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ON A GENERALISATION OF THE BANACH INDICATRIX THEOREM

BY

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Abstract. We prove that for any regulated function $f : [a,b] \to \mathbb{R}$ and $c \geq 0$, the infimum of the total variations of functions approximating f with accuracy $c/2$ is equal to $\int_{\mathbb{R}} n_c^y \, dy$, where n_c^y is the number of times f crosses the interval $[y, y+c]$.

1. Introduction, definitions, notation. Let $a < b$ be fixed reals. The Banach Indicatrix Theorem describes an interesting relationship between the total variation of a continuous function $f : [a,b] \to \mathbb{R}$, defined as

$$\mathrm{TV}(f,[a,b]) := \sup_n \sup_{a \le t_0 < t_1 < \cdots < t_n \le b} \sum_{i=1}^n |f(t_i) - f(t_{i-1})|$$

and the numbers of level crossings of the function f, defined as

$$N^y(f) := \#\{x \in [a,b] : f(x) = y\}.$$

($\#A$ denotes here the cardinality of the set A.) The function $\mathbb{R} \ni y \mapsto N^y(f) \in \{0,1,2,\ldots\} \cup \{+\infty\}$ is a Baire function of class ≤ 2 and it is called the *Banach indicatrix*. The aforementioned relationship has the form

$$\mathrm{TV}(f,[a,b]) = \int_{\mathbb{R}} N^y(f) \, \mathrm{d}y.$$

It was proven by Banach [1, Théorème 1.2], and he is usually given full credit for it. Vitali [15] published the result (in the same journal) one year later, and Sergeĭ M. Lozinskiĭ [13], [14] generalised it to include the case of *regulated* functions, i.e. functions $f : [a,b] \to \mathbb{R}$ admitting right limits $f(t+)$ at any point $t \in [a,b)$ and left limits $f(t-)$ at any point $t \in (a,b]$. For a modern exposition of the proof of the Banach Indicatrix Theorem for continuous functions see [2, Theorem 4.2.7] or [4, Theorem (3.i)].

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Unfortunately, when $\mathrm{TV}(f,[a,b])=+\infty$ this result seems to be useless. The purpose of this paper is to state and prove a meaningful generalisation of the Banach Indicatrix Theorem for *any* real regulated function, possibly with infinite total variation, in terms of *interval crossings* rather than level crossings. Together with the numbers of interval and level crossings of the function f, we will consider the numbers of interval and level upcrossings (or crossings from below) and the numbers of interval and level downcrossings (or crossings from above). The numbers of level upcrossings and downcrossings are related to the *positive* and *negative* variations of a real function:

$$\mathrm{UTV}(f,[a,b]):=\sup_n \sup_{a\le t_0<t_1<\cdots<t_n\le b} \sum_{i=1}^n (f(t_i)-f(t_{i-1}))_+,$$

$$\mathrm{DTV}(f,[a,b]):=\sup_n \sup_{a\le t_0<t_1<\cdots<t_n\le b} \sum_{i=1}^n (f(t_i)-f(t_{i-1}))_-,$$

where $x_+=\max\{x,0\}$, $x_-=\max\{-x,0\}$. Recall the Jordan decomposition:

$$\mathrm{TV}(f,[a,b])=\mathrm{UTV}(f,[a,b])+\mathrm{DTV}(f,[a,b]).$$

It is possible to prove an analog of the Banach result for negative and positive variations and the numbers of level upcrossings and downcrossings (see [4, Section 9]). To be more precise, let us start with several definitions.

Definition 1.1. Given a function $f:[a,b]\to\mathbb{R}$, for $c\ge 0$ we set $\sigma_0^c=a$, and for $n=0,1,\ldots,$

$$\tau_n^c=\inf\{t>\sigma_n^c : t\le b,\ f(t)>y+c\},\quad \sigma_{n+1}^c=\inf\{t>\tau_n^c : t\le b,\ f(t)<y\}.$$

Next, we set

$$d_c^y(f,[a,b]):=\max\{n:\sigma_n^c\le b\}.$$

Similarly we introduce

Definition 1.2. Given a function $f:[a,b]\to\mathbb{R}$, for $c\ge 0$ we set $\tilde\sigma_0^c=a$, and for $n=0,1,\ldots,$

$$\tilde\tau_n^c=\inf\{t>\tilde\sigma_n^c : t\le b, f(t)<y\},\ \tilde\sigma_{n+1}^c=\inf\{t>\tilde\tau_n^c : t\le b, f(t)>y+c\}.$$

Next, we set

$$u_c^y(f,[a,b]):=\max\{n:\tilde\sigma_n^c\le b\}. \tag{1}$$

In all definitions we take the convention that $\inf\emptyset=+\infty$.

The number $d_c^y(f,[a,b])$ can be viewed as the number of times the graph of f "downcrosses" (on $[a,b]$) the closed value interval $[y,y+c]$, while the number $u_c^y(f,[a,b])$ can be viewed as the number of times the graph of f "upcrosses" the value interval $[y,y+c]$.

At last, for f and the interval $[a,b]$ as in the preceding two definitions, we define the number of times the graph of f crosses (on $[a,b]$) the value interval $[y, y+c]$ as

$$n_c^y(f,[a,b]) := d_c^y(f,[a,b]) + u_c^y(f,[a,b]).$$

REMARK 1.3. The precise definition of the number of level (up-, down-) crossings may be obtained by setting $c = 0$ in the preceding definitions. We define

$$D^y(f,[a,b]) := d_0^y(f,[a,b]), \qquad U^y(f,[a,b]) := u_0^y(f,[a,b]).$$

Finally, we define

$$N^y(f,[a,b]) := D^y(f,[a,b]) + U^y(f,[a,b]).$$

REMARK 1.4. Let us notice that the number $N^y(f,[a,b])$ of level crossings introduced in the previous remark may differ from the Banach indicatrix $N^y(f)$ (when f is continuous) or its generalisation for regulated f, introduced by Lozinskiĭ [13], which we will also denote by $N^y(f)$. However, it is not difficult to prove that the set $\{y \in \mathbb{R} : N^y(f) \neq N^y(f,[a,b])\}$ is countable. Indeed, both numbers coincide for any real $y \notin f([a,b])$ and for any $y \in f([a,b]) \setminus \{f(a), f(b)\}$ which is not a local maximum or minimum of f. It remains to prove that the set of local maxima and minima is countable. Indeed, for any local maximum y there exist two rational numbers o and $r > 0$ such that $[o-r, o+r] \subset [a,b]$ and $y = \max_{t\in[o-r,o+r]} f(t)$. Since the mapping $y \mapsto (o,r)$ is injective, the set of local maxima is countable. Similarly one proves that the set of local minima is countable.

Our numbers $N^y(f,[a,b])$, $U^y(f,[a,b])$ and $D^y(f,[a,b])$ coincide with the numbers N_e, N_+ and N_- introduced by Cesari [4, p. 329].

Naturally, the just defined numbers of interval or level crossings may be infinite; however, if f is regulated then for any $c > 0$ and $y \in \mathbb{R}$, $n_c^y(f,[a,b])$ is finite (see [6, Theorem 2.1]). The fact that a regulated function on a compact interval crosses any non-degenerate value interval only finitely many times is closely related to the fact that for any regulated function its truncated variation on the truncation level $c > 0$ (on a compact interval) is always finite. For $f : [a,b] \to \mathbb{R}$, its *truncated variation* on $[a,b]$ with the truncation parameter $c > 0$ is defined as

$$\text{(2)} \qquad \mathrm{TV}^c(f,[a,b]) := \sup_n \sup_{a\le t_0<t_1<\cdots<t_n\le b} \sum_{i=1}^n \max\{|f(t_i) - f(t_{i-1})| - c, 0\}.$$

Similarly, we define the *upward* and *downward truncated variations* (with truncation parameter $c > 0$) of f respectively as

$$\text{(3)}\quad \mathrm{UTV}^c(f,[a,b]) := \sup_n \sup_{a\le t_0<t_1<\cdots<t_n\le b} \sum_{i=1}^n \max\{f(t_i)-f(t_{i-1})-c,0\},$$

$$\text{(4)}\quad \mathrm{DTV}^c(f,[a,b]) := \sup_n \sup_{a\le t_0<t_1<\cdots<t_n\le b} \sum_{i=1}^n \max\{f(t_{i-1})-f(t_i)-c,0\}.$$

It is possible to prove that $f:[a,b]\to\mathbb{R}$ is regulated iff $\mathrm{TV}^c(f,[a,b])<\infty$ for any $c>0$ (see [10, Fact 2.2]).

REMARK 1.5. For $g,h:[a,b]\to\mathbb{R}$ let us denote

$$\|g-h\|_\infty := \sup_{t\in[a,b]} |g(t)-h(t)|.$$

The (downward-, upward-) truncated variation has an interesting variational property. It is possible to prove (see [12, Theorem 4]) that $\mathrm{TV}^c(f,[a,b])$ is the *attainable* infimum of total variations of functions uniformly approximating regulated f with accuracy $c/2$,

$$\mathrm{TV}^c(f,[a,b]) = \inf\{\mathrm{TV}(g,[a,b]) :\ g:[a,b]\to\mathbb{R},\ \|f-g\|_\infty \le c/2\}.$$

Similarly, $\mathrm{UTV}^c(f,[a,b])$ is the *attainable* infimum of positive variations of functions uniformly approximating f with accuracy $c/2$,

$$\mathrm{UTV}^c(f,[a,b]) = \inf\{\mathrm{UTV}(g,[a,b]) :\ g:[a,b]\to\mathbb{R},\ \|f-g\|_\infty \le c/2\},$$

and

$$\mathrm{DTV}^c(f,[a,b]) = \inf\{\mathrm{DTV}(g,[a,b]) :\ g:[a,b]\to\mathbb{R},\ \|f-g\|_\infty \le c/2\}.$$

From [12, Theorem 4] and the Jordan decomposition it also follows that

$$\text{(5)}\qquad \mathrm{TV}^c(f,[a,b]) = \mathrm{UTV}^c(f,[a,b]) + \mathrm{DTV}^c(f,[a,b]).$$

Now we are ready to state the main result of this article.

THEOREM 1. *Let $f:[a,b]\to\mathbb{R}$ be a regulated function. For any $c>0$ the mappings $y\mapsto u_c^y(f,[a,b])$ and $y\mapsto d_c^y(f,[a,b])$ are pointwise limits of step functions, and the following equalities hold:*

$$\text{(6)}\qquad \mathrm{UTV}^c(f,[a,b]) = \int_{\mathbb{R}} u_c^y(f,[a,b])\,dy,$$

$$\text{(7)}\qquad \mathrm{DTV}^c(f,[a,b]) = \int_{\mathbb{R}} d_c^y(f,[a,b])\,dy,$$

$$\text{(8)}\qquad \mathrm{TV}^c(f,[a,b]) = \int_{\mathbb{R}} n_c^y(f,[a,b])\,dy.$$

From this theorem, Remark 1.4 and the classical monotone convergence theorem for the Lebesgue integral, letting $c\downarrow 0$ we easily obtain Lozinskiĭ's result as well as Cesari's result [4, Section 9] for positive and negative variations.

REMARK 1.6. The natural question arises whether it is possible to obtain further generalisations of Theorem 1. For example, if one wants to distinguish the importance of some levels from others by assigning to each level $x \in \mathbb{R}$ its weight $m(x)$ then one should consider the integral

$$\int_{\mathbb{R}} n_c^y(f,[a,b])m(y)\,dy.$$

A similar generalisation for $N^y(f)$ is known in the form of the change of variables formula under minimal assumptions (see [7]).

2. Proof of Theorem 1. The proof will go along similar lines to the proof of [12, Theorem 8]. Another possible method of the proof of the following, slightly weaker estimates:

$$\mathrm{UTV}^c(f,[a,b]) = \int_{\mathbb{R}} u_c^y(f,[a,b])\,dy \le \mathrm{UTV}^c(f,[a,b]) + c,$$

$$\mathrm{DTV}^c(f,[a,b]) = \int_{\mathbb{R}} d_c^y(f,[a,b])\,dy \le \mathrm{DTV}^c(f,[a,b]) + c,$$

$$\mathrm{TV}^c(f,[a,b]) = \int_{\mathbb{R}} n_c^y(f,[a,b])\,dy \le \mathrm{TV}^c(f,[a,b]) + 2c$$

was outlined (for càdlàg f) and then used in [11] to prove limit theorems for numbers of interval crossings for diffusions and semimartingales. This method utilized the fact that for any $c > 0$ and any starting value $x \in [f(a) - c/2, f(a) + c/2]$ there exists a function $f^{c,x} : [a,b] \to \mathbb{R}$ such that $\|f^{c,x} - f\|_\infty \le c/2$, $f^{c,x}(a) = x$ and $f^{c,x}$ has the minimal total variation on any interval $[a,t]$, $t \in [a,b]$, among all functions approximating f with accuracy $c/2$ and attaining at a the value x. This function coincides with the solution of the so called Skorokhod problem on $[-c/2, c/2]$ for f and has an interesting property: the number of times $f^{c,x}$ crosses (from above or from below) the level $y + c/2$ is almost the same as the number of times f crosses the interval $[y, y+c]$ (see [11, Lemmas 3.3, 3.4]). For other problems similar to the Skorokhod problem, like the play operator or taut strings, see for example [12] or [8].

Proof of Theorem 1. First, as in [12], we will prove Theorem 1 for the family of step functions and then we will utilise the fact that each regulated function is a uniform limit of step functions.

STEP 1: *Proof for step functions.* First we will assume that

$$f(t) = \sum_{k=0}^{n} f_{2k} 1_{\{t(2k)\}}(t) + \sum_{k=0}^{n-1} f_{2k+1} 1_{(t(2k+1);t(2k+2))}(t),$$

where $a = t(0) = t(1) < t(2) = t(3) < \cdots < t(2n-2) = t(2n-1) < t(2n) = b$.

Let $f_{(0)} \le f_{(1)} \le \cdots \le f_{(2n)}$ be the non-decreasing rearrangement of the sequence f_i, $i = 0, 1, \ldots, 2n$. For any real y_1, y_2 such that $f_{(i_1-1)} < y_1 \le y_2 < f_{(i_1)}$ and $f_{(i_2-1)} < y_1 + c \le y_2 + c < f_{(i_2)}$ for some $i_1, i_2 = 1, \ldots, 2n$, we have $u_c^{y_1}(f, [a,b]) = u_c^{y_2}(f, [a,b])$ and $d_c^{y_1}(f, [a,b]) = d_c^{y_2}(f, [a,b])$, thus $y \mapsto u_c^y(f, [a,b])$ and $y \mapsto d_c^y(f, [a,b])$ are step functions.

Now we will prove (6). The (upward-, downward-) truncated variations of f and the numbers of interval (up-, down-) crossings of f are equal to the (discrete versions of the) truncated variations and the numbers of interval crossings of the function $p : \{0, 1, \ldots, 2n\} \to \mathbb{R}$ defined as

$$p(i) = f_i \quad \text{for } i = 0, 1, \ldots, 2n.$$

The truncated variation of p is simply defined as

$$\mathrm{TV}^c(p, [a,b]) := \max_{m \le 2n} \max_{0 \le i_0 < i_1 < \cdots < i_m \le 2n} \sum_{j=1}^{m} \max\{|p(i_j) - p(i_{j-1})| - c, 0\}$$

and analogously one defines the upward- and downward-truncated variations of p. The definitions of the numbers of interval up- and down-crossings of p are obvious modifications of Definitions 1.2 and 1.1.

Thus, it is enough to prove the assertion for the (upward-, downward-) truncated variation of p and the numbers of interval (up-, down-) crossings of p.

First, for $i = 0, 1, \ldots, 2n - 1$ we define

$$I_U(i) = \min\Big\{ j \in \{i+1, \ldots, 2n\} : f_j > \min_{i \le k < j} f_k + c \Big\},$$

$$I_D(i) = \min\Big\{ j \in \{i+1, \ldots, 2n\} : f_j < \max_{i \le k < j} f_k - c \Big\},$$

with the convention that $\min \emptyset = \infty$.

Now we are ready to compare the upward-truncated variation of p with the integrated numbers of interval crossings. Assume that $I_U(0) \le I_D(0)$ (the case $I_U(0) \ge I_D(0)$ is symmetric).

If $I_U(0) = \infty$ then also $I_D(0) = \infty$ and the function p crosses no interval of the form $[y, y+c]$, and its oscillation

$$\|p\|_{\mathrm{osc}} := \max_{0 \le i < j \le 2n} |p(j) - p(i)|$$

is smaller than or equal to c. Hence

$$\mathrm{TV}^c(p, [0, 2n]) = 0$$

and

$$n_c^y(p, [0, 2n]) = 0 \quad \text{for all } y \in \mathbb{R}.$$

Thus the assertion follows.

Now assume that $I_U(0) < +\infty$, thus $I_U(0) \le 2n$, and define sequences $(I_{D,k})_{k=-1}^{\infty}$, $(I_{U,k})_{k=0}^{\infty}$ in the following way: $I_{D,-1} = 0$, $I_{U,0} = I_U(0)$, and for

$k = 0, 1, \ldots,$

$$I_{D,k} = \begin{cases} I_D(I_{U,k}) & \text{if } I_{U,k} \le 2n-1, \\ \infty & \text{otherwise,} \end{cases} \quad I_{U,k+1} = \begin{cases} I_U(I_{D,k}) & \text{if } I_{D,k} \le 2n-1, \\ \infty & \text{otherwise.} \end{cases}$$

For $k = 0, 1, \ldots$ such that $I_{D,k-1} \le 2n$ denote

$$m_k = \min_{I_{D,k-1} \le j \le \min\{I_{U,k}-1, 2n\}} f_j,$$

and for $k = 0, 1, \ldots$ such that $I_{U,k} \le 2n$ set

$$M_k = \max_{I_{U,k} \le j \le \min\{I_{D,k}-1, 2n\}} f_j.$$

We see that on the interval $[I_{D,-1}, \min\{I_{D,0}, 2n\}]$ the function p crosses the value interval $[y, y+c]$ from below exactly once iff $y \in (m_0, M_0 - c)$. It does not cross the interval $[y, y+c]$ if $y \le m_0$ or $y \ge M_0 - c$. See Figure 1. Next, if $I_{D,0} \le 2n$ then the function p crosses no value interval

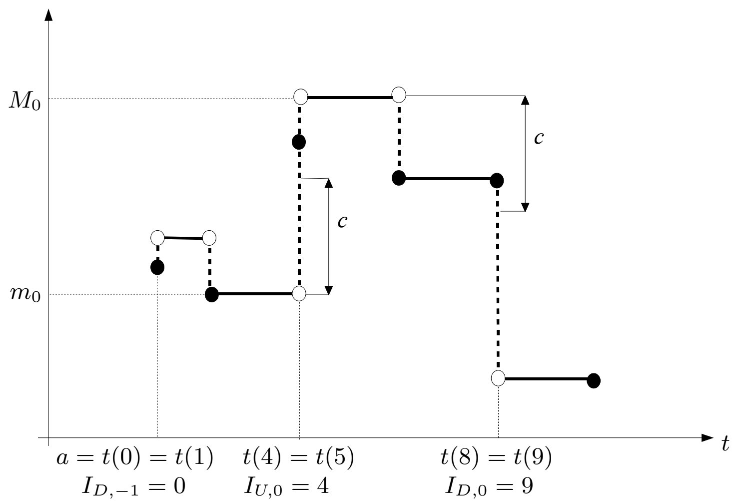

Fig. 1. Times $t(0), t(1), \ldots$ and values of m_0 and M_0 for a typical step path (the graph of the path is represented by the thick solid line and black dots)

$[y, y+c]$ from below on $[I_{D,0}, \min\{I_{U,1} - 1, 2n\}]$ but, as before, if $I_{U,1} \le 2n$ then it crosses the value interval $[y, y+c]$ from below exactly once on $[I_{D,0}, \min\{I_{D,1}, 2n\}]$ iff $y \in (m_1, M_1 - c)$. It does not cross the value interval $[y, y+c]$ on $[I_{D,0}, \min\{I_{D,1}, 2n\}]$ if $y \le m_1$ or $y \ge M_1 - c$. Continuing this reasoning in the same way we find that for any $k = 0, 1, \ldots,$ if $I_{U,k} \le 2n$ then on the interval $[I_{D,k-1}, \min\{I_{D,k}, 2n\}]$ we have

$$M_k - m_k - c = \int_{m_k}^{M_k - c} 1\, dy = \int_{\mathbb{R}} u_c^y(p, [I_{D,k-1}, \min\{I_{D,k}, 2n\}])\, dy.$$

Next, denote

$$K = \max\big\{k \in \{0, 1, \dots\} : I_{U,k} \le 2n\big\}.$$

We will prove that in fact

$$\sum_{k=0}^{K} \{M_k - m_k - c\} = \int_{\mathbb{R}} u_c^y(p, [0, 2n])\, dy. \tag{9}$$

This may be easily proven using e.g. induction on K. From the equality

$$M_k - m_k - c = \int_{\mathbb{R}} 1_{(m_k, M_k - c)}(y)\, dy$$

it follows that to prove (9) it is sufficient to prove that for any real y,

$$u_c^y(p, [0, 2n]) = \#\big\{k \in \{0, 1, \dots, K\} : y \in (m_k, M_k - c)\big\}. \tag{10}$$

For $K = 0$ this was already shown. Next, let $K > 0$ and for $y \in \mathbb{R}$ let

$$k_0(y) = \max\{k : (k = -1) \text{ or } (k \in \{0, 1, \dots, K-1\} \text{ and } y \in (m_k, M_k - c))\}.$$

By the induction hypothesis (10) this means that there are no i, j with $I_{D,k_0(y)} \le i < j \le I_{D,K-1} - 1$ such that

$$p(i) < y < p(j) - c.$$

But this also means that

$$\begin{aligned} u_c^y(p, [0, 2n]) &= u_c^y(p, [0, I_{D,K-1}]) + u_c^y(p, [I_{D,K-1}, \min\{I_{D,K}, 2n\}]) \\ &= \#\big\{k \in \{0, 1, \dots, K-1\} : y \in (m_k, M_k - c)\big\} + 1_{(m_K, M_K - c)}(y) \\ &= \#\big\{k \in \{0, 1, \dots, K\} : y \in (m_k, M_k - c)\big\}, \end{aligned}$$

which completes the induction proof of (9).

Now we will prove the equality

$$\sum_{k=0}^{K} \{M_k - m_k - c\} = \mathrm{UTV}^c(p, [0, 2n]). \tag{11}$$

We define the following function $p^c : \{0, 1, \dots, 2n\} \to \mathbb{R}$:

$$p^c(i) = \begin{cases} m_0 + c/2 & \text{if } i < I_{U,0}, \\ \max_{j \in \{I_{U,k}, I_{U,k}+1, \dots, i\}} f_j - c/2 & \text{if } I_{U,k} \le i < I_{D,k} \text{ for some } k = 0, 1, \dots, \\ \min_{j \in \{I_{D,k}, I_{D,k}+1, \dots, i\}} f_j + c/2 & \text{if } I_{D,k} \le i < I_{U,k+1} \text{ for some } k = 0, 1, \dots. \end{cases}$$

From the definition of $I_{U,k}$, $I_{D,k}$, $k = 0, 1, \dots$, (and the assumption $I_{U,0} \le I_{D,0}$) it follows that the function p^c approximates p with accuracy $c/2$, $\|p - p^c\|_\infty \le c/2$ (see also [12, proof of Theorem 8]). From this and the equality

$$p(j) - p(i) - \{p(j) - p^c(j)\} - \{p^c(i) - p(i)\} = p^c(j) - p^c(i) \quad \text{for } i, j \in [0, 2n],$$

it follows that

$$(p(j)-p(i)-c)_+ \le (p^c(j)-p^c(i))_+ \quad \text{for } i,j \in [0,2n]$$

and

$$\text{UTV}^c(p,[0,2n]) \le \text{UTV}(p^c,[0,2n]) \tag{12}$$

(recall also Remark 1.5). Moreover, since for k such that $I_{U,k} < +\infty$, p^c is non-decreasing on $[I_{U,k}, \min\{I_{D,k}-1, 2n\}]$, and for k such that $I_{D,k} < +\infty$, p^c is non-increasing on $[I_{D,k}, \min\{I_{U,k+1}-1, 2n\}]$, we see that

$$\text{UTV}(p^c,[0,2n]) = \sum_{k=0}^{K}\{M_k - m_k - c\}. \tag{13}$$

On the other hand, since $\text{UTV}^c(p,\cdot)$ is a superadditive functional of the interval, i.e. for $0 < i < 2n$,

$$\text{UTV}^c(p,[0,2n]) \ge \text{UTV}^c(p,[0,i]) + \text{UTV}^c(p,[i,2n]),$$

we have

$$\begin{aligned}\text{UTV}^c(p,[0,2n]) &\ge \sum_{k=0}^{K-1} \text{UTV}^c(p,[I_{D,k-1}, I_{D,k}]) + \text{UTV}^c(p,[I_{D,K-1}, 2n]) \\ &\ge \sum_{k=0}^{K}\{M_k - m_k - c\}.\end{aligned} \tag{14}$$

Now, from (12)–(14) we get (11). Finally, from (9) and (11) we obtain

$$\text{UTV}^c(p,[0,2n]) = \int_{\mathbb{R}} u_c^y(p,[0,2n])\,dy,$$

which may be translated into

$$\text{UTV}^c(f,[a,b]) = \int_{\mathbb{R}} u_c^y(f,[a,b])\,dy.$$

Similarly, considering interval downcrossings and the downward-truncated variation of p on the intervals $[I_{U,k}, I_{U,k+1}]$, $k = 0,1,\ldots$, one obtains

$$\text{DTV}^c(p,[0,2n]) = \int_{\mathbb{R}} d_c^y(p,[0,2n])\,dy,$$

which translates into

$$\text{DTV}^c(f,[a,b]) = \int_{\mathbb{R}} d_c^y(f,[a,b])\,dy.$$

Finally, from the definition of n_c^y and the equality

$$\text{TV}^c(f,[a,b]) = \text{UTV}^c(f,[a,b]) + \text{DTV}^c(f,[a,b])$$

(recall (5) in Remark 1.5) we get

$$\mathrm{TV}^c(f,[a,b]) = \int_{\mathbb{R}} n_c^y(f,[a,b])\,dy.$$

STEP 2: *Proof for arbitrary regulated functions.* We will use [12, Lemma 21] and the fact that any regulated function $f:[a,b]\to\mathbb{R}$ is a uniform limit of step functions (see e.g. [5, Theorem 7.6.1]).

Let $\varepsilon\in(0,c/2)$ and $f^\varepsilon:[a,b]\to\mathbb{R}$ be a step function such that $\|f-f^\varepsilon\|_\infty\le\varepsilon$. From the very definition of the number of interval upcrossings we get

$$u_c^y(f,[a,b]) \le u_{c-2\varepsilon}^{y+\varepsilon}(f^\varepsilon,[a,b]). \tag{15}$$

Indeed, each time f upcrosses the value interval $[y,y+c]$ there exist two times s,t such that $a\le s<t\le b$ and $f(s)<y$, and $f(t)>y+c$. From this and $\|f-f^\varepsilon\|_\infty\le\varepsilon$ we immediately get

$$f^\varepsilon(s)\le f(s)+\varepsilon<y+\varepsilon \text{ and } f^\varepsilon(t)\ge f(t)-\varepsilon>y+c-\varepsilon=(y+\varepsilon)+c-2\varepsilon,$$

thus times s,t also correspond to the upcrossing by f^ε of the value interval $[y+\varepsilon,y+c-\varepsilon]$. Similarly, considering the times when f^ε upcrosses the value interval $[y-\varepsilon,y+c+\varepsilon]$ one proves that

$$u_{c+2\varepsilon}^{y-\varepsilon}(f^\varepsilon,[a,b]) \le u_c^y(f,[a,b]). \tag{16}$$

Letting $\varepsilon\to0$ we see that $y\mapsto u_c^y(f,[a,b])$ is a pointwise limit of step functions (since $y\mapsto u_{c-2\varepsilon}^{y+\varepsilon}(f^\varepsilon,[a,b])$ and $y\mapsto u_{c+2\varepsilon}^{y-\varepsilon}(f^\varepsilon,[a,b])$ are step functions). Now, from (15), (16) and Step 1 we get

$$\mathrm{UTV}^{c+2\varepsilon}(f^\varepsilon,[a,b]) \le \int_{\mathbb{R}} u_c^y(f,[a,b])\,dy \le \mathrm{UTV}^{c-2\varepsilon}(f^\varepsilon,[a,b]).$$

Letting $\varepsilon\to0$ and applying [12, Lemma 21] we get

$$\mathrm{UTV}^c(f,[a,b]) = \int_{\mathbb{R}} u_c^y(f,[a,b])\,dy.$$

Analogous equalities (7) and (8) for $\mathrm{DTV}^c(f,[a,b])$ and $\mathrm{TV}^c(f,[a,b])$ may be justified in the same way. ∎

3. Example of application. In this section we will give an example of application of Theorem 1. Let B_t, $t\ge0$, be a standard Brownian motion and $W_t=B_t+\mu t$, $t\ge0$, be a standard Brownian motion with drift μ. The following formula for the Laplace transform of the function $t\mapsto\mathbb{E}\mathrm{UTV}^c(W,[0,t])$ was proven in [9, (25)]: if ν is a complex number with $\Re(\nu)<0$, then

$$\int_0^{+\infty} e^{\nu t}\mathbb{E}\mathrm{UTV}^c(W,[0,t])\,dt = \frac{e^{\mu c}\sqrt{\mu^2-2\nu}}{2\nu^2\sinh(c\sqrt{\mu^2-2\nu})}. \tag{17}$$

Using Theorem 1 we will prove this formula when $v = -\nu$ is a positive real number. Notice that if $v = -\nu$ is a positive real number and τ is an exponential random variable with mean $1/v$, independent of B, then

$$\int_0^{+\infty} e^{\nu t}\mathbb{E}\mathrm{UTV}^c(W,[0,t])\,dt = \frac{1}{v}\mathbb{E}\mathrm{UTV}^c(W,[0,\tau]).$$

Thus, to prove formula (17) it is sufficient to prove that

$$\mathbb{E}\mathrm{UTV}^c(W,[0,\tau]) = \frac{e^{\mu c}\sqrt{\mu^2+2v}}{2v\sinh(c\sqrt{\mu^2+2v})}.$$

First, for $y\in\mathbb{R}$ let us calculate $\mathbb{E}u_c^y(W,[0,\tau])$. For $y\in\mathbb{R}$ let τ_y be the first hitting time of y by W, i.e.

$$\tau_y = \inf\{t\geq 0 : W_t = y\}.$$

For $y\geq 0$ the event $u_c^y(W,[0,\tau])\geq 1$ is equivalent to $\{\tau_{y+c}\leq\tau\}$, and by [3, formula 1.1.2, p. 250] we have

$$\begin{aligned}(18)\qquad \mathbb{P}(u_c^y(W,[0,\tau])\geq 1) &= \mathbb{P}(\tau_{y+c}\leq\tau)\\ &= \mathbb{P}\Big(\sup_{0\leq s\leq\tau} W_s\geq y+c\Big) = e^{\mu(y+c)-(y+c)\sqrt{\mu^2+2v}}.\end{aligned}$$

Now, let $y<0$. By [3, formula 1.2.2, p. 251] we have

$$\mathbb{P}(\tau_y\leq\tau) = \mathbb{P}\Big(\inf_{0\leq s\leq\tau} W_s\leq y\Big) = e^{\mu y+y\sqrt{\mu^2+2v}}.$$

Next, for $y<0$ the event $u_c^y(W,[0,\tau])\geq 1$ is equivalent to $\{\tau_y\leq\tau\}\cap\{\sup_{\tau_y\leq s\leq\tau}(W_s-W_{\tau_y})\geq c\}$ and by the strong Markov property of the Brownian motion and the lack of memory of the exponential distribution we get

$$\mathbb{P}\Big(\sup_{\tau_y\leq s\leq\tau}(W_s-W_{\tau_y})\geq c\;\Big|\;\tau_y\leq\tau\Big) = \mathbb{P}\Big(\sup_{0\leq s\leq\tau} W_s\geq c\Big),$$

thus

$$\begin{aligned}(19)\qquad \mathbb{P}(u_c^y(W,[0,\tau])\geq 1) &= \mathbb{P}(\tau_y\leq\tau)\mathbb{P}(\tau_c\leq\tau)\\ &= e^{\mu y+y\sqrt{\mu^2+2v}}e^{\mu c-c\sqrt{\mu^2+2v}} = e^{\mu(y+c)+(y-c)\sqrt{\mu^2+2v}}.\end{aligned}$$

Now, for $y\in\mathbb{R}$ define the stopping time

$$\upsilon_y = \inf\{t\geq 0 : u_c^y(W,[0,t])\geq 1\}.$$

Again using the strong Markov property of the Brownian motion and the lack of memory of the exponential distribution we get

$$\begin{aligned}\mathbb{P}(u_c^y(W,[0,\tau])\geq 2) &= \mathbb{P}(\upsilon_y\leq\tau)\mathbb{P}(u_c^y(W,[\upsilon_y,\tau])\geq 1\mid\upsilon_y\leq\tau)\\ &= \mathbb{P}(u_c^y(W,[0,\tau])\geq 1)\mathbb{P}(u_c^{-c}(W,[0,\tau])\geq 1)\\ &= \mathbb{P}(u_c^y(W,[0,\tau])\geq 1)e^{-2c\sqrt{\mu^2+2v}}.\end{aligned}$$

Similarly, for all $n = 1, 2, \dots,$

$$\mathbb{P}(u_c^y(W,[0,\tau]) \geq n) = \mathbb{P}(u_c^y(W,[0,\tau]) \geq 1)e^{-2c(n-1)\sqrt{\mu^2+2v}}. \tag{20}$$

From (20) we get

$$\begin{aligned} \mathbb{E}u_c^y(W,[0,\tau]) &= \mathbb{P}(u_c^y(W,[0,\tau]) \geq 1)\sum_{n=1}^{\infty} e^{-2c(n-1)\sqrt{\mu^2+2v}} \\ &= \mathbb{P}(u_c^y(W,[0,\tau]) \geq 1)\frac{1}{1-e^{-2c\sqrt{\mu^2+2v}}}. \end{aligned} \tag{21}$$

Finally, from Theorem 1, (18), (19) and (21) we have

$$\begin{aligned} \mathbb{E}\mathrm{UTV}^c(W,[0,\tau]) &= \mathbb{E}\int_{\mathbb{R}} u_c^y(W,[0,\tau])\,dy = \int_{\mathbb{R}} \mathbb{E}u_c^y(W,[0,\tau])\,dy \\ &= \int_0^{+\infty} \frac{e^{\mu(y+c)-(y+c)\sqrt{\mu^2+2v}}}{1-e^{-2c\sqrt{\mu^2+2v}}}\,dy + \int_{-\infty}^{0} \frac{e^{\mu(y+c)+(y-c)\sqrt{\mu^2+2v}}}{1-e^{-2c\sqrt{\mu^2+2v}}}\,dy \\ &= \frac{e^{\mu c-c\sqrt{\mu^2+2v}}}{1-e^{-2c\sqrt{\mu^2+2v}}}\left(\frac{1}{\sqrt{\mu^2+2v}-\mu}+\frac{1}{\sqrt{\mu^2+2v}+\mu}\right) \\ &= \frac{e^{\mu c}}{e^{c\sqrt{\mu^2+2v}}-e^{-c\sqrt{\mu^2+2v}}}\,\frac{2\sqrt{\mu^2+2v}}{2v} = \frac{e^{\mu c}\sqrt{\mu^2+2v}}{2v\sinh(c\sqrt{\mu^2+2v})}. \end{aligned}$$

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