[\mathcal{S},\mathcal{S}]_{\mathrm{can}}+[\mathcal{S}',\mathcal{S}']_{\mathrm{can}}}{2}\tag{2.2}$$

on $\mathsf{P}^1\times\mathsf{P}^1$. If K is archimedean, then the function $\mathsf{d}(z,w)$ on $\mathsf{P}^1\times\mathsf{P}^1$ is nothing but the chordal metric $[z,w]$ on $\mathsf{P}^1\cong\mathbb{P}^1$. If K is non-archimedean, the function d extends $[z,w]$ to P^1 as a metric on P^1 (see [8, §4.7], [2, §2.7]), and is called the *small model metric* on P^1.

Although the difference $\mathcal{S}-\mathcal{S}'$ between $\mathcal{S},\mathcal{S}'\in\mathsf{P}^1$ is defined only if both $\mathcal{S},\mathcal{S}'$ are in K, set

$$|\mathcal{S}-\mathcal{S}'|_\infty:=\frac{[\mathcal{S},\mathcal{S}']_{\mathrm{can}}}{[\mathcal{S},\infty]_{\mathrm{can}}\cdot[\mathcal{S}',\infty]_{\mathrm{can}}}\tag{2.3}$$

on $\mathsf{P}^1 \times \mathsf{P}^1$, under the convention $0/0^2 = \infty$. If K is archimedean, then the restriction $|z - w|_\infty$ to $K \times K$ is precisely the euclidean metric $|z - w|$ on $K \cong \mathbb{C}$. If K is non-archimedean, $|\mathcal{S} - \mathcal{S}'|_\infty$ is the unique (jointly) upper semicontinuous and separately continuous extension to $\mathsf{P}^1 \times \mathsf{P}^1$ of the distance function $(z, w) \mapsto |z - w| = [z, w]/([z, \infty] \cdot [w, \infty])$ on $K \times K$, and is called the (original) *Hsia kernel* on P^1 (see [2, §4.4]).

LEMMA 2.1. *On* $\mathbb{P}^1 \times \mathsf{P}^1$, $\mathsf{d}(z, \mathcal{S}) \geq [z, \mathcal{S}]_{\mathrm{can}}/2$.

Proof. There is nothing to show for archimedean K, so suppose that K is non-archimedean. For every $z \in \mathbb{P}^1$ and every $\mathcal{S} \in \mathsf{P}^1$, by the definition (2.2) of the metric d, we have $\mathsf{d}(z, \mathcal{S}) = [z, \mathcal{S}]_{\mathrm{can}} - [\mathcal{S}, \mathcal{S}]_{\mathrm{can}}/2$, and by the definition (2.1) of the kernel $[\mathcal{S}, \mathcal{S}']_{\mathrm{can}}$, we have

$$
\begin{aligned}
[z, \mathcal{S}]_{\mathrm{can}} &= \begin{cases} \operatorname{diam}(z \wedge \mathcal{S}) & \text{if } \mathcal{S}_{\mathrm{can}} \succeq \mathcal{S} \text{ and } \mathcal{S}_{\mathrm{can}} \succeq z, \\ \operatorname{diam} \mathcal{S}_{\mathrm{can}} & \text{if } \mathcal{S}_{\mathrm{can}} \succeq \mathcal{S} \text{ and } \mathcal{S}_{\mathrm{can}} \not\succeq z, \\ 1/\operatorname{diam}(\mathcal{S}_{\mathrm{can}} \wedge z) & \text{if } \mathcal{S}_{\mathrm{can}} \not\succeq \mathcal{S} \text{ and } \mathcal{S}_{\mathrm{can}} \wedge \mathcal{S} \succeq \mathcal{S}_{\mathrm{can}} \wedge z, \\ 1/\operatorname{diam}(\mathcal{S}_{\mathrm{can}} \wedge \mathcal{S}) & \text{if } \mathcal{S}_{\mathrm{can}} \not\succeq \mathcal{S} \text{ and } \mathcal{S}_{\mathrm{can}} \wedge z \succeq \mathcal{S}_{\mathrm{can}} \wedge \mathcal{S} \end{cases} \\
&\geq \begin{cases} \operatorname{diam} \mathcal{S} & \text{if } \mathcal{S}_{\mathrm{can}} \succeq \mathcal{S}, \\ (\operatorname{diam}(\mathcal{S}_{\mathrm{can}} \wedge \mathcal{S}))/(\operatorname{diam}(\mathcal{S}_{\mathrm{can}} \wedge \mathcal{S}))^2 & \text{if } \mathcal{S}_{\mathrm{can}} \not\succeq \mathcal{S} \end{cases} \\
&\geq \begin{cases} \operatorname{diam} \mathcal{S} & \text{if } \mathcal{S}_{\mathrm{can}} \succeq \mathcal{S}, \\ (\operatorname{diam} \mathcal{S})/(\operatorname{diam}(\mathcal{S}_{\mathrm{can}} \wedge \mathcal{S}))^2 & \text{if } \mathcal{S}_{\mathrm{can}} \not\succeq \mathcal{S} \end{cases} \\
&= [\mathcal{S}, \mathcal{S}]_{\mathrm{can}},
\end{aligned}
$$

which completes the proof. ∎

2.3. The isometry group U_K on $(\mathsf{P}^1, \mathsf{d})$. The action on $\mathbb{P}^1$ of a linear fractional transformation $h \in \mathrm{PGL}(2, K)$ uniquely extends to P^1 as a continuous automorphism of P^1, and induces the pullback h^* and the push-forward $h_* = (h^{-1})^*$ on the space of all continuous functions on P^1 and, by duality, on the space of all probability Radon measures on P^1 (see e.g. [2, §2.3]). Let U_K be either the subgroup $\mathrm{PSU}(2, K)$ (for archimedean $K \cong \mathbb{C}$) or the subgroup $\mathrm{PGL}(2, \mathcal{O}_K)$ (for non-archimedean K) in $\mathrm{PGL}(2, K)$. Each $h \in U_K$ acts on $(\mathbb{P}^1, [z, w])$ isometrically (for non-archimedean K, see e.g. [3, §1]). Hence each $h \in U_K$ not only satisfies $[h(\mathcal{S}), h(\mathcal{S}')]_{\mathrm{can}} = [\mathcal{S}, \mathcal{S}']_{\mathrm{can}}$ on $\mathsf{P}^1 \times \mathsf{P}^1$ by the separate continuity of both sides on $\mathsf{P}^1 \times \mathsf{P}^1$ and the density of $\mathbb{P}^1$ in P^1, but also acts on $(\mathsf{P}^1, \mathsf{d})$ isometrically (recall the definition (2.2) of d). Moreover, for every $h \in U_K$, we have $h^* \Omega_{\mathrm{can}} = \Omega_{\mathrm{can}}$ on P^1; indeed, fixing $\mathcal{S} \in \mathsf{P}^1$, we have

$$
\begin{aligned}
h^* \Omega_{\mathrm{can}} &= h^* \delta_{\mathcal{S}} - \Delta h^* \log[\cdot, \mathcal{S}]_{\mathrm{can}} = h^* \delta_{\mathcal{S}} - \Delta \log[h(\cdot), h(h^{-1}(\mathcal{S}))]_{\mathrm{can}} \\
&= \delta_{h^{-1}(\mathcal{S})} - \Delta \log[\cdot, h^{-1}(\mathcal{S})]_{\mathrm{can}} = \delta_{h^{-1}(\mathcal{S})} - (\delta_{h^{-1}(\mathcal{S})} - \Omega_{\mathrm{can}}) = \Omega_{\mathrm{can}} \quad \text{on } \mathsf{P}^1
\end{aligned}
$$

(for the functoriality $h^* \Delta = \Delta h^*$ for non-archimedean K, see e.g. [2, §9]).

LEMMA 2.2. *For every normalized weight g on P^1 and every $h \in U_K$, $g \circ h$ is also a normalized weight on P^1.*

Proof. We first compute

$$\mu^{g\circ h} = \Delta(g \circ h) + \Omega_{\mathrm{can}} = h^*(\Delta g) + \Omega_{\mathrm{can}} = h^*\mu^g - h^*\Omega_{\mathrm{can}} + \Omega_{\mathrm{can}} = h^*\mu^g,$$

which is a probability Radon measure on P^1, so $g \circ h$ is a continuous weight on P^1. Next, from $[h(\mathcal{S}), h(\mathcal{S}')]_{\mathrm{can}} = [\mathcal{S}, \mathcal{S}']_{\mathrm{can}}$ on $\mathsf{P}^1 \times \mathsf{P}^1$ and the characterization of μ^g (resp. $\mu^{g\circ h}$) as the (unique) g-equilibrium (resp. $g \circ h$-equilibrium) mass distribution on P^1, we have

$$V_{g\circ h} = \int_{\mathsf{P}^1\times\mathsf{P}^1} \Phi_{g\circ h}\, d((h^*\mu^g) \times (h^*\mu^g)) = \int_{\mathsf{P}^1\times\mathsf{P}^1} \Phi_g\, d(\mu^g \times \mu^g) = V_g = 0,$$

which completes the proof. ∎

2.4. A regularization of an effective divisor on P^1. First, for every $z \in K$ and every $\epsilon > 0$, we define the ϵ-*regularization* of the Dirac measure δ_z on P^1 to be the probability Radon measure

$$\begin{aligned}[z]_\epsilon &:= \Delta \log \max\{\epsilon, |\cdot - z|_\infty\} + \delta_\infty \\ &= \begin{cases} \delta_{\pi_\epsilon(z)} & \text{for non-archimedean } K, \\ m_{\{w\in K:|w-z|=\epsilon\}} & \text{for archimedean } K \end{cases}\end{aligned}$$

on P^1; for non-archimedean K, the continuous mapping $\pi_\epsilon : \mathbb{P}^1 \to \mathsf{H}^1$ is defined so that for every $z \in K$, $\pi_\epsilon(z) \in \mathsf{H}^1$ is represented by the constant sequence (B_n) of disks $B_n \equiv \{w \in K : |w - z| \le \epsilon\}$ in K, and for archimedean K, $m_{\partial\mathbb{D}(z,\epsilon)}$ is the 1-dimensional Lebesgue measure on the circle $\partial\mathbb{D}(z, \epsilon) := \{w \in K : |w - z| = \epsilon\}$ in $K \cong \mathbb{C}$ normalized as $m_{\partial\mathbb{D}(z,\epsilon)}(\partial\mathbb{D}(z, \epsilon)) = 1$. Next, with the involution $\iota(z) := 1/z \in U_K$ on P^1, for every $\epsilon > 0$, set

$$[\infty]_\epsilon := \iota_*[0]_\epsilon.$$

Then for every $z \in \mathbb{P}^1$ and every $\epsilon > 0$, $[z]_\epsilon$ has no atoms on $\mathbb{P}^1$. The following justifies the terminology (cf. [8, Lemme 4.8], [12, §2.1]).

LEMMA 2.3. *For every $z \in \mathbb{P}^1$ and every $\epsilon > 0$, the chordal potential $U^{\#}_{[z]_\epsilon}(\cdot) := \int_{\mathsf{P}^1} \log[\cdot, \mathcal{S}']_{\mathrm{can}}\, d[z]_\epsilon(\mathcal{S}')$ of the ϵ-regularization $[z]_\epsilon$ of δ_z on P^1 is a continuous function on P^1.*

Proof. Fix $z \in \mathbb{P}^1$ and $\epsilon > 0$. For non-archimedean K, if $z \in K$, then $U^{\#}_{[z]_\epsilon}(\cdot) \equiv \log[\cdot, \pi_\epsilon(z)]_{\mathrm{can}}$ is continuous on P^1 by $\pi_\epsilon(z) \in \mathsf{H}^1$ and the separate continuity of $[\mathcal{S}, \mathcal{S}']_{\mathrm{can}}$ on $\mathsf{P}^1 \times \mathsf{P}^1$. For archimedean $K \cong \mathbb{C}$, if $z \in K$, then

$$\begin{aligned}
w \mapsto U^{\#}_{[z]_\epsilon}(w) &= \int_0^{2\pi} \log[w, z+\epsilon e^{i\theta}]\,\frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} \log|(w-z)-\epsilon e^{i\theta}|\,\frac{d\theta}{2\pi} - \log\sqrt{1+|w|^2} - \int_0^{2\pi} \log\sqrt{1+|z+\epsilon e^{i\theta}|^2}\,\frac{d\theta}{2\pi} \\
&\equiv \log(\max\{|w-z|,\epsilon\})/\sqrt{1+|w|^2}) + U^{\#}_{[z]_\epsilon}(\infty)
\end{aligned}$$

is continuous on $\mathsf{P}^1 \cong \mathbb{P}^1$. Finally, no matter whether K is archimedean or non-archimedean, $U^{\#}_{[\infty]_\epsilon} \equiv U^{\#}_{[0]_\epsilon} \circ \iota$ is continuous on P^1 by the above continuity of $U^{\#}_{[0]_\epsilon}$ on P^1 and the continuity of $\iota : \mathsf{P}^1 \to \mathsf{P}^1$. ∎

2.5. The g-Fekete sum with respect to an effective divisor on $\mathbb{P}^1$. Every effective divisor $\mathcal{Z}$ on $\mathbb{P}^1 = \mathbb{P}^1(K)$ can be regarded as a positive and discrete Radon measure

$$\sum_{w \in \operatorname{supp} \mathcal{Z}} (\operatorname{ord}_w \mathcal{Z}) \cdot \delta_w$$

on P^1, which is denoted by the same $\mathcal{Z}$. Let $\operatorname{diag}_{\mathbb{P}^1}$ be the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$.

For a continuous weight g on P^1, the *g-Fekete sum* with respect to an effective divisor $\mathcal{Z}$ on $\mathbb{P}^1$ is defined by

$$\begin{aligned}
(\mathcal{Z},\mathcal{Z})_g &:= \int_{(\mathsf{P}^1\times\mathsf{P}^1)\setminus \operatorname{diag}_{\mathbb{P}^1}} \Phi_g \, d(\mathcal{Z}\times\mathcal{Z}) \\
&= \sum_{z\in\operatorname{supp}\mathcal{Z}} \sum_{w\in\operatorname{supp}\mathcal{Z}\setminus\{z\}} (\operatorname{ord}_z \mathcal{Z})(\operatorname{ord}_w \mathcal{Z}) \cdot \Phi_g(z,w) \in \mathbb{R},
\end{aligned}$$

where the sign convention is opposite to those of Favre–Rivera-Letelier's Dirichlet forms in [8], and is compatible with the log of the original Fekete product [10, 11] in Section 1.

Every non-empty finite subset F in $\mathbb{P}^1$ is canonically regarded as the effective divisor $\mathcal{Z}_F$ on $\mathbb{P}^1$ such that $\operatorname{supp}\mathcal{Z}_F = F$ and that $\operatorname{ord}_w \mathcal{Z}_F = 1$ for every $w \in F$. For a continuous weight g on P^1 and a non-empty finite subset F in $\mathbb{P}^1$, we also define the g-Fekete sum with respect to F by

$$(F,F)_g := (\mathcal{Z}_F,\mathcal{Z}_F)_g = \sum_{z\in F}\sum_{w\in F\setminus\{z\}} \Phi_g(z,w),$$

which satisfies

$$(F,F)_g = 2\cdot \sum_{w\in F\setminus\{z\}} \Phi_g(z,w) + (F\setminus\{z\}, F\setminus\{z\})_g \tag{2.4}$$

for every $z \in F$. Recall Lemma 2.2 here.

Lemma 2.4. *For every normalized weight g on P^1, every effective divisor $\mathcal{Z}$ on $\mathbb{P}^1$, and every $h \in U_K$,*

$$(\mathcal{Z},\mathcal{Z})_g = (h^*\mathcal{Z}, h^*\mathcal{Z})_{g\circ h}.$$

Proof. From $[h^{-1}(\mathcal{S}), h^{-1}(\mathcal{S}')]_{\mathrm{can}} = [\mathcal{S},\mathcal{S}']_{\mathrm{can}}$ on $\mathsf{P}^1 \times \mathsf{P}^1$ for every $h \in U_K$, we obtain

$$\begin{aligned}(\mathcal{Z},\mathcal{Z})_g &= \int_{(\mathsf{P}^1\times\mathsf{P}^1)\setminus\mathrm{diag}_{\mathbb{P}^1}} \Phi_g \, d(\mathcal{Z}\times\mathcal{Z}) \\ &= \int_{(\mathsf{P}^1\times\mathsf{P}^1)\setminus\mathrm{diag}_{\mathbb{P}^1}} \big(\log[h^{-1}(\mathcal{S}), h^{-1}(\mathcal{S}')]_{\mathrm{can}} - g(\mathcal{S}) - g(\mathcal{S}')\big)\, d(\mathcal{Z}\times\mathcal{Z})(\mathcal{S},\mathcal{S}') \\ &= \int_{(\mathsf{P}^1\times\mathsf{P}^1)\setminus\mathrm{diag}_{\mathbb{P}^1}} \Phi_{g\circ h}\, d((h^*\mathcal{Z})\times(h^*\mathcal{Z})) = (h^*\mathcal{Z}, h^*\mathcal{Z})_{g\circ h},\end{aligned}$$

which completes the proof. ∎

2.6. Estimates of regularized Fekete sums. For every $\epsilon > 0$ and every effective divisor $\mathcal{Z}$ on $\mathbb{P}^1$, the ϵ-regularization of $\mathcal{Z}$ is defined by $\mathcal{Z}_\epsilon := \sum_{w\in\mathcal{Z}}(\mathrm{ord}_w \mathcal{Z})\cdot[w]_\epsilon$ on P^1, and for every continuous weight g on P^1, the *ϵ-regularized g-Fekete sum* with respect to $\mathcal{Z}$ is

$$(\mathcal{Z}_\epsilon,\mathcal{Z}_\epsilon)_g := \int_{\mathsf{P}^1\times\mathsf{P}^1} \Phi_g \, d(\mathcal{Z}_\epsilon\times\mathcal{Z}_\epsilon) \in \mathbb{R}.$$

Fact 2.5. If the continuous weight g is a normalized weight on P^1, then for every $\epsilon > 0$ and every effective divisor $\mathcal{Z}$ on $\mathbb{P}^1$, the continuity of $U^{\#}_{[z]_\epsilon}(\cdot)$ on P^1 (Lemma 2.3) implies the *negativity*

$$(\mathcal{Z}_\epsilon,\mathcal{Z}_\epsilon)_g \le V_g = 0 \tag{2.5}$$

(cf. [8, §2.5 et §4.5]), and if in addition $\epsilon \in (0,1]$, then also

$$\begin{aligned}(\mathcal{Z}_\epsilon,\mathcal{Z}_\epsilon)_g \ge (\mathcal{Z},\mathcal{Z})_g &+ 2\sum_{w\in\mathrm{supp}\,\mathcal{Z}\setminus\{\infty\}} (\mathrm{ord}_w \mathcal{Z})^2 \cdot \log[w,\infty] \\ &- 2\sum_{w\in\mathrm{supp}\,\mathcal{Z}} (\mathrm{ord}_w \mathcal{Z})^2 g(w) \\ &+ (\log\epsilon)\cdot(\mathcal{Z}\times\mathcal{Z})(\mathrm{diag}_{\mathbb{P}^1}) - 2(\deg\mathcal{Z})^2\cdot\hat{\eta}_{g,\mathrm{supp}\,\mathcal{Z}}(\epsilon),\end{aligned} \tag{2.6}$$

where for every non-empty subset F in $\mathbb{P}^1$, recalling the definition (1.5) of the restricted modulus of continuity $\eta_{g,F} : [0,1] \to \mathbb{R}_{\ge 0}$ of g around F with respect to d, the function $\hat{\eta}_{g,F} : [0,1] \to \mathbb{R}_{\ge 0}$ is defined by

$$[0,1] \ni \epsilon \mapsto \hat{\eta}_{g,F}(\epsilon) := \eta_{g,F}(\epsilon) + \begin{cases} 0 & \text{for non-archimedean } K, \\ \epsilon & \text{for archimedean } K \end{cases} \tag{2.7}$$

(cf. [8, §2.6 et §4.7]).

We will include a deduction of (2.6) in Section 4.

3. Proofs of Theorems 1 and 2. Let K be an algebraically closed field that is complete with respect to a non-trivial absolute value, and let g be a normalized weight on P^1. Recall the definition (2.7) of $\hat{\eta}_{g,F}$ (and (1.5) of $\eta_{g,F}$). The following could be regarded as a refinement of Favre–Rivera-Letelier [8, Propositions 2.8 et 4.9].

THEOREM 2. *Let K be an algebraically closed field of possibly positive characteristic that is complete with respect to a non-trivial and possibly non-archimedean absolute value, and let g be a normalized weight on P^1. Then for every non-empty finite subset F in $\mathbb{P}^1$ and every $\epsilon \in (0,1]$,*

$$(F,F)_g \le (\#F)\log(\epsilon^{-1}) + 2(\#F)^2 \cdot \hat{\eta}_{g,F}(\epsilon) + 2(\#F)\sup_{\mathsf{P}^1}|g|. \tag{3.1}$$

REMARK 3.1. By [8, Propositions 2.8 et 4.9] mentioned above, we could only assert that for every non-empty and finite subset F in $\mathbb{P}^1 \setminus \{\infty\}$ and every $\epsilon \in (0,1]$,

$$\begin{aligned}(F,F)_g \le (\#F)\log(\epsilon^{-1}) + 2(\#F)^2\hat{\eta}_{g,F}(\epsilon) + 2(\#F)\sup_{\mathsf{P}^1}|g| \\ - 2\sum_{w\in F}\log[w,\infty].\end{aligned} \tag{3.2}$$

We note that the term $-2\sum_{w\in F}\log[w,\infty]$ is strictly positive. Even to obtain the sharper (3.1) in the case $F \subset \mathbb{P}^1 \setminus \{\infty\}$, we needed (2.5) and (2.6) applicable to F possibly containing ∞, and the formula (2.4).

Proof of Theorem 2. Fix $\epsilon \in (0,1]$. Let us show (3.1) for every non-empty finite subset F in $\mathbb{P}^1$ by induction on $\#F$. First of all, for every singleton F in $\mathbb{P}^1$, we have $(F,F)_g = 0$, so (3.1) holds. Let $N \in \mathbb{N}$ be more than 1, and suppose that (3.1) holds for every $F \subset \mathbb{P}^1$ satisfying $\#F = N-1$.

Fix $F \subset \mathbb{P}^1$ satisfying $\#F = N$. If $\infty \in F$, then $\#(F\setminus\{\infty\}) = N-1$. By the upper and lower estimates (2.5) and (2.6) of $((\mathcal{Z}_F)_\epsilon, (\mathcal{Z}_F)_\epsilon)_g$, we have

$$\begin{aligned}
2\cdot\sum_{w\in F\setminus\{\infty\}}\Phi_g(w,\infty) &= 2\cdot\sum_{w\in F\setminus\{\infty\}}(\log[w,\infty] - g(w) - g(\infty))\\
&= 2\cdot\sum_{w\in F\setminus\{\infty\}}\log[w,\infty] - 2\Big(\sum_{w\in F\setminus\{\infty\}} g(w) + (N-1)\cdot g(\infty)\Big)\\
&\le \Big(-(F,F)_g + 2\sum_{w\in F} g(w) - N\cdot\log\epsilon + 2N^2\cdot\hat{\eta}_{g,F}(\epsilon)\Big) \quad \text{(by (2.5) and (2.6))}\\
&\quad - 2\Big(\sum_{w\in F\setminus\{\infty\}} g(w) + (N-1)\cdot g(\infty)\Big)\\
&= -(F,F)_g + N\cdot\log(\epsilon^{-1}) + 2N^2\cdot\hat{\eta}_{g,F}(\epsilon) - 2(N-2)g(\infty),
\end{aligned}$$

which together with the formula (2.4) on $(F,F)_g$ for $z=\infty$ yields

$$2(F,F)_g \le N\cdot\log(\epsilon^{-1}) + 2N^2\cdot\hat{\eta}_{g,F}(\epsilon) + 2N\cdot\sup_{\mathsf{P}^1}|g| + (F\setminus\{\infty\},F\setminus\{\infty\})_g,$$

and we also note that $\eta_{g,F\setminus\{\infty\}}(\epsilon)\le\eta_{g,F}(\epsilon)$. Hence the induction assumption applied to $(F\setminus\{\infty\},F\setminus\{\infty\})_g$ completes the proof of (3.1) in this case. If $\infty\notin F$, then there is $h\in U_K$ satisfying $\infty\in h^{-1}(F)$. Then $\#(h^{-1}(F))=N$. By (Lemma 2.2 and) Lemma 2.4, we have $(F,F)_g=(h^{-1}(F),h^{-1}(F))_{g\circ h}$, and by $\mathsf{d}(h(\mathcal{S}),h(\mathcal{S}'))=\mathsf{d}(\mathcal{S},\mathcal{S}')$ on $\mathsf{P}^1\times\mathsf{P}^1$, we also have $\eta_{g,F}\equiv\eta_{g\circ h,h^{-1}(F)}$ on $[0,1]$. Hence (3.1) applied to $(h^{-1}(F),h^{-1}(F))_{g\circ h}$ completes the proof of (3.1) in this case. ∎

Proof of Theorem 1. Suppose that $g|\mathbb{P}^1$ is $1/\kappa$-Hölder continuous on $(\mathbb{P}^1,[z,w])$ for some $\kappa\ge 1$ so that $C:=\sup_{z,w\in\mathbb{P}^1,\,z\ne w}|g(z)-g(w)|/[z,w]^{1/\kappa}\in\mathbb{R}_{\ge 0}$. Recall the definition of $C'\in\mathbb{R}_{\ge 0}$ in Theorem 1.

LEMMA 3.2. *On $\mathbb{P}^1\times\mathsf{P}^1$, $|g(z)-g(\mathcal{S})|\le C'\cdot\mathsf{d}(z,\mathcal{S})^{1/\kappa}$.*

Proof. If K is archimedean, then there is nothing to show by $\mathsf{P}^1\cong\mathbb{P}^1$, $\mathsf{d}(z,w)=[z,w]$ on $\mathbb{P}^1\times\mathbb{P}^1$, and $C':=C$. Suppose that K is non-archimedean. Then $C':=C\cdot 2^{1/\kappa}$, and for every $z\in\mathbb{P}^1$, by the continuity of g and $[z,\cdot]_{\mathrm{can}}$ on P^1 and the density of $\mathbb{P}^1$ in P^1, we have $|g(z)-g(\cdot)|\le C[z,\cdot]_{\mathrm{can}}^{1/\kappa}$ on P^1, and in turn $|g(z)-g(\cdot)|\le(C\cdot 2^{1/\kappa})\cdot\mathsf{d}(z,\cdot)^{1/\kappa}$ on P^1 by Lemma 2.1. ∎

Once Lemma 3.2 is at our disposal, for every non-empty finite subset F in $\mathbb{P}^1$ we have $\eta_{g,F}(\epsilon)\le C'\epsilon^{1/\kappa}$ on $[0,1]$, so by setting $\epsilon=(\#F)^{-\kappa}\in(0,1]$ in (3.1), we obtain (3.1′). Now the proof of Theorem 1 is complete. ∎

4. On deduction of (2.6). Let K be an algebraically closed field that is complete with respect to a non-trivial absolute value $|\cdot|$. Let g be a continuous weight on P^1 and, for every non-empty finite subset F in $\mathbb{P}^1$, recall the definitions (1.5) and (2.7) of the functions $\eta_{g,F}:[0,1]\to\mathbb{R}_{\ge 0}$ and $\hat{\eta}_{g,F}:[0,1]\to\mathbb{R}_{\ge 0}$. Let us see that for every $\epsilon\in(0,1]$ and every $z,w\in\mathbb{P}^1$,

$$\text{(4.1)}\qquad \int_{\mathsf{P}^1\times\mathsf{P}^1}\Phi_g\,d([z]_\epsilon\times[w]_\epsilon) \ge \begin{cases}\Phi_g(z,w)-2\hat{\eta}_{g,\{z,w\}}(\epsilon) & \text{if } z\ne w,\\ \log\epsilon+2\log[z,\infty]-2g(z)-2\hat{\eta}_{g,\{z\}}(\epsilon) & \text{if } z=w\in K,\\ \log\epsilon-2g(\infty)-2\hat{\eta}_{g,\{\infty\}}(\epsilon) & \text{if } z=w=\infty.\end{cases}$$

Once (4.1) is at our disposal, (2.6) will follow by a computation similar to that in [16, proof of Lemma 6.1].

Remark 4.1. An estimate similar to (4.1) was obtained in [16, Lemma 3.2], where for archimedean K, the definition of $[z]_\epsilon$ was slightly different (or, more precisely, $[z]_\epsilon$ was defined to be more smooth for archimedean K).

Proof of (4.1). We recall that for every $\epsilon > 0$ and every $z, w \in K$,

$$\int_{\mathsf{P}^1\times\mathsf{P}^1} \log|\mathcal{S}-\mathcal{S}'|_\infty \, d([z]_\epsilon \times [w]_\epsilon)(\mathcal{S},\mathcal{S}') \geq \begin{cases} \log|z-w| & \text{if } z \neq w, \\ \log\epsilon & \text{if } z = w \end{cases} \tag{4.2}$$

(see Favre–Rivera-Letelier [8, Lemme 4.11] and Fili–Pottmeyer [12, Lemma 4] for non-archimedean and archimedean K, respectively), and for every $\epsilon \in (0,1]$ and every $z \in K$, we have

$$\operatorname{supp}[z]_\epsilon \subset \{\mathcal{S} \in \mathsf{P}^1 : |\mathcal{S} - z|_\infty \leq \epsilon\} \subset \{\mathcal{S} \in \mathsf{P}^1 : \mathsf{d}(\mathcal{S}, z) \leq \epsilon\}. \tag{4.3}$$

By the first inclusion of (4.3) and the density of $\mathbb{P}^1$ in P^1, a direct computation shows that for every $z \in K$ and every $\epsilon \in (0,1]$,

$$\sup_{\mathcal{S}\in\operatorname{supp}[z]_\epsilon} \bigl|\log[\mathcal{S},\infty]_{\mathrm{can}} - \log[z,\infty]\bigr| \leq \begin{cases} \epsilon & \text{for archimedean } K, \\ 0 & \text{for non-archimedean } K. \end{cases} \tag{4.4}$$

By (4.2)–(4.4) (and the definitions (1.3) and (2.3) of Φ_g and $|\mathcal{S}-\mathcal{S}'|_\infty$, respectively), we immediately deduce (4.1) for every $z, w \in K$ and every $\epsilon \in (0,1]$, and also by $[\iota(\mathcal{S}), \iota(\mathcal{S}')]_{\mathrm{can}} = [\mathcal{S},\mathcal{S}']_{\mathrm{can}}$ on $\mathsf{P}^1 \times \mathsf{P}^1$, for every $\epsilon \in (0,1]$ we have

$$\begin{aligned} &\int_{\mathsf{P}^1\times\mathsf{P}^1} \Phi_g \, d([\infty]_\epsilon \times [\infty]_\epsilon) \\ &\geq \int_{\mathsf{P}^1\times\mathsf{P}^1} \log[\mathcal{S},\mathcal{S}']_{\mathrm{can}} \, d([0]_\epsilon \times [0]_\epsilon)(\mathcal{S},\mathcal{S}') - 2g(\infty) - 2\eta_{g,\{\infty\}}(\epsilon) \\ &\geq \log\epsilon + 2\log[0,\infty] - 2g(\infty) - 2\hat{\eta}_{g,\{\infty\}}(\epsilon) = \log\epsilon - 2g(\infty) - 2\hat{\eta}_{g,\{\infty\}}(\epsilon). \end{aligned}$$

There remains the case where $z = \infty$ and $w \in K$. By $\mathsf{d}(\iota(\mathcal{S}), \iota(\mathcal{S}')) = \mathsf{d}(\mathcal{S},\mathcal{S}')$ on $\mathsf{P}^1 \times \mathsf{P}^1$ and (4.3), for every $\epsilon \in (0,1]$ and every $z \in K$, we obtain

$$\begin{aligned} &\int_{\mathsf{P}^1\times\mathsf{P}^1} \Phi_g \, d([\infty]_\epsilon \times [z]_\epsilon) \\ &\quad\geq \int_{\mathsf{P}^1\times\mathsf{P}^1} \log[\mathcal{S},\mathcal{S}']_{\mathrm{can}} \, d([\infty]_\epsilon \times [z]_\epsilon)(\mathcal{S},\mathcal{S}') - g(\infty) - g(z) - 2\eta_{g,\{\infty,z\}}(\epsilon); \end{aligned}$$

moreover, (i) for archimedean $K \cong \mathbb{C}$, also by $\iota^2 = \mathrm{Id}$ on P^1, we have

$$\begin{aligned} &\int_{\mathsf{P}^1\times\mathsf{P}^1} \log[\mathcal{S},\mathcal{S}']_{\mathrm{can}} \, d([\infty]_\epsilon \times [z]_\epsilon)(\mathcal{S},\mathcal{S}') \\ &= \int_{\mathsf{P}^1\times\mathsf{P}^1} \log[\mathcal{S},\mathcal{S}']_{\mathrm{can}} \, d([0]_\epsilon \times \iota_*[z]_\epsilon)(\mathcal{S},\mathcal{S}') \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathsf{P}^1\times\mathsf{P}^1} \log|\mathcal{S}-\mathcal{S}'|_\infty \, d([0]_\epsilon \times \iota_*[z]_\epsilon)(\mathcal{S},\mathcal{S}') \\
&\quad + \int_{\mathsf{P}^1} \log[\mathcal{S},\infty]_{\mathrm{can}} \, d[0]_\epsilon(\mathcal{S}) + \int_{\mathsf{P}^1} \log[\mathcal{S}',0]_{\mathrm{can}} \, d[z]_\epsilon(\mathcal{S}') \\
&\ge \int_{\mathsf{P}^1} \log[\mathcal{S},\infty]_{\mathrm{can}} \, d[0]_\epsilon(\mathcal{S}) + \int_{\mathsf{P}^1} \log[\mathcal{S}',\infty]_{\mathrm{can}} \, d[z]_\epsilon(\mathcal{S}') \\
&\ge \log[0,\infty] + \log[z,\infty] - 2\epsilon = \log[z,\infty] - 2\epsilon,
\end{aligned}
$$

the former inequality in which holds because

$$
\begin{aligned}
&\int_{\mathsf{P}^1\times\mathsf{P}^1} \log|\mathcal{S}-\mathcal{S}'|_\infty \, d([0]_\epsilon \times \iota_*[z]_\epsilon)(\mathcal{S},\mathcal{S}') \\
&= \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^{2\pi} \log\left|\epsilon e^{i\theta} - \frac{1}{z+\epsilon e^{i\phi}}\right| \frac{d\theta}{2\pi} = \int_0^{2\pi} \max\left\{\log\left|\frac{1}{z+\epsilon e^{i\phi}}\right|, \log\epsilon\right\} \frac{d\phi}{2\pi} \\
&\ge -\int_0^{2\pi} \log|(z+\epsilon e^{i\phi})-0| \frac{d\phi}{2\pi} = -\int_{\mathsf{P}^1} \log|\mathcal{S}'-0|_\infty \, d[z]_\epsilon(\mathcal{S}') \\
&= -\int_{\mathsf{P}^1} \log[\mathcal{S}',0]_{\mathrm{can}} \, d[z]_\epsilon(\mathcal{S}') + \int_{\mathsf{P}^1} \log[\mathcal{S}',\infty]_{\mathrm{can}} \, d[z]_\epsilon(\mathcal{S}'),
\end{aligned}
$$

and (ii) for non-archimedean K, also by the definition (2.1) of $[\mathcal{S},\mathcal{S}']_{\mathrm{can}}$, we have

$$
\begin{aligned}
&\int_{\mathsf{P}^1\times\mathsf{P}^1} \log[\mathcal{S},\mathcal{S}']_{\mathrm{can}} \, d([\infty]_\epsilon \times [z]_\epsilon)(\mathcal{S},\mathcal{S}') \\
&\qquad = \log[\iota(\pi_\epsilon(0)),\pi_\epsilon(z)]_{\mathrm{can}} \\
&\qquad \left(= \begin{cases} \log \operatorname{diam} \mathcal{S}_{\mathrm{can}} = 1 & \text{if } |z| \le 1 \\ \log[\pi_\epsilon(0),\iota(\pi_\epsilon(z))]_{\mathrm{can}} \ge \log[0,\iota(z)] & \text{otherwise} \end{cases} \right) \\
&\qquad \ge \log[\infty,z].
\end{aligned}
$$

Hence $\int_{\mathsf{P}^1\times\mathsf{P}^1} \Phi_g \, d([\infty]_\epsilon \times [z]_\epsilon) \ge \Phi_g(\infty,z) - 2\hat{\eta}_{g,\{\infty,z\}}(\epsilon)$ for every $\epsilon \in (0,1]$ and every $z \in K$, and the proof of (4.1) is complete. ∎

5. On the optimality of (1.4). Let K be an algebraically closed field that is complete with respect to a non-trivial absolute value $|\cdot|$. We include a few examples of normalized weights g on P^1 for which (1.4) is optimal.

For archimedean K, the function $z \mapsto \log\max\{1,|z|\} + \log[z,\infty]$ on $K \cong \mathbb{C}$ extends to a normalized weight g_0 on $\mathsf{P}^1 \cong \mathbb{P}^1$, which is Lipschitz, i.e., $1/\kappa$-Hölder continuous on $(\mathbb{P}^1,[z,w])$ for $\kappa = 1$, by the piecewise smoothness of g_0 on $\mathbb{P}^1$. Moreover, for every $N \in \mathbb{N}$, we have $g_0 \equiv \log[\cdot,\infty]$ on $F_N :=$

$\{e^{2i\pi k/N} : k \in \{0,1,\ldots,N-1\}\}$ and

$$\frac{\sum_{z\in F_N}\sum_{w\in F_N\setminus\{z\}}\Phi_{g_0}(z,w)}{(\#F_N)\log(\#F_N)} = \frac{\sum_{k=0}^{N-1}\sum_{j\in\{0,1,\ldots,N-1\}\setminus\{k\}}\log|e^{2i\pi k/N}-e^{2i\pi j/N}|}{N\log N} = \frac{\sum_{k=0}^{N-1}\log\left|(z^N-1)'|_{z=e^{2i\pi k/N}}\right|}{N\log N} = \frac{N\log|N|}{N\log N} = 1,$$

which implies that (1.4) is optimal for g_0.

For non-archimedean K, we can fix $d\in\mathbb{N}$ such that $d>1$ and $|d|=1$ (note that if $k\in\mathbb{N}$ satisfies $|k|<1$, then $|k+1|=\max\{|k|,|1|\}=1$). Fix $\lambda\in K$ such that $|\lambda|>d^{d/(d-1)}$ (>1) and set $f_\lambda(z):=z^d+\lambda\in K[z]$. Then $g_{f_\lambda}(\mathcal{S})=-\lim_{n\to\infty}(\log[f_\lambda^n(\mathcal{S}),\infty]_{\mathrm{can}})/d^n+\log[\mathcal{S},\infty]_{\mathrm{can}}$ on P^1.

For every $z\in K$ and every $w\in K\setminus\{z\}$, we have $(f_\lambda(z)-f_\lambda(w))/(z-w)=\sum_{j=0}^{d-1}z^jw^{d-1-j}$, so that by the strong triangle inequality, we have

$$\frac{|f_\lambda(z)-f_\lambda(w)|}{|z-w|}\le\max\{|z|,|w|\}^{d-1}. \tag{5.1}$$

By $|\lambda|>1$ (and the strong triangle inequality), we also have $f_\lambda(\{z\in K : |z|\le|\lambda|^{1/d}\})\subset\{z\in K:|z|\le|\lambda|\}$, $f_\lambda(\{z\in K:|z|<|\lambda|^{1/d}\})\subset\{z\in K: |z|=|\lambda|\}\subset\{z\in K:|z|>|\lambda|^{1/d}\}$, and on $\{z\in K:|z|>|\lambda|^{1/d}\}$, $f_\lambda(z)=z^d$ (so $|f_\lambda(z)|=|z|^d>|z|>|\lambda|^{1/d}>1$). In particular,

$$\sup_{z\in K,\,n\in\mathbb{N}}\frac{\max\{1,|z|\}}{\max\{1,|f_\lambda^n(z)|\}}\le|\lambda|^{1/d}, \tag{5.2}$$

and for every $n\in\mathbb{N}$, setting $P_n:=\{z\in K : f_\lambda^n(z)=z\}$, we obtain $P_n\subset\{z\in K:|z|=|\lambda|^{1/d}\}$, $\#P_n=d^n$, and $g_{f_\lambda}\equiv\log[\cdot,\infty]$ on P_n, and also by the chain rule and $|d|=1$, we have $|(f_\lambda^n)'|\equiv(|d|\,|\lambda|^{(d-1)/d})^n=(|\lambda|^{(d-1)/d})^n$ (>1) on P_n.

Hence, for every $n\in\mathbb{N}$, using also the strong triangle inequality, we get

$$\frac{\sum_{z\in P_n}\sum_{w\in P_n\setminus\{z\}}\Phi_{g_{f_\lambda}}(z,w)}{(\#P_n)\log(\#P_n)} = \frac{\sum_{w\in P_n}\log\left|(f_\lambda^n(z)-z)'|_{z=w}\right|}{d^n\cdot\log(d^n)} = \frac{\log((|\lambda|^{(d-1)/d})^n)}{n\log d} = \frac{\log(|\lambda|^{(d-1)/d})}{\log d}.$$

On the other hand, for every $n\in\mathbb{N}$, every $z\in K$ and every $w\in K\setminus\{z\}$, setting

$$t:=\min\bigl\{j\in\{0,1,\ldots,n-1\}:\max\{|f^j(z)|,|f^j(w)|\}>|\lambda|^{1/d}\bigr\}$$

(under the convention that $\min\emptyset=-\infty$) and $L_t:=\max\{|f^t(z)|,|f^t(w)|\}$ unless $t=-\infty$, recalling the definition of $[z,w]$ in Notation 1.1, and using

(5.1) and (5.2), we obtain

$$\begin{aligned}&\frac{[f_\lambda^n(z), f_\lambda^n(w)]}{[z,w]}\\ &\le \Big(\prod_{j=0}^{n-1}\max\{|f_\lambda^j(z)|,|f_\lambda^j(w)|\}\Big)^{d-1}\cdot\frac{\max\{1,|z|\}}{\max\{1,|f_\lambda^n(z)|\}}\cdot\frac{\max\{1,|w|\}}{\max\{1,|f_\lambda^n(w)|\}}\\ &\le \begin{cases} ((|\lambda|^{1/d})^n)^{(d-1)}\cdot(|\lambda|^{1/d})^2 & \text{if } t=-\infty,\\ \dfrac{(\max\{|z|,|w|\}^{(d^n-1)/(d-1)})^{(d-1)}\cdot\max\{|z|,|w|\}}{\max\{|z|,|w|\}^{d^n}}\cdot|\lambda|^{1/d} & \text{if } t=0,\\ ((|\lambda|^{1/d})^{t-1})^{d-1}\cdot\dfrac{(L_t^{(d^{n-t}-1)/(d-1)})^{(d-1)}\cdot L_t}{L_t^{d^{n-t}}}\cdot|\lambda|^{1/d} & \\ & \text{if } t\in\{1,\dots,n-1\}\end{cases}\\ &\le ((|\lambda|^{1/d})^n)^{(d-1)}\cdot(|\lambda|^{1/d})^2,\end{aligned}$$

so that for every $n\in\mathbb{N}$, using also $|\lambda|>d^{d/(d-1)}$, we deduce that

$$\sup_{z,w\in\mathbb{P}^1,\, z\neq w}\frac{[f_\lambda^n(z), f_\lambda^n(w)]}{[z,w]}\le((|\lambda|^{1/d})^n)^{d-1}\cdot(|\lambda|^{1/d})^2>d^n.$$

In particular, by Fact 1.2, $g_{f_\lambda}|\mathbb{P}^1$ is $1/\kappa$-Hölder continuous on $(\mathbb{P}^1,[z,w])$ for every $\kappa>(\log(|\lambda|^{(d-1)/d}))/\log d$. Hence (1.4) is optimal for g_{f_λ}.

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