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# *SYMMETRIES AND ERGODIC PROPERTIES IN QUANTUM PROBABILITY*

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**Abstract.** We deal with the general structure of (noncommutative) stochastic processes by using the standard techniques of operator algebras. Any stochastic process is associated to a state on a universal object, i.e. the free product $C^*$-algebra, in a natural way. In this setting, one recovers the classical (i.e. commutative) probability scheme and many others, like those associated to the monotone, boolean and $q$-deformed canonical commutation relations including the Bose/Fermi and Boltzmann cases. Natural symmetries like stationarity and exchangeability, as well as the ergodic properties of the stochastic processes are reviewed in detail for many interesting cases arising from quantum physics and probability.

## 1. Introduction.

The concept of *quantum probability* has been introduced in the mid-seventies in the pioneering works of L. Accardi, R. L. Hudson, K. R. Parthasarathy, and many other scientists [1, 5, 17, 22]. Since then, natural applications to various fields of mathematics and physics have been found. We mention the seminal investigation by D. V. Voiculescu involving the *free probability* and its applications to non-hyperfinite type $\mathrm{II}_1$ von Neumann factors [28], as well as the intersections with harmonic analysis first found by Bożejko [9]. We also point out the remarkable connections, recently investigated in [6, 23], between quantum groups introduced by Woronowicz [31] and quantum probability.

The present paper mainly deals with the general structure of stochastic processes and their natural symmetries, like stationarity and exchangeability, by using the standard techniques of operator algebras [14, 15, 16]. Although some of the pivotal results have been obtained in the above mentioned papers, our aim is to describe new ones and present the matter in a unified approach. Thus, the notes appear as an expository-research paper. We first show it is possible to view in a unified way the stochastic processes

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with sample space the unital, not necessarily commutative, $C^*$-algebra $\mathfrak{A}$ and with index set $J$. Indeed, we show that the collection of these stochastic processes is in one-to-one correspondence with the state space of the free product $C^*$-algebra $*_J\,\mathfrak{A}$. In particular, the subclasses of exchangeable or stationary stochastic processes correspond to the convex compact (provided the processes are identity-preserving) sets of symmetric (i.e. invariant under finite permutations) and shift-invariant states, respectively. The *algebraic probability space* $(\mathfrak{B},\varphi)$ (see e.g. [23]) associated to the stochastic process described by the state $\omega$ is then recovered by the Gelfand–Naimark–Segal (GNS for short) representation $(\mathcal{H}_\omega,\pi_\omega,\Omega_\omega)$, as

$$(\mathfrak{B},\varphi) := \big(\overline{\pi_\omega(*_J\,\mathfrak{A})}, \langle \pi_\omega(\,\cdot\,)\Omega_\omega\,\Omega_\omega\rangle\big),$$

where the closure is meant in norm or in the weak operator topology for the $C^*$ and $W^*$ cases, respectively. This provides a generalisation to the quantum case of the Kolmogorov Extension-Reconstruction Theorem. It follows that quantum probability, considered as a universal scheme, has a significant complexity of its own.

In our opinion, the main advantages of this unified description are the following.

First, it is possible to study all known cases directly on a suitable quotient algebra of a single one. Any quotient is obtained by factoring the free product $C^*$-algebra by the ideal generated by a “concrete” commutator. This allows us to cover, e.g., the cases of $q$-canonical commutation relations, which includes the Bose/Fermi and the Boltzmann (i.e. free) case, or equally well the boolean and monotone cases. The list is far from being complete, since one can add the commutative scheme which arises from the abelianisation of the free product.

Second, the natural symmetries of stochastic processes like stationarity and exchangeability, as well as their ergodic properties, can be handled using the standard results of ergodic theory (see e.g. [13, Chapter 4]).

As an example, we mention the equivalence between some factorisation rules naturally emerging in quantum statistical mechanics and the property of convergence to equilibrium for stationary and symmetric states (cf. Theorem 3.4). The features above are the content of Section 2.

The subsequent sections of the paper are devoted to reviewing the applications of the general results of Section 2 to the ergodic properties of stochastic processes arising from genuine quantum cases. The main results are complemented by sketches of proofs for the convenience of the reader.

In Section 3, we first connect some ergodic/clustering properties of a given stochastic process with some algebraic properties of the corresponding state, namely the *product state* or the *block singleton* conditions (see Definition 3.3). This leads to clarifying the role played by the block singleton condition in quantum probability as the right noncommutative analogue of the product state condition. Indeed, under the invariance conditions of stationarity or exchangeability, it is shown that the states realising the block singleton condition are exactly those satisfying the noncommutative analogue of convergence to equilibrium (3.3). Moreover, the product state condition turns out to be equivalent to the ergodic property of weak clustering (3.2). Such results find a natural application to the so called Haagerup states [21] on the group $C^*$-algebra of the free group on infinitely many generators, which appear in free probability.

Section 4 is devoted to examples. We review in a self-contained form de Finetti-type results and ergodic properties for stationary and symmetric states in some concrete $C^*$-algebras, plenty of them coming from physical investigations. The cases of $q$-deformed, $-1 < q < 1$ [11, 15], Bose [16, 26], Fermi [14, 15], boolean [9, 15, 16] and monotone [16, 24] processes are described in detail.

**2. Exchangeable and stationary stochastic processes.** Fix a $C^*$-algebra $\mathfrak{A}$ and an index set $J$. We suppose without further mention that $\mathfrak{A}$ is unital and all morphisms preserve the algebraic structure, including the $*$-operation. For simplicity we may think about $J = \mathbb{Z}$ to achieve the two-sided shift, but general index sets are allowed too, as also are nonunital sample algebras $\mathfrak{A}$, or non-identity-preserving stochastic processes.

A *dynamical system* based on a group $G$ is a pair $(\mathfrak{A}, \alpha)$, where $\mathfrak{A}$ is a $C^*$-algebra and $\alpha : G \ni g \mapsto \alpha_g \in \mathrm{Aut}(\mathfrak{A})$ is an action of $G$ on $\mathfrak{A}$. To achieve dissipative dynamics, one needs to consider merely completely positive linear maps $\alpha_g$. This is the case when just a monoid naturally acts on $\mathfrak{A}$. Nevertheless, in the present paper we consider only dynamical systems based on automorphisms. The *fixed point subalgebra* is defined as $\mathfrak{A}^G := \{a \in \mathfrak{A} \mid \alpha_g(a) = a$ for all $g \in G\}$. We denote by $\mathcal{S}_G(\mathfrak{A})$ the $*$-weakly compact convex subset of $G$-invariant states, and by $\mathcal{E}(\mathcal{S}_G(\mathfrak{A}))$ the collection of its extremal (i.e. ergodic) states. For $(\mathfrak{A}, \alpha)$ as above and an invariant state $\varphi$ on $\mathfrak{A}$, $(\pi_\varphi, \mathcal{H}_\varphi, U_\varphi, \Omega_\varphi)$ is the GNS covariant quadruple canonically associated to $\varphi$ (see e.g. [13]). As usual, $\mathfrak{Z}_\varphi := \pi_\varphi(\mathfrak{A})'' \wedge \pi_\varphi(\mathfrak{A})'$ is the centre of $\pi_\varphi(\mathfrak{A})''$.

In these notes, we mainly deal with two groups: the group $\mathbb{P}_J$ of permutations of $J$ leaving fixed all of its elements but a finite number, and the group $\mathbb{Z}$. In the latter case, the action of $\mathbb{Z}$ is generated by a single automorphism (i.e. the shift) $\alpha \in \mathrm{Aut}(\mathfrak{A})$. A state is called *symmetric* or *shift invariant* if it belongs to $\mathcal{S}_{\mathbb{P}_J}(\mathfrak{A})$ or $\mathcal{S}_{\mathbb{Z}}(\mathfrak{A})$, respectively.

Consider the unital free product $C^*$-algebra $*_{\mathbb{Z}} \mathfrak{A}$ based on a single $C^*$-algebra $\mathfrak{A}$ (see e.g. [29]). For $j \in \mathbb{Z}$, denote by $i_j : \mathfrak{A} \to *_{\mathbb{Z}} \mathfrak{A}$ the canonical injections. Then both $\mathbb{P}_{\mathbb{Z}}$ and $\mathbb{Z}$ naturally act on $*_{\mathbb{Z}} \mathfrak{A}$ by considering permutations of indices and the shift of indices.

Proposition 2.1. *We have* $\mathcal{S}_{\mathbb{P}_{\mathbb{Z}}}(*_{\mathbb{Z}}\mathfrak{A})\subset\mathcal{S}_{\mathbb{Z}}(*_{\mathbb{Z}}\mathfrak{A})$.

*Proof.* For $\{j_1,\dots,j_n\}\subset\mathbb{Z}$ with possibly repeated indices such that the contiguous ones are different (i.e. $j_i\neq j_{i+1}, i=1,\dots,n-1$), the elements $X:=i_{j_1}(A_1)\cdots i_{j_n}(A_n)$ with $A_j\in\mathfrak{A}$ generate $*_{\mathbb{Z}}\mathfrak{A}$. By a standard approximation argument, we reduce ourselves to such generators. For $X$ and the corresponding sequence of indices as above, there exists a finite interval $J_X=[k,l]\subset\mathbb{Z}$ with $\{j_1,\dots,j_n\}\subset J_X$. In addition, there exists a cycle $\gamma_X\in\mathbb{P}_{\mathbb{Z}}$ such that $[k+1,l+1]=\gamma_X(J_X)$. For $\varphi\in\mathcal{S}_{\mathbb{P}_{\mathbb{Z}}}(*_{\mathbb{Z}}\mathfrak{A})$, after denoting by $\alpha$ and $\alpha_g$ the one-step shift and the action of $\mathbb{P}_{\mathbb{Z}}$, respectively, we get

$$\varphi(\alpha(X))=\varphi(\alpha_{\gamma_X}(X))=\varphi(X).\ ■$$

One of the fundamental achievements in the theory of stochastic processes (classical or not) provides sufficient conditions to construct a process starting from the knowledge of a collection of finite-dimensional distributions. In the abelian case they are summarised in the Kolmogorov Reconstruction Theorem, whereas a quantum generalisation is provided by the GNS construction.

Fix $n\in\mathbb{N}$, let $\{j_1,\dots,j_n\}\subset J$ with contiguous indices different, and let $\{A_1,\dots,A_n\}\subset\mathfrak{A}$. The finite joint distributions of the random variables $j_1(A_1),\dots,j_n(A_n)$ are the values $p_{j_1,\dots,j_n}(A_1,\dots,A_n)$ which arise from multilinear functionals $\{p_{j_1,\dots,j_n}\}_{j_1,\dots,j_n\in J}$ on $\mathfrak{A}$. They satisfy natural positivity and consistency conditions:

(i) $p_{j_n,\dots,j_2,j_1,j_2,\dots,j_n}(A_n^*,\dots,A_1^*A_1,\dots,A_n)\geq 0$ (*positivity*),

(ii) $p_{j_1,\dots,j_{k-1},j_k,j_{k+1},\dots,j_n}(A_1,\dots,A_{k-1},\mathbb{1},A_{k+1},\dots,A_n)$
$=p_{j_1,\dots,j_{k-1},j_{k+1},\dots,j_n}(A_1,\dots,A_{k-1},A_{k+1},\dots,A_n)$ (*consistency*).

In the classical case, i.e. $\mathfrak{A}=C(I)$, the algebra of continuous functions on the compact space $I$, the above properties reduce to the Kolmogorov requirements. Thus, one can construct a probability measure $\mu$ on the Tikhonov product $\prod_J I$ of $J$ copies of $I$. In the quantum setting, they allow one to consider the GNS representation (defined up to unitary equivalence), and so give rise to general stochastic processes as defined below. In order to avoid technicalities, we assume (by definition) that the process under consideration is directly realised on a Hilbert space, corresponding to $L^2(\prod_J I,\mu)$ in the classical situation.

Definition 2.2. A (realisation of the) *stochastic process* labelled by the index set $J$ is a quadruple $(\mathfrak{A},\mathcal{H},\{\iota_j\}_{j\in J},\Omega)$, where $\mathfrak{A}$ is a $C^*$-algebra, $\mathcal{H}$ is a Hilbert space, the $\iota_j$'s are $*$-homomorphisms of $\mathfrak{A}$ in $\mathcal{B}(\mathcal{H})$, and $\Omega\in\mathcal{H}$ is a unit vector, cyclic for the von Neumann algebra $M:=\bigvee_{j\in J}\iota_j(\mathfrak{A})$ naturally acting on $\mathcal{H}$. The process is said to be *unital* if $\iota_j(\mathbb{1}_{\mathfrak{A}})=I_{\mathcal{H}}$ for all $j\in J$.

The process is said to be *exchangeable*, resp. *stationary*, when $J$ is $\mathbb{Z}$, resp. $\mathbb{N}$, and for any $n \in \mathbb{N}$, $j_1, \dots, j_n \in J$ and $A_1, \dots, A_n \in \mathfrak{A}$,

$$\langle \iota_{j_1}(A_1) \cdots \iota_{j_n}(A_n)\Omega, \Omega\rangle = \langle \iota_{g(j_1)}(A_1) \cdots \iota_{g(j_n)}(A_n)\Omega, \Omega\rangle$$

for $g \in \mathbb{P}_J$, resp. $g(j_l) = j_l + 1$.

It is easy to see that the quadruple $(\mathfrak{A}, \mathcal{H}, \{\iota_j\}_{j\in J}, \Omega)$ uniquely realises, up to unitary equivalence, the stochastic process by which we mean that it determines all its joint finite distributions:

$$p_{j_1,\dots,j_n}(A_1, \dots, A_n) := \langle \iota_{j_1}(A_1) \cdots \iota_{j_n}(A_n)\Omega, \Omega\rangle.$$

From now on, if not otherwise specified, we only deal with unital stochastic processes.

Consider the unital free product $C^*$-algebra $*_J\,\mathfrak{A}$. One can see that a stochastic process uniquely defines a state $\varphi \in \mathcal{S}(*_J\,\mathfrak{A})$ and vice versa.

THEOREM 2.3. *The unitary equivalence class determined by the quadruple* $(\mathfrak{A}, \mathcal{H}, \{\iota_j\}_{j\in J}, \Omega)$ *uniquely defines a state* $\varphi \in \mathcal{S}(*_J\,\mathfrak{A})$*, and a representation* $\pi$ *of* $*_J\,\mathfrak{A}$ *on the Hilbert space* $\mathcal{H}$ *such that* $(\pi, \mathcal{H}, \Omega)$ *is the GNS representation of the state* $\varphi$*. Conversely, each state* $\varphi \in \mathcal{S}(*_J\,\mathfrak{A})$ *defines a stochastic process.*

*This one-to-one correspondence sends exchangeable and stationary processes* (*provided* $J = \mathbb{Z}$ *for the latter*) *to symmetric and shift invariant states, respectively.*

*Proof.* Take a quadruple $(\mathfrak{A}, \mathcal{H}, \{\iota_j\}_{j\in J}, \Omega)$ and consider the universal property of the free product $C^*$-algebra $*_J\,\mathfrak{A}$ together with the corresponding $*$-monomorphisms $i_j : \mathfrak{A} \to *_J\,\mathfrak{A}$, $j \in J$. Then there exists a $C^*$-homomorphism $\pi : *_J\,\mathfrak{A} \to \mathcal{B}(\mathcal{H})$ that is a representation making the diagram

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, "i_j"] \arrow[d, "\iota_j"'] & *_J\,\mathfrak{A} \arrow[dl, "\pi"] \\
\mathcal{B}(\mathcal{H}) &
\end{tikzcd}$$

commutative for each $j \in J$. It is easily seen that

$$\varphi(X) := \langle \pi(X)\Omega, \Omega\rangle, \qquad X \in *_J\,\mathfrak{A},$$

defines a state whose GNS representation is precisely $(\pi, \mathcal{H}, \Omega)$ (see [15, Theorem 3.3]). Conversely, for each state $\varphi \in \mathcal{S}(*_J\,\mathfrak{A})$ with GNS representation $(\pi_\varphi, \mathcal{H}_\varphi, \Omega_\varphi)$, one can define a collection of $*$-homomorphisms $\iota_j : \mathfrak{A} \to \mathcal{B}(\mathcal{H}_\varphi)$ by $\iota_j := \pi_\varphi \circ i_j$ for $j \in J$. It is straightforward to check that the quadruple $(\mathfrak{A}, \mathcal{H}_\varphi, \{\iota_j\}_{j\in J}, \Omega_\varphi)$ is a stochastic process according to Definition 2.2.

Finally, one can see as in [15, Theorem 3.3] that exchangeable and stationary stochastic processes correspond to symmetric and shift invariant states, respectively. ■

Definition 2.4. If $\mathfrak{A}$ is abelian, then the stochastic process is called *commutative* or *classical* if, for the homomorphisms $\iota_j$ in Definition 2.2,

$$\iota_{j_k}(A)\iota_{j_l}(B) = \iota_{j_l}(B)\iota_{j_k}(A), \qquad j_k, j_l \in J,\ A, B \in \mathfrak{A}.$$

One immediately sees that Definition 2.4 covers all stochastic processes arising in classical probability.

Consider the *free abelian product* unital $C^*$-algebra $\mathbf{ab}_J\,\mathfrak{A}$ of a single, not necessarily abelian $C^*$-algebra $\mathfrak{A}$. It is the universal object among the $C^*$-algebras, for morphisms with commuting ranges. In other words, if $\{\rho_j\}_{j\in J}$ is a collection of $*$-homomorphisms such that

$$\rho_{j_1}(A_1)\rho_{j_2}(A_2) = \rho_{j_2}(A_2)\rho_{j_1}(A_1), \qquad A_1, A_2 \in \mathfrak{A},\ j_1, j_2 \in J,\ j_1 \neq j_2,$$

then $\mathbf{ab}_J\,\mathfrak{A}$ is the universal (unital) $C^*$-algebra making the diagrams

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, "r_j"] \arrow[d, "\rho_j"'] & \mathbf{ab}_J\,\mathfrak{A} \arrow[dl, "\mathrm{P}"] \\
\mathfrak{B} &
\end{tikzcd}$$

commutative for all $j \in J$, where each $r_j$ is the canonical embedding. Such a universal object can be described in the following way.

Consider the norm closed two-sided ideal

$$\mathfrak{I} := \overline{\mathrm{span}\{a[i_1(A_1), i_2(A_2)]b \mid a, b \in *_J\,\mathfrak{A},\ A_1, A_2 \in \mathfrak{A},\ i_1, i_2 \in J,\ i_1 \neq i_2\}}^{\|\ \|},$$

which is the smallest ideal containing all commutators in $*_J\,\mathfrak{A}$ of the form $[i_1(A_1), i_2(A_2)]$, $i_1 \neq i_2$. Thus, $\mathbf{ab}_J\,\mathfrak{A} = *_J\,\mathfrak{A}/\!\sim$, where $\sim$ is the relation associated to the ideal $\mathfrak{I}$ above. We denote by $P : *_J\,\mathfrak{A} \to \mathbf{ab}_J\,\mathfrak{A}$ the associated quotient map.

For a fixed unital $C^*$-algebra $\mathfrak{A}$ and finite subsets $I, I_1, I_2 \subset J$, define the $|I|$-fold projective $C^*$-tensor product (cf. [27, Section IV.4])

$$\mathfrak{A}_I := \underbrace{\mathfrak{A} \otimes_{\max} \cdots \otimes_{\max} \mathfrak{A}}_{|I|\ \text{times}},$$

together with the canonical embedding

$$\mathfrak{A}_{I_1} \sim \mathfrak{A}_{I_1} \otimes \mathbb{1}_{\mathfrak{A}_{I_2\setminus I_1}} \subset \mathfrak{A}_{I_2}, \qquad I_1 \subset I_2.$$

It is then possible to form the $C^*$-inductive limit (cf. [30, Section L.2])

$$\otimes_J^{\max}\,\mathfrak{A} := \varinjlim_{I\uparrow J} \mathfrak{A}_I.$$

We point out that all the above considerations can be extended to the nonunital case described in [15, Section 3] either by using an approximate unity which always exists in any $C^*$-algebra, or by adding a unity to $\mathfrak{A}$. The reader is referred to [13, Section 2.2.3] or [27, Section IV.4].

Remark 2.5. By [27, Proposition IV.4.7], we have

$$\mathbf{ab}_J\,\mathfrak{A} \sim \otimes^{\max}_J \mathfrak{A}.$$

If in addition $\mathfrak{A}$ is commutative, i.e. $\mathfrak{A} \sim C(I)$ for a compact space $I$, then

$$\mathbf{ab}_J\,\mathfrak{A} \sim C\Big(\prod_J I\Big)$$

as $C(I)$ is nuclear. We refer the reader to [8, Theorem 36.1] for the explicit construction of the probability measure corresponding to the stochastic process under consideration.

Recall that an abelian stochastic process uniquely determines a state $\varphi \in \mathcal{S}(*_J\,\mathfrak{A})$. The next result shows how to realise commutative stochastic processes in this picture.

Proposition 2.6. *For an abelian $C^*$-algebra $\mathfrak{A}$ of samples, a stochastic process $(\mathfrak{A}, \mathcal{H}, \{\iota_j\}_{j\in J}, \Omega)$, and the corresponding state $\varphi \in \mathcal{S}(*_J\,\mathfrak{A})$ according to Theorem* 2.3, *the following are equivalent:*

(i) *$\varphi$ is the pull back on $*_J\,\mathfrak{A}$ of a state $\omega \in \mathcal{S}(\mathbf{ab}_J\,\mathfrak{A})$, i.e. $\varphi = \omega \circ P$;*
(ii) *the stochastic process $(\mathfrak{A}, \mathcal{H}, \{\iota_j\}_{j\in J}, \Omega)$ is commutative.*

*Proof.* We treat the unital case. In the absence of unity we can recover the same result arguing as in [15].

(i)⇒(ii). If $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ is the GNS representation of $\omega$, then the corresponding GNS representation of $\varphi$ and $*$-homomorphisms $\iota_j$, $j \in J$, are given by $(\pi_\omega \circ P, \mathcal{H}_\omega, \Omega_\omega)$ and $\pi_\omega \circ P \circ i_j$, respectively. We compute, for each $j_1 \neq j_2$,

$$\begin{aligned}\iota_{j_1}(A_1)\iota_{j_2}(A_2) &= \pi_\omega\big(P(i_{j_1}(A_1))\big)\pi_\omega\big(P(i_{j_2}(A_2))\big)\\ &= \pi_\omega\big(P(i_{j_1}(A_1))P(i_{j_2}(A_2))\big) = \pi_\omega\big(P(i_{j_2}(A_2))P(i_{j_1}(A_1))\big)\\ &= \pi_\omega\big(P(i_{j_2}(A_2))\big)\pi_\omega\big(P(i_{j_1}(A_1))\big) = \iota_{j_2}(A_2)\iota_{j_1}(A_1).\end{aligned}$$

(ii)⇒(i). As is shown in Theorem 2.3,

$$\varphi = \langle \pi(\,\cdot\,)\Omega, \Omega\rangle, \tag{2.1}$$

where $\pi$ is the unique homomorphism making the diagram

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "i_j"] \arrow[d, "\iota_j"'] & *_J\,\mathfrak{A} \arrow[dl, "\pi"] \\ M & \end{tikzcd}$$

commutative for all $j \in J$, and $M = \bigvee_{j\in J} \iota_j(\mathfrak{A})$ is the von Neumann algebra acting on $\mathcal{H}$ generated by all images $\iota_j(\mathfrak{A})$, $j \in J$. Since $M$ is abelian, for each $j \in J$ and the embeddings $r_j : \mathfrak{A} \to \mathbf{ab}_J\,\mathfrak{A}$, $i_j : \mathfrak{A} \to *_J\,\mathfrak{A}$, the universal properties applied to $*_J\,\mathfrak{A}$ and $\mathbf{ab}_J\,\mathfrak{A}$, respectively, give $P \circ i_j = r_j$ and the existence of a unique $\sigma : \mathbf{ab}_J\,\mathfrak{A} \to M$ such that $\sigma \circ r_j = \iota_j$ for all $j \in J$. Then $\sigma \circ P \circ i_j = \iota_j$, and consequently $\pi = \sigma \circ P$. If $\omega := \langle \sigma(\,\cdot\,)\Omega, \Omega\rangle \in \mathcal{S}(\mathbf{ab}_J\,\mathfrak{A})$, then (2.1) gives

$$\varphi = \langle \pi(\,\cdot\,)\Omega, \Omega\rangle = \langle \sigma \circ P(\,\cdot\,)\Omega, \Omega\rangle = \omega \circ P. \ \blacksquare$$

We have just shown that the general quantum scenario described in the first part of the section includes, as a particular case, the classical scheme. The latter is indeed achieved by the commutative diagrams

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, "i_j"] \arrow[d, "\beta_j"'] & *_J\,\mathfrak{A} \arrow[dl, "\Phi"] \\
\mathfrak{B} &
\end{tikzcd} \tag{2.2}$$

for $j \in J$ where $\mathfrak{B} = \mathbf{ab}_J\,\mathfrak{A}$, and where $\beta_j = r_j$ $(j \in J)$, and $\Phi = P$ are the canonical embeddings of $\mathfrak{A}$ in $\mathbf{ab}_J\,\mathfrak{A}$ and the canonical projection of $*_J\,\mathfrak{A}$ onto its abelianised $\mathbf{ab}_J\,\mathfrak{A}$, respectively. Thus, any $\varphi \in \mathcal{S}(*_J\,\mathfrak{A})$ realising a classical stochastic process (cf. Theorem 2.3) is obtained from a state $\omega$ on $\mathbf{ab}_J\,\mathfrak{A}$ through a pull back relation.

The basic idea yielding the above result is taking a suitable quotient of the free $C^*$-algebra. Hence, it appears clear that in the general case there are several ways to consider processes, each of them arising from factoring $*_J\,\mathfrak{A}$ by two-sided ideals generated by suitable commutators, and the commutative diagram (2.2) can be seen as the most general situation describing quantum stochastic processes. Specifically, we mention the so called $q$-deformed relations for $q \in [-1, 1]$, with pivotal examples given by $q = \pm 1$ corresponding to the Bose/Fermi cases, and $q = 0$ corresponding to the Boltzmann case describing the group reduced $C^*$-algebra of the free group on infinitely many generators [12] (see Section 4.1 for further details). Other noteworthy cases are the monotone [16] and boolean [15, 20] cases. As possible areas of future investigation, we also mention the cases arising from the more general setting of interacting Fock spaces (see e.g. [3]).

Concerning the Bose case (cf. Section 4.2), consider the infinite tensor product $C^*$-algebra $\otimes_J^{\min}\mathfrak{A}$ as in [26] for the not necessarily abelian algebra of samples $\mathfrak{A}$, together with the canonical projection $\Phi : *_J\,\mathfrak{A} \to \otimes_J^{\min}\mathfrak{A}$ recovered by universality from the embeddings $t_j : \mathfrak{A} \to \otimes_J^{\min}\mathfrak{A}$, $j \in J$. As the $t_j$ have commuting ranges, $\Phi$ factors through the canonical projection $\Psi : \mathbf{ab}_J\,\mathfrak{A} \to \otimes_J^{\min}\mathfrak{A}$, i.e. $\Phi = \Psi \circ P$. Accordingly, the stochastic process determined by a state $\omega \in \mathcal{S}(\otimes_J^{\min}\mathfrak{A})$ such that $\varphi = \omega \circ \Phi$, factors through $\mathbf{ab}_J\,\mathfrak{A}$ as $\varphi = \omega \circ \Psi \circ P$. We then have the following

Remark 2.7. Each stochastic process on $\otimes_J^{\min}\mathfrak{A}$ comes from a stochastic process on the free abelianised product $\mathbf{ab}_J\,\mathfrak{A}\sim\otimes_J^{\max}\mathfrak{A}$ uniquely determined by the state $\omega\circ\Psi\in\mathcal{S}(\mathbf{ab}_J\,\mathfrak{A})$. The same construction holds true for stochastic processes on any other infinite tensor product $\otimes_J^{\gamma}\mathfrak{A}$ based on the $C^*$-cross norm $\|\ \|_\gamma$ (see [27, Section IV.4]).

We end the section by recalling the definition, useful in what follows, of the *tail algebra* $\mathfrak{Z}_\varphi^\perp$ for the stochastic process $(\mathfrak{A},\mathcal{H},(\iota_j)_{j\in J},\Omega)$, with corresponding state $\varphi\in\mathcal{S}(*_J\,\mathfrak{A})$:

$$\mathfrak{Z}_\varphi^\perp:=\bigwedge_{I\subset J,\,I\,\text{finite}}\Big(\bigcup_{\substack{K\cap I=\emptyset\\ K\,\text{finite}}}\Big(\bigvee_{k\in K}\iota_k(\mathfrak{A})\Big)\Big)''.$$

In statistical mechanics this is known as the *algebra at infinity* (see e.g. [13]).

**3. Ergodic properties of stochastic processes.** The present section is devoted to natural ergodic properties of stochastic processes. Here, we also make a direct link between algebraic relations and ergodic conditions.

Let $(\mathfrak{A},\alpha)$ be a $C^*$-dynamical system with $\mathcal{S}_G(\mathfrak{A})=\{\omega\}$. Such a system is said to be *uniquely ergodic*. When $G=\mathbb{Z}$, one can see that unique ergodicity is equivalent to

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}f(\alpha^k(a))=f(\mathbb{1})\omega(a),\qquad a\in\mathfrak{A},\ f\in\mathfrak{A}^*,\tag{3.1}$$

or again to

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\alpha^k(a)=\omega(a)\mathbb{1},\qquad a\in\mathfrak{A},$$

pointwise, in norm. Some natural generalisations of such a strong ergodic property can be achieved by replacing the ergodic average (3.1) with

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}|f(\alpha^k(a))-f(\mathbb{1})\omega(a)|=0,\qquad a\in\mathfrak{A},\ f\in\mathfrak{A}^*,$$

or simply

$$\lim_{n\to\infty}f(\alpha^n(a))=f(\mathbb{1})\omega(a),\qquad a\in\mathfrak{A},\ f\in\mathfrak{A}^*,$$

for some state $\omega\in\mathcal{S}(\mathfrak{A})$ which is necessarily invariant. In this case, $(\mathfrak{A},\alpha)$ is called *uniquely weak mixing* or *uniquely mixing*, respectively. For all these cases, $\mathfrak{A}^{\mathbb{Z}}=\mathbb{C}\mathbb{1}$, and the (unique) invariant conditional expectation onto the fixed point subalgebra is precisely $E(a)=\omega(a)\mathbb{1}$.

Another natural generalisation is to look at the fixed point subalgebra whenever it is nontrivial, and at the unique invariant conditional expectation onto such a subalgebra, $E^{\mathbb{Z}}:\mathfrak{A}\to\mathfrak{A}^{\mathbb{Z}}$, provided it exists. The unique ergodicity, weak mixing, and mixing with respect to the fixed point subalgebra (also known as $E^{\mathbb{Z}}$-ergodicity, $E^{\mathbb{Z}}$-weak mixing and $E^{\mathbb{Z}}$-mixing, $E^{\mathbb{Z}}$ being the invariant conditional expectation onto $\mathfrak{A}^{\mathbb{Z}}$ which necessarily exists) are given by definition, for $a\in\mathfrak{A}$ and $f\in\mathfrak{A}^*$, by

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}f(\alpha^k(a))=f(E^{\mathbb{Z}}(a)),\\&\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}|f(\alpha^k(a))-f(E^{\mathbb{Z}}(a))|=0,\\&\lim_{n\to\infty}f(\alpha^n(a))=f(E^{\mathbb{Z}}(a)).\end{aligned}$$

Fix a dynamical system $(\mathfrak{A},\alpha)$ based on the group $G$. For each $\varphi\in\mathcal{S}(\mathfrak{A})$, we set $L^\infty(\mathfrak{A},\varphi):=\pi_\varphi(\mathfrak{A})''$ and $L^2(\mathfrak{A},\varphi):=\pi_\varphi(\mathfrak{A}):=\mathcal{H}_\varphi$. One can define a bounded linear map $T:L^\infty(\mathfrak{A},\varphi)\to L^2(\mathfrak{A},\varphi)$ by

$$TX:=X\Omega_\varphi,\qquad X\in L^\infty(\mathfrak{A},\varphi).$$

Suppose $\varphi\in\mathcal{S}_G(\mathfrak{A})$. We denote by $\alpha$ the actions of $G$ on both $L^\infty(\mathfrak{A},\varphi)$ and $L^2(\mathfrak{A},\varphi)$, defined by $\alpha_g(X):=\mathrm{ad}_{U_\varphi(g)}(X)$ and $\alpha_g(\xi):=U_\varphi(g)\xi$. These actions are compatible with the map $T$:

$$T\,\mathrm{ad}_{U_\varphi(g)}(X)=U_\varphi(g)TX,\qquad g\in G,\ X\in L^\infty(\mathfrak{A},\varphi).$$

The sets of invariant elements are denoted respectively by

$$\begin{aligned}L^\infty(\mathfrak{A},\varphi)^G&:=\pi_\varphi(\mathfrak{A})''\wedge\{U_\varphi(G)\}',\\L^2(\mathfrak{A},\varphi)^G&:=\{\xi\in\mathcal{H}_\varphi\mid U_\varphi(g)\xi=\xi\text{ for all }g\in G\}.\end{aligned}$$

Definition 3.1. A state $\varphi\in\mathcal{S}_G(\mathfrak{A})$ is said to be *weakly clustering* if

$$\lim_{\Lambda\uparrow G}\frac{1}{|\Lambda|}\sum_{g\in\Lambda}\varphi(A\alpha_g(B))=\varphi(A)\varphi(B),\qquad A,B\in\mathfrak{A},\tag{3.2}$$

and it has the *property of convergence to equilibrium* if

$$\lim_{\Lambda\uparrow G}\frac{1}{|\Lambda|}\sum_{g\in\Lambda}\varphi(A\alpha_g(B)C)=\varphi(AC)\varphi(B),\qquad A,B,C\in\mathfrak{A},\tag{3.3}$$

along the net $\{\Lambda\subset G\mid|\Lambda|<\infty\}$.

When $G=\mathbb{Z}$, i.e. when one deals with the shift, one usually restricts oneself to $\Lambda_n:=[0,n-1]$. Condition (3.2) is obviously weaker than (3.3). We can see that they are equivalent provided the support of $\varphi$ belongs to the centre of the bidual, i.e. $s(\varphi)\in Z(\mathfrak{A}^{**})$. They are also equivalent for (graded) asymptotically abelian states, that is, for systems possibly including fermions (cf. [7, 18]), and in the classical case. Furthermore, a weakly clustering state $\varphi \in \mathcal{S}_G(\mathfrak{A})$ is automatically ergodic (i.e. extremal invariant), that is, $\varphi \in \mathcal{E}(\mathcal{S}_G(\mathfrak{A}))$. The converse holds true in the particular case of $G$-abelian states (see [13, Section 4.3]).

Our aim now is to prove that condition (3.3) is the right one ensuring convergence to equilibrium in the quantum case. Thus, consider a physical system in a state $\omega$ invariant for the (discrete) dynamics. Then we have a dynamical system $(\mathfrak{A}, \alpha, \omega)$ based on a single automorphism $\alpha$ on $\mathfrak{A}$. Such a localised perturbation usually produces a state $\varphi$ which is normal with respect to the reference state $\omega$. Namely, $\varphi \in \mathcal{F}_{\pi_\omega}(\mathfrak{A})$ where, for each representation $(\pi, \mathcal{H}_\pi)$ of $\mathfrak{A}$,

$$\mathcal{F}_\pi(\mathfrak{A}) = \{\varphi \in \mathcal{S}(\mathfrak{A}) \mid \varphi = \operatorname{Tr}(\pi(\,\cdot\,)T) \text{ for some } T \in \mathcal{T}(\mathcal{H}_\pi)_{+,1}\}$$

is the folium generated by $\pi$, with Tr and $\mathcal{T}(\mathcal{H}_\pi)_{+,1}$ being the canonical trace on $\mathcal{H}_\pi$ and the positive normalised trace class operators on it, respectively. Convergence to equilibrium simply means

$$\lim_n \frac{1}{n} \sum_{k=0}^{n-1} \varphi(\alpha^n(A)) = \omega(A), \qquad A \in \mathfrak{A},\ \varphi \in \mathcal{F}_{\pi_\omega}(\mathfrak{A}), \tag{3.4}$$

that is, $\mathcal{F}_{\pi_\omega}(\mathfrak{A})$ is contained in the *basin of attraction* of $\omega$ in the whole $\mathcal{S}(\mathfrak{A})$. By a standard approximation argument, we can restrict ourselves to the generators of $\mathcal{F}_{\pi_\omega}(\mathfrak{A})$ of the form

$$\varphi_B(A) := \frac{\omega(B^*AB)}{\omega(B^*B)}, \qquad A \in \mathfrak{A},$$

provided $\omega(B^*B) \neq 0$. It is then evident that (3.4) simply means

$$\lim_n \frac{1}{n} \sum_{k=0}^{n-1} \omega(B^*\alpha^n(A)B) = \omega(B^*B)\omega(A), \qquad A, B \in \mathfrak{A},$$

which turns out to be equivalent to (3.3) by polarisation.

We report a version of a pivotal result in noncommutative ergodic theory, well known to experts. To avoid technicalities, from now on we restrict ourselves to $\mathbb{P}_J$ or $\mathbb{Z}$ if not otherwise specified. Recall that for a $C^*$-algebra $\mathfrak{A}$, for $\varphi \in \mathcal{S}(\mathfrak{A})$ with support $s(\varphi) \in \mathfrak{A}^{**}$ and the corresponding GNS representation $(\pi_\varphi, \mathcal{H}_\varphi, \Omega_\varphi)$, one has $s(\varphi) \in Z(\mathfrak{A}^{**})$ if and only if $\Omega_\varphi$ is cyclic for $\pi_\varphi(\mathfrak{A})'$.

THEOREM 3.2. *Let $(\mathfrak{A}, \alpha)$ be a dynamical system where $G$ is $\mathbb{P}_J$ or $\mathbb{Z}$. With the net generated by all finite subgroups $\mathbb{P}_I$, $\{I \subset J \mid |I| < \infty\}$ for $\mathbb{P}_J$, or the sequence $[0, n-1]$, $n \in \mathbb{N}$, for $\mathbb{Z}$, and $\varphi \in \mathcal{S}_G(\mathfrak{A})$, consider the following assertions:*

(i) *$\varphi$ is weakly clustering,*
(ii) *$L^2(\mathfrak{A}, \varphi)^G = \mathbb{C}\Omega_\varphi$,*

(iii) *$\varphi$ has the property of convergence to equilibrium,*
(iv) $L^\infty(\mathfrak{A},\varphi)^G=\mathbb{C}I$.

*Then we have* (iv)⇒(iii)⇒(ii)⇔(i)*, and all conditions are equivalent if* $s(\varphi)\in Z(\mathfrak{A}^{**})$.

*Proof.* The equivalence (i)⇔(ii) is nothing but the Mean Ergodic Theorem of J. von Neumann (see e.g. [13, Section 4.3] and [14, Proposition 3.1]). The equivalence (ii)⇔(iv) is well known (cf. [13, Section 4.3]), provided $s(\varphi)\in Z(\mathfrak{A}^{**})$.

We now discuss (iv)⇒(iii). By reasoning as in [13, Theorem 4.3.20], first note that (iv) implies that $s(\varphi)\in Z(\mathfrak{A}^{**})$. In addition, (iv) implies (ii), which is equivalent to (i). The latter turns out to be equivalent to (iii) as $s(\varphi)\in Z(\mathfrak{A}^{**})$. ■

In the following definition, we present the algebraic properties of a stochastic process on $\mathfrak{A}$, or equivalently of the corresponding state on $*_J\mathfrak{A}$, which will be linked with the above ergodic conditions. To treat also the shift on $\mathbb{Z}$, we specialise to $*_\mathbb{Z}\mathfrak{A}$.

Definition 3.3. The state $\varphi\in\mathcal{S}(*_\mathbb{Z}\mathfrak{A})$ is said to satisfy the *product state condition* if

$$\varphi(A_1A_2)=\varphi(A_1)\varphi(A_2)$$

whenever $A_k\in\mathrm{alg}\{i_{j_k}(\mathfrak{A})\mid j_k\in I_k\}$, $I_k\subset J$, $k=1,2$, and $I_1\cap I_2=\emptyset$.

The state $\varphi$ satisfies the *block singleton condition* [2] if

$$\varphi(A_1A_2A_3)=\varphi(A_1A_3)\varphi(A_2)$$

whenever $A_k\in\mathrm{alg}\{i_{j_k}(\mathfrak{A})\mid j_k\in I_k\}$, $I_k\subset J$, $k=1,2,3$, and $(I_1\cup I_3)\cap I_2=\emptyset$.

The next result connects the above conditions with those in Theorem 3.2. We refer the reader to [15] for further details.

Theorem 3.4. *For $G$ as in Theorem 3.2 and $\varphi\in\mathcal{S}_G(*_\mathbb{Z}\mathfrak{A})$, the following assertions hold true:*

(i) *$\varphi$ satisfies the product state condition if and only if it is weakly clustering,*
(ii) *$\varphi$ is a block singleton state if and only if it has the property of convergence to equilibrium.*

*Proof.* We handle the case of permutations (cf. [15]) corresponding to (ii), and leave to the reader the easier situation (i). The analogous case of the shift is in [2].

Suppose that $\varphi\in\mathcal{S}_{\mathbb{P}_\mathbb{Z}}(*_\mathbb{Z}\mathfrak{A})$ has the property of convergence to equilibrium. Fix words $u,v,w\in *_\mathbb{Z}\mathfrak{A}$ whose supports satisfy $I_v\cap(I_u\cup I_w)=\emptyset$. Let $I\supset I_u,I_v,I_w$ be an arbitrarily large but finite part of $J$, and consider the set $B\subset\mathbb{P}_I$ of permutations leaving $I_u\cup I_w$ pointwise fixed. Since $\varphi$ is symmetric, we get

$$\begin{aligned}\varphi(uvw) &= \frac{1}{|B|}\sum_{g\in B}\varphi(\alpha_g(uvw)) = \frac{1}{|B|}\sum_{g\in B}\varphi(u\alpha_g(v)w)\\ &= \frac{|\mathbb{P}_I|}{|B|}\bigg(\frac{1}{|\mathbb{P}_I|}\sum_{g\in\mathbb{P}_I}\varphi(u\alpha_g(v)w)\bigg) - \frac{1}{|B|}\sum_{g\in\mathbb{P}_I\backslash B}\varphi(u\alpha_g(v)w).\end{aligned}$$

By [14, Lemma 3.3], $|B|/|\mathbb{P}_I| \to 1$ and $|B^c|/|B| \to 0$ as $I \uparrow J$. By taking the limit of both sides, the l.h.s. does not depend on $J$, whereas the r.h.s. converges to $\varphi(uw)\varphi(v)$ by the property of convergence to equilibrium. Thus, $\varphi$ satisfies the block singleton condition.

Conversely, suppose $\varphi \in \mathcal{S}_{\mathbb{P}_\mathbb{Z}}(*_\mathbb{Z}\mathfrak{A})$ is a block singleton state. By density, we can restrict ourselves to elementary words. Fix $u, v, w \in *_\mathbb{Z}\mathfrak{A}$ with supports $I_u, I_v, I_w$, respectively. If $I$ is a finite part of $J$, define

$$A := \{g \in \mathbb{P}_I \mid (I_u \cup I_w) \cap I_{\alpha_g(v)} = \emptyset\}.$$

By applying the block singleton condition, we get

$$\begin{aligned}&\bigg|\frac{1}{|\mathbb{P}_I|}\sum_{g\in\mathbb{P}_I}\varphi(u\alpha_g(v)w) - \varphi(uw)\varphi(v)\bigg|\\ &\quad\leq\bigg|\frac{1}{|\mathbb{P}_I|}\sum_{g\in A}\varphi(u\alpha_g(v)w) - \varphi(uw)\varphi(v)\bigg| + \bigg|\frac{1}{|\mathbb{P}_I|}\sum_{g\in\mathbb{P}_I\backslash A}\varphi(u\alpha_g(v)w)\bigg|\\ &\quad\leq\bigg|\bigg(\frac{|A|}{|\mathbb{P}_I|} - 1\bigg)\varphi(uw)\varphi(v)\bigg| + \|u\|\,\|v\|\,\|w\|\frac{|A^c|}{|\mathbb{P}_I|},\end{aligned}$$

where $A^c := \mathbb{P}_I\backslash A$. Letting $I \uparrow J$, again by [14, Lemma 3.3] one finds that $|A|/|\mathbb{P}_I| \to 1$ and $|A^c|/|\mathbb{P}_I| \to 0$. Thus, $\varphi$ satisfies condition (3.3). ■

As an application of the above results, one gets some ergodic properties for the so-called Haagerup states [21] naturally arising in free probability. For $\lambda \in (0, \infty)$, those are defined on the group $C^*$-algebra $C^*(\mathbb{F}_\infty)$ of the free group $\mathbb{F}_\infty$ on infinitely many generators, as $\varphi_\lambda(w) := e^{-\lambda|w|}$, where $w \in C^*(\mathbb{F}_\infty)$ is a reduced word and $|w|$ is its length. The case $\lambda = \infty$ corresponds to the tracial state, the unique one passing to the quotient given by the reduced group algebra $C^*_{\mathrm{red}}(\mathbb{F}_\infty)$ and considered below. The Haagerup states are automatically symmetric by construction, and satisfy the product state condition: $\varphi_\lambda(vw) = \varphi_\lambda(v)\varphi_\lambda(w)$ when $I_v \cap I_w = \emptyset$. But they do not fulfil the block singleton condition if $\lambda \in (0, \infty)$. In fact, for elementary generators $g_i, g_j$ with $i \neq j$,

$$\varphi_\lambda(g_ig_jg_i^{-1}) = e^{-3\lambda} \neq e^{-\lambda} = \varphi_\lambda(g_j)\varphi_\lambda(\mathbb{1}) = \varphi_\lambda(g_j)\varphi_\lambda(g_ig_i^{-1}).$$

COROLLARY 3.5. *For the Haagerup states* $\varphi_\lambda \in \mathcal{S}(C^*(\mathbb{F}_\infty))$, *one has* $s(\varphi_\lambda) \notin Z(C^*(\mathbb{F}_\infty)^{**})$ *for all* $\lambda \in (0, \infty)$.

*Proof.* As $\varphi_\lambda$ satisfies the product state condition, by Theorem 3.4 it is weakly clustering. Suppose that $s(\varphi_\lambda) \in Z(C^*(\mathbb{F}_\infty)^{**})$. By Theorem 3.2, it has the property of convergence to equilibrium. Then it is a block singleton state again by Theorem 3.4, which is a contradiction. ∎

**4. Examples.** In this section we deal with stochastic processes directly built on concrete $C^*$-algebras.

**4.1. $q$-deformed commutation relations.** We briefly recall the $q$-deformed commutation relations, $q \in [-1, 1]$:

$$a_q(i)a_q^\dagger(j) - qa_q^\dagger(j)a_q(i) = \delta_{ij}\mathbb{1}, \quad i, j \in \mathbb{Z}. \tag{4.1}$$

The above commutation rule can be represented concretely as creators and annihilators on the $q$-deformed Fock spaces (see e.g. [12]). The remarkable cases of Bose (CCR), Fermi (CAR) and Boltzmann (Free) relations are realised for $q = \pm 1$ and $q = 0$, respectively. We first treat the case $q \in (-1, 1)$.

Let $\mathfrak{R}_q$ and $\mathfrak{G}_q$ be the concrete unital $C^*$-algebras acting on the $q$-Fock space generated by the annihilators $\{a_q(i) \mid i \in \mathbb{Z}\}$, and by their selfadjoint part $\{a_q(i) + a_q^\dagger(i) \mid i \in \mathbb{Z}\}$, respectively, for $a_q(i) := a_q(e_i)$ and $e_i$ the elements of the canonical basis of $\ell^2(\mathbb{Z})$. The group $\mathbb{P}_\mathbb{Z}$ of permutations, and the group $\mathbb{Z}$ of powers of the shift naturally act on both $\mathfrak{R}_q$ and $\mathfrak{G}_q$ as Bogolyubov automorphisms implemented by the unitaries $Ue_i := e_{i+1}$ and $U_g e_i := e_{g(i)}$ on $\ell^2(\mathbb{Z})$ (see [16, Proposition 3.1]). We denote by $\mathfrak{A}_q$ one of those concrete $C^*$-algebras, and by $G$ and $\alpha$ those groups and their actions, respectively. The vacuum expectation state is given by $\omega_q := \langle \cdot\, \Omega_q, \Omega_q\rangle$, $\Omega_q$ being the vacuum vector in the $q$-Fock space.

We record the following strong ergodic result, and refer the reader to [16] for its proof.

Theorem 4.1. *Fix $q \in (-1, 1)$ and consider a countable set $\{g_k\}_{k\in\mathbb{N}} \subset G$. Then*

$$\lim_n \left\| \frac{1}{n}\sum_{k=1}^n \alpha_{g_k}(A) - \omega_q(A)I \right\| = 0, \qquad A \in \mathfrak{A}_q.$$

*In addition,*

$$\mathcal{S}_{\mathbb{P}_\mathbb{Z}}(\mathfrak{A}_q) = \mathcal{S}_\mathbb{Z}(\mathfrak{A}_q) = \{\omega_q\}.$$

*The $C^*$-dynamical system $(\mathfrak{A}_q, \alpha)$ based on the action of $\mathbb{Z}$ as powers of the shift is uniquely mixing with $\omega_q$ as the unique invariant state.*

As the case $q = 0$ corresponds to the reduced group algebra of the free group $\mathfrak{G}_0 \sim C^*_{\text{red}}(\mathbb{F}_\infty)$ (cf. [29]), with $\omega_0$ corresponding to the canonical trace, we find that $\omega_0$ is the unique state on $C^*_{\text{red}}(\mathbb{F}_\infty)$ invariant for both the permutations moving only finitely many generators, and the shift. In addition, it is the unique invariant state on $C^*(\mathbb{F}_\infty)$ coming from the natural quotient

$$C^*_{\mathrm{red}}(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty)/\ker\lambda,$$

$\lambda$ being the (left) regular representation of $\mathbb{F}_\infty$.

**4.2. Bose case.** As the creators satisfying (4.1) cannot be bounded if $q = 1$, we handle the boson case by using the Weyl algebra (formally by exponentiating the field operators, see e.g. [13]). In this situation, $\mathfrak{R}_1$ must be replaced by the Weyl algebra $W(C_{00}(\mathbb{Z}))$, where $C_{00}(\mathbb{Z})$ is the pre-Hilbert space of all finitely supported complex sequences on $\mathbb{Z}$. The algebra generated by the selfadjoint parts of annihilators leads to abelian processes and is treated in the standard probability literature. It is well known that

$$W(C_{00}(\mathbb{Z})) \sim \otimes^{\min}_{\mathbb{Z}} W(\mathbb{C}),$$

the infinite tensor product of infinitely many copies of $W(\mathbb{C})$. By Størmer's results [26], one can obtain an ergodic decomposition of symmetric states as in the de Finetti Theorem:

(i) $\mathcal{S}_{\mathbb{P}_\mathbb{Z}}(W(C_{00}(\mathbb{Z})))$ is a mixture (i.e. direct integral) of states which are an infinite product state of a single state on $W(\mathbb{C})$, the latter providing the ergodic states $\mathcal{E}(\mathcal{S}_{\mathbb{P}_\mathbb{Z}}(W(C_{00}(\mathbb{Z}))))$.

By using the results in [4], it is not hard to show that

(ii) $\mathcal{S}_{\mathbb{P}_\mathbb{Z}}(W(C_{00}(\mathbb{Z}))) \subsetneq \mathcal{S}_\mathbb{Z}(W(C_{00}(\mathbb{Z})))$.

One can also prove the following version of the de Finetti Theorem:

(iii) a process on the Weyl algebra is exchangeable if and only if it is conditionally independent and identically distributed with respect to the tail algebra.

Finally, as in [15, Theorem 5.3] one can establish a quantum analogue of the Hewitt and Savage Lemma:

(iv) for states $\varphi \in \mathcal{S}_{\mathbb{P}_\mathbb{Z}}(W(C_{00}(\mathbb{Z})))$, the *tail algebra* $\mathfrak{Z}^\perp_\varphi$ coincides with the symmetric part of the centre $\mathfrak{Z}^{\mathbb{P}_\mathbb{Z}}_\varphi$.

**4.3. Fermi case.** For the unital $C^*$-algebra $\mathfrak{R}_{-1}$ generated by the annihilators (i.e. the CAR algebra), the same results listed above for the Bose case hold true. The reader is referred to [14, 15], and to the examples relative to Fermi Markov states in [19, Section 6] for item (ii) above. As any symmetric state on the CAR algebra is automatically shift invariant, it is even (cf. [13]). Then the analogue of (iv) above (the Hewitt and Savage Lemma for the Bose case) assumes the following form in the CAR case (cf. [15, Theorem 5.3]):

(iv$'$) for states $\varphi \in \mathcal{S}_{\mathbb{P}_{\mathbb{Z}}}(\mathfrak{R}_{-1})$, the tail algebra $\mathfrak{Z}^{\perp}_{\varphi}$ coincides with the even portion of the symmetric part of the centre.

**4.4. Boolean case.** Let $\mathcal{H}$ be a complex Hilbert space. The boolean Fock space over $\mathcal{H}$ is given by $\Gamma(\mathcal{H}) := \mathbb{C}\oplus\mathcal{H}$ and $(1,0)$ is the vacuum vector. On $\Gamma(\mathcal{H})$ we define the creation and annihilation operators by setting, for $f,g \in \mathcal{H}$ and $\alpha \in \mathbb{C}$,

$$a^{\dagger}(f)(\alpha\oplus g) := 0\oplus\alpha f, \quad a(f)(\alpha\oplus g) := \langle g,f\rangle_{\mathcal{H}}\oplus 0.$$

For $\mathcal{H} = \ell^2(\mathbb{Z})$, it is seen that the concrete unital $C^*$-algebra $\mathfrak{B}$ (called the *boolean algebra*) generated by the annihilators coincides with that generated by their selfadjoint parts (see e.g. [15]). In addition,

$$\mathfrak{B} = \mathcal{K}(\ell^2(\{\#\}\cup\mathbb{Z})) + \mathbb{C}I.$$

Here, $a_i := a(e_i) = \varepsilon_{\#,i}$ is the standard matrix unit, $\#$ corresponds to the subspace in $\Gamma(\ell^2(\mathbb{Z}))$ generated by the vacuum $e_{\#}$, and $\mathcal{K}$ stands for compact operators. If $\omega_{\#}$ denotes the vacuum state and $\omega_{\infty}$ the state at infinity

$$\omega_{\infty}(A+cI) := c, \qquad A\in\mathcal{K}(\ell^2(\{\#\}\cup\mathbb{Z})),\ c\in\mathbb{C},$$

we get the following structure for symmetric and stationary states (cf. [15, 16]):

Theorem 4.2. *For the shift invariant and symmetric states, we get*

$$\mathcal{S}_{\mathbb{P}_{\mathbb{Z}}}(\mathfrak{B}) = \mathcal{S}_{\mathbb{Z}}(\mathfrak{B}) = \{(1-\gamma)\omega_{\#}+\gamma\omega_{\infty} \mid \gamma\in[0,1]\}.$$

The well established structure of $\mathfrak{B}$ allows one to completely determine the fixed point algebras for the action of the shift and the permutations:

$$\mathfrak{B}^{\mathbb{P}_{\mathbb{Z}}} = \mathfrak{B}^{\mathbb{Z}} = \mathbb{C}P_{\#}\oplus\mathbb{C}P_{\#}^{\perp},$$

$P_{\#}$ being the orthogonal projection onto $\mathbb{C}e_{\#}$.

Consider the dynamical system $(\mathfrak{B},\alpha)$ based on the boolean algebra and the shift, together with the unique invariant conditional expectation $E$ onto $\mathfrak{B}^{\mathbb{P}_{\mathbb{Z}}}$ given by

$$E(A+bI) := \langle Ae_{\#},e_{\#}\rangle P_{\#} + bI, \qquad A\in\mathcal{K}(\ell^2(\{\#\}\cup\mathbb{Z})),\ b\in\mathbb{C}. \tag{4.2}$$

Proposition 4.3 ([16, Proposition 7.2]). *The $C^*$-dynamical system $(\mathfrak{B},\alpha)$ is $E^{\mathbb{Z}}$-mixing with $E=E^{\mathbb{Z}}$ the unique invariant conditional expectation onto the fixed point subalgebra given in* (4.2).

Notice that the conditional expectation in (4.2) is also the unique invariant one for the natural action of $\mathbb{P}_{\mathbb{Z}}$. Denoting that action again by $\alpha$, one can show that

$$\lim_{J\uparrow\mathbb{Z}}\frac{1}{|J|!}\sum_{g\in\mathbb{P}_J}\alpha_g(A) = E(A), \qquad A\in\mathfrak{B},$$

where $\{J \mid J\subset\mathbb{Z}\}$ is the directed net of all finite subsets of $\mathbb{Z}$.

Moreover, we record the following assertions proved in [20]:

(i) a boolean process is exchangeable if and only if it is conditionally independent and identically distributed with respect to the tail algebra (as in the classical case);

(ii) for $\omega \in \mathcal{S}_{\mathbb{P}_{\mathbb{Z}}}(\mathfrak{B})$, if $\mathfrak{B}_\omega := \pi_\omega(\mathfrak{B})''$ and $\mathfrak{Z}^\perp_\omega$, $\mathfrak{B}^{\mathbb{P}_{\mathbb{Z}}}_\omega$, $\mathfrak{B}^{\mathbb{Z}}_\omega$ denote the tail algebra, the symmetric algebra and the stationary algebra, respectively, we get

$$\mathfrak{Z}^\perp_\omega = \mathfrak{B}^{\mathbb{P}_{\mathbb{Z}}}_\omega \subsetneq \mathfrak{B}^{\mathbb{Z}}_\omega.$$

As a consequence, the equality above transfers the Hewitt–Savage Lemma to the boolean situation, whereas the last inclusion entails that the Olshen Theorem [25] does not hold for boolean stochastic processes.

**4.5. Monotone case.** We outline the structure of stationary monotone processes corresponding to states on the concrete unital monotone $C^*$-algebra, and in addition on the subalgebra generated by the selfadjoint parts of annihilators.

As in [10], for $k \geq 1$, denote $I_k := \{(i_1, \dots, i_k) \mid i_1 < \dots < i_k,\, i_j \in \mathbb{Z}\}$, and for $k = 0$, set $I_0 := \{\emptyset\}$, $\emptyset$ being the empty sequence. The Hilbert space $\mathcal{H}_k := \ell^2(I_k)$ is precisely the $k$-particle space for monotone quantisation. In particular, the 0-particle space $\mathcal{H}_0 = \ell^2(\emptyset)$ is identified with the complex scalar field $\mathbb{C}$. The monotone Fock space is $\mathcal{F}_m = \bigoplus_{k=0}^\infty \mathcal{H}_k$.

For an increasing sequence $\alpha = (i_1, \dots, i_k)$ of integers, we denote by $e_\alpha$ the generic element of the canonical basis of $\mathcal{F}_m$. There is a natural order structure on such sequences. Indeed, if $\alpha = (i_1, \dots, i_k)$, $\beta = (j_1, \dots, j_l)$, we say $\alpha < \beta$ if $i_k < j_1$. The monotone creation and annihilation operators are respectively given, for any $i \in \mathbb{Z}$, by

$$a_i^\dagger e_{(i_1,\dots,i_k)} := \begin{cases} e_{(i,i_1,\dots,i_k)} & \text{if } i < i_1, \\ 0 & \text{otherwise}, \end{cases}$$

$$a_i e_{(i_1\dots,i_k)} := \begin{cases} e_{(i_2,\dots,i_k)} & \text{if } k \geq 1 \text{ and } i = i_1, \\ 0 & \text{otherwise} \end{cases}$$

where $a_i := a(e_i)$. One can prove that $\|a_i^\dagger\| = \|a_i\| = 1$ and that $a_i^\dagger$ and $a_i$ are mutually adjoint. Moreover, we have

$$a_i^\dagger a_j^\dagger = a_j a_i = 0 \quad \text{if } i \geq j,$$
$$a_i a_j^\dagger = 0 \quad \text{if } i \neq j,$$

and the commutation relation

$$a_i a_i^\dagger = I - \sum_{k \leq i} a_k^\dagger a_k, \tag{4.3}$$

with the sum meant in the strong operator topology (cf. [16, Proposition 3.2]). The $C^*$-algebra $\mathfrak{M}$ acting on $\mathcal{F}_m$ is the unital $C^*$-algebra generated by the annihilators $\{a_i \mid i \in \mathbb{Z}\}$. It was proven in [16] that the selfadjoint parts of annihilators, $\{a_i + a_i^+ \mid i \in \mathbb{Z}\}$, also generate the same unital $C^*$-algebra as the annihilators (just as for booleans). Thus, we can restrict ourselves to $\mathfrak{M}$. Because of the order structure, the group $\mathbb{P}_\mathbb{Z}$ of permutations does not naturally act on $\mathfrak{M}$. So we mainly focus on the action of the shift.

The results we enumerate below heavily rely on writing the algebraic part of the algebra (i.e. the part algebraically generated by the annihilators) in terms of reduced words in quasi-Wick order (see [16, Section 5]). If $\mathfrak{M}_0$ is the concrete unital $*$-algebra generated by the monotone annihilators, a word $X$ in $\mathfrak{M}_0$ is said to have a $\lambda$-*form* if there are $m, n \in \{0, 1, 2, \ldots\}$ and $i_1 < \cdots < i_m$, $j_1 > \cdots > j_n$ such that

$$X = a^\dagger_{i_1} \cdots a^\dagger_{i_m} a_{j_1} \cdots a_{j_n},$$

with $X = I$ the empty word corresponding to $m = n = 0$. Its length is $l(X) = m + n$. In addition, $X$ is said to have a $\pi$-*form* if there are $m, n \in \{0, 1, 2, \ldots\}$, $k \in \mathbb{Z}$, $i_1 < \cdots < i_m$, $j_1 > \cdots > j_n$ such that $i_m < k > j_1$ and

$$X = a^\dagger_{i_1} \cdots a^\dagger_{i_m} a_k a^\dagger_k a_{j_1} \cdots a_{j_n}.$$

Its length is $l(X) = m + 2 + n$. As is seen in [16], the words in $\lambda$-form and in $\pi$-form are reduced, and in addition each element in $\mathfrak{M}_0$ can be expressed as a finite linear combination of $\lambda$-forms and/or $\pi$-forms. One could expect that the set of words in $\lambda$-form and $\pi$-form are linearly independent. However, this is not true. As an example, the reduced $\pi$-form $a^\dagger_i a_j a^\dagger_j a_l$, $i < j > l$, can be written as a sum of $\lambda$-forms:

$$a^\dagger_i a_j a^\dagger_j a_l = a^\dagger_i a_l - \sum_{k=(i \vee l)+1}^{j} a^\dagger_i a^\dagger_k a_k a_l,$$

where $i \vee l := \max\{i, l\}$, as one can straightforwardly see by using (4.3).

The $\lambda$-forms and $\pi$-forms structure of the algebra yields a "splitting" representation of $\mathfrak{M}$, which turns out to describe the convex set of stationary states. In more detail, if $\mathcal{M}_0 := \operatorname{span}\big\{X \in \mathfrak{M}_0 \mid l(X) > 0\big\}$ and $\mathcal{M} := \overline{\mathcal{M}_0}$, where the closure is meant in the norm topology, one finds (cf. [16, Corollary 5.10])

$$\mathfrak{M} = \mathcal{M} + \mathbb{C}I.$$

Furthermore, if $\omega$ denotes the vacuum expectation, and $\omega_\infty \in \mathcal{S}(\mathfrak{M})$ is

$$\omega_\infty(X + cI) := c, \qquad X \in \mathcal{M},\, c \in \mathbb{C},$$

the monotone stationary states are exactly those lying in the segment linking $\omega$ and $\omega_\infty$.

Theorem 4.4 (cf. [16]). *We have*

$$\mathcal{S}_{\mathbb{Z}}(\mathfrak{M}) = \{(1-\gamma)\omega + \gamma\omega_\infty \mid \gamma \in [0,1]\}.$$

Thus, similar to the boolean case, the stationary states give rise to the simplest nontrivial simplex, with ergodic points given by the vacuum and the state at infinity $\omega_\infty$. However, in contrast to what happens for booleans, monotone stationary stochastic processes do not satisfy any strong ergodic property like unique ergodicity or unique (weak) mixing (see [16]).

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