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On the sign changes in a weighted divisor problem

by

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1. Introduction

1.1. Dirichlet divisor problem. Let $d(n)$ be the Dirichlet divisor function, and let $D(x) = \sum_{n\le x} d(n) = \sum_{n_1 n_2 \le x} 1$ denote the summatory function. In 1849, Dirichlet proved that

$$D(x) = x\log x + (2\gamma - 1)x + O(\sqrt{x}),$$

where γ is the Euler constant.

Let

$$\Delta(x) = D(x) - x\log x - (2\gamma - 1)x$$

be the error term in the asymptotic formula for $D(x)$. Dirichlet's *divisor problem* consists in determining the smallest α for which $\Delta(x) \ll x^{\alpha+\varepsilon}$ holds for any $\varepsilon > 0$. Clearly, Dirichlet's result implies that $\alpha \le 1/2$. Since then, there have been many improvements on this estimate. The best result to date, given by Huxley [7, 8], reads

$$\Delta(x) \ll x^{\frac{131}{416}} \log^{\frac{26947}{8320}} x. \tag{1.1}$$

It is widely conjectured that $\alpha = 1/4$ is admissible and is the best possible.

Since $\Delta(x)$ exhibits considerable fluctuations, one natural way to study the upper bounds is to consider the moments.

In 1904, Voronoi [15] showed that

$$\int_1^T \Delta(x)\,dx = \frac{T}{4} + O(T^{3/4}).$$

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Later, in 1922 Cramér [3] proved the mean square formula

$$\int_1^T \Delta(x)^2\,dx = cT^{3/2} + O(T^{5/4+\varepsilon}), \qquad \forall\varepsilon > 0,$$

where c is a positive constant. In 1983, Ivić [9] used the method of large values to prove that

$$\int_1^T |\Delta(x)|^A\,dx \ll T^{1+A/4+\varepsilon}, \qquad \forall\varepsilon > 0, \tag{1.2}$$

for each fixed $0 \le A \le 35/4$. The range of A can be extended to $262/27$ by the estimate (1.1). In 1992, Tsang [13] obtained the asymptotic formula

$$\int_1^T \Delta(x)^k\,dx = c_k T^{1+k/4} + O(T^{1+k/4-\delta_k}) \quad \text{for } k = 3, 4, \tag{1.3}$$

with positive constants c_3, c_4, and $\delta_3 = 1/14$, $\delta_4 = 1/23$. Ivić and Sargos [10] improved the values δ_3, δ_4 to $\delta_3' = 7/20$, $\delta_4' = 1/12$, respectively. Heath-Brown [5] proved in 1992 that for any integer $k < A$, where A satisfies (1.2), the limit

$$c_k = \lim_{X\to\infty} X^{-1-k/4} \int_1^X \Delta(x)^k\,dx$$

exists. Then, there followed a series of investigations on explicit asymptotic formulas of the type (1.3) for larger values of k. In 2004, Zhai [16] established asymptotic formulas for $3 \le k \le 9$.

At the beginning of the 20th century, Voronoi [15] proved the remarkable exact formula

$$\Delta(x) = -\frac{2}{\pi}\sqrt{x}\sum_{n=1}^{\infty} \frac{d(n)}{\sqrt{n}} K_1(4\pi\sqrt{nx}) + \frac{\pi}{2} Y_1(4\pi\sqrt{nx}),$$

where K_1, Y_1 are the Bessel functions, and the series on the right-hand side is boundedly convergent for x lying in each fixed closed interval.

Heath-Brown and Tsang [6] studied the sign changes of $\Delta(x)$. They proved that for a suitable constant $C > 0$, $\Delta(x)$ changes sign on the interval $[T, T + C\sqrt{T}]$ for every sufficiently large T. Here the length $\sqrt{T}$ is almost best possible since they proved that in the interval $[T, 2T]$ there are many subintervals of length $\gg \sqrt{T}\log^{-5} T$ such that $\Delta(x)$ does not change sign in any of these subintervals.

1.2. A weighted divisor problem. Recently, Berndt et al. [1, 2] considered a weighted divisor function $\sum'_{mn\le x} \cos(2\pi m\theta_1)\sin(2\pi n\theta_2)$, where the prime $'$ on the summation sign indicates that if x is an integer, then only 1/2 of the last terms are counted. They got an analogue of Voronoi's formula as follows.

Let J_1 be the ordinary Bessel function. If $0 < \theta_1, \theta_2 < 1$ and $x > 0$, then

$$\sum_{mn\le x}{}' \cos(2\pi m\theta_1)\sin(2\pi n\theta_2)$$
$$= -\frac{\cot(\pi\theta_2)}{4} + \frac{\sqrt{x}}{4}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\bigg\{\frac{J_1(4\pi\sqrt{(m+\theta_1)(n+\theta_2)x})}{\sqrt{(m+\theta_1)(n+\theta_2)}}$$
$$+\frac{J_1(4\pi\sqrt{(m+1-\theta_1)(n+\theta_2)x})}{\sqrt{(m+1-\theta_1)(n+\theta_2)}} - \frac{J_1(4\pi\sqrt{(m+\theta_1)(n+1-\theta_2)x})}{\sqrt{(m+\theta_1)(n+1-\theta_2)}}$$
$$-\frac{J_1(4\pi\sqrt{(m+1-\theta_1)(n+1-\theta_2)x})}{\sqrt{(m+1-\theta_1)(n+1-\theta_2)}}\bigg\}.$$

Denote

$$S(x; a_1/q_1, a_2/q_2) = \sum_{mn\le x}{}' \cos(2\pi m a_1/q_1)\sin(2\pi n a_2/q_2).$$

In [11], we found for $x \ge 1$, $1 \le a_i \le q_i$, $(a_i, q_i) = 1$ $(i = 1, 2)$ and a large number $T \ge 1$ that

$$S(q_1q_2x; a_1/q_1, a_2/q_2) \ll q_1q_2x^{\frac{131}{416}}(\log x)^{\frac{26947}{8320}},$$

and

$$\int_1^T S(q_1q_2x; a_1/q_1, a_2/q_2)\,dx \ll q_1q_2T^{3/4}. \tag{1.4}$$

If $T \gg (q_1q_2)^{\varepsilon}$ is large enough, then for $2 \le k \le 9$ we proved

$$\int_1^T S(q_1q_2x; a_1/q_1, a_2/q_2)^k\,dx = (q_1q_2)^k C_k \int_1^T x^{k/4}\,dx + o((q_1q_2)^k T^{1+k/4}), \tag{1.5}$$

where the C_k are explicit constants.

Here we study $S(x; a_1/q_1, a_2/q_2)$ further and give some more estimates for it.

Notation. For a real number t, let $[t]$ be the largest integer no greater than t, $\{t\} = t - [t]$, $\psi(t) = \{t\} - 1/2$, $\|t\| = \min(\{t\}, 1 - \{t\})$, $e(t) = e^{2\pi i t}$. Moreover, $f \asymp g$ means that both $f \ll g$ and $f \gg g$. Throughout this paper, ε denotes sufficiently small positive constants, and $\mathcal{L}$ denotes $\log T$.

2. Main results. In this paper, we will discuss the sign changes of $S(x; a_1/q_1, a_2/q_2)$ and prove

Theorem 2.1. *Let $c_1 > 0$ be a sufficiently small constant and $c_2 > 0$ be a sufficiently large constant, $q_1 \ge 2$, $q_2 \ge 3$, $1 \le a_i \le q_i$ and $(a_i, q_i) = 1$*

$(i = 1, 2)$. *For any function* $f : (0, \infty) \to \mathbb{R}$ *with* $|f(t)| \le c_1 t^{1/4}$, *the function* $S(t; a_1/q_1, a_2/q_2) + f(t)$ *changes sign at least once in the interval* $[T, T + c_2\sqrt{q_1 q_2 T}\,]$ *for every sufficiently large* $T \ge (q_1 q_2)^{1+\varepsilon}$. *In particular, there exist* $t_1, t_2 \in [T, T + c_2\sqrt{q_1 q_2 T}]$ *such that*

$$S(t_1; a_1/q_1, a_2/q_2) \ge c_1 t_1^{1/4} \quad \text{and} \quad S(t_2; a_1/q_1, a_2/q_2) \le -c_1 t_2^{1/4}.$$

THEOREM 2.2. *There exist positive absolute constants* c_3, c_4, c_5 *such that, for any large* $T \ge (q_1 q_2)^{1+\varepsilon}$, *there are at least* $c_3\sqrt{T}\mathcal{L}^7$ *disjoint subintervals of length* $c_4\sqrt{T}\mathcal{L}^{-7}$ *in* $[T, 2T]$ *such that* $\pm S(t; a_1/q_1, a_2/q_2) > c_5 (q_1 q_2)^{3/4} t^{1/4}$ *whenever* t *lies in any of these subintervals. Moreover,*

$$\operatorname{meas}\{t \in [T, 2T] : \pm S(t; a_1/q_1, a_2/q_2) > c_5 (q_1 q_2)^{3/4} t^{1/4}\} \gg T.$$

We also study the Ω-result for the error term in the asymptotic formula (1.5) for odd k by using Theorem 2.2. Define

$$\mathcal{F}_k(q_1 q_2 x; a_1/q_1, a_2/q_2) := (q_1 q_2)^{-k} \int_1^T S(q_1 q_2 x; a_1/q_1, a_2/q_2)^k \, dx - C_k T^{1+k/4}.$$

THEOREM 2.3. *We have*

$$\mathcal{F}_k(q_1 q_2 T; a_1/q_1, a_2/q_2) = \Omega(T^{1/2+k/4}\mathcal{L}^{-7})$$

for any fixed odd integer $k \ge 3$ *and every sufficiently large* $T \ge (q_1 q_2)^{\varepsilon}$.

REMARK 2.1. Although at present we can only prove (1.5) for $2 \le k \le 9$, Theorem 2.3 holds for any odd $k \ge 2$.

REMARK 2.2. We can get the same or similar conclusions for the sums $\sum'_{mn \le x} \sin(2\pi n a_1/q_1) \sin(2\pi m a_2/q_2)$, $\sum'_{mn \le x} \cos(2\pi n a_1/q_1) \cos(2\pi m a_2/q_2)$ with the same approach.

3. Voronoi-type formula for $S(x; a_1/q_1, a_2/q_2)$. In [11], we proved an analogue of Voronoi's formula for $S(q_1 q_2 x; a_1/q_1, a_2/q_2)$. Set

$$E(n; H, J) = \bigl\{\{h, l\} \mid hl = n,\ 1 \le h \le H,\ h \le l \le 2^{J+1} h\bigr\}.$$

Denote

$$\begin{aligned}\Delta d_2(n; a_1, q_1, a_2, q_2) &= d(n; a_1, q_1, a_2, q_2) + d(n; -a_1, q_1, a_2, q_2) \\ &\quad - d(n; a_1, q_1, -a_2, q_2) - d(n; -a_1, q_1, a_2, q_2),\end{aligned}$$

$$\begin{aligned}&\Delta d_{2,1}(n, H, J; a_1, q_1, a_2, q_2) \\ &\quad = \sideset{}{'}\sum_{\substack{\{h,l\} \in E(n;H,J) \\ h \equiv a_2 \,(\mathrm{mod}\, q_2) \\ l \equiv a_1 \,(\mathrm{mod}\, q_1)}} 1 + \sideset{}{'}\sum_{\substack{\{h,l\} \in E(n;H,J) \\ h \equiv a_2 \,(\mathrm{mod}\, q_2) \\ l \equiv -a_1 \,(\mathrm{mod}\, q_1)}} 1 - \sideset{}{'}\sum_{\substack{\{h,l\} \in E(n;H,J) \\ h \equiv -a_2 \,(\mathrm{mod}\, q_2) \\ l \equiv a_1 \,(\mathrm{mod}\, q_1)}} 1 - \sideset{}{'}\sum_{\substack{\{h,l\} \in E(n;H,J) \\ h \equiv -a_2 \,(\mathrm{mod}\, q_2) \\ l \equiv -a_1 \,(\mathrm{mod}\, q_1)}} 1,\end{aligned}$$

$$\Delta d_{2,2}(n,H,J;a_1,q_1,a_2,q_2) = \sideset{}{'}\sum_{\substack{\{h,l\}\in E(n;H,J)\\ h\equiv a_1 \,(\mathrm{mod}\, q_1)\\ l\equiv a_2 \,(\mathrm{mod}\, q_2)}} 1 + \sideset{}{'}\sum_{\substack{\{h,l\}\in E(n;H,J)\\ h\equiv -a_1 \,(\mathrm{mod}\, q_1)\\ l\equiv a_2 \,(\mathrm{mod}\, q_2)}} 1 - \sideset{}{'}\sum_{\substack{\{h,l\}\in E(n;H,J)\\ h\equiv a_1 \,(\mathrm{mod}\, q_1)\\ l\equiv -a_2 \,(\mathrm{mod}\, q_2)}} 1 - \sideset{}{'}\sum_{\substack{\{h,l\}\in E(n;H,J)\\ h\equiv -a_1 \,(\mathrm{mod}\, q_1)\\ l\equiv -a_2 \,(\mathrm{mod}\, q_2)}} 1.$$

Let

$$J = \left[\frac{\mathcal{L} + 2\log q_1q_2 - 4\log\mathcal{L}}{\log 2}\right],$$

$H \ge 2$ be a parameter, and $T^{\varepsilon} < y \le \min(H^2, (q_1q_2)^2T)\mathcal{L}^{-4}$. Suppose $T \le x \le 2T$. Then

$$S(q_1q_2x; a_1/q_1, a_2/q_2) = R_0(x;y) + R_{12}(x;y,H) + R_{21}(x;y,H) + G_{12}(x;H) + G_{21}(x;H) + O(q_1q_2\mathcal{L}^3), \tag{3.1}$$

where

$$R_0(x;y) = \frac{q_1q_2x^{1/4}}{4\sqrt{2}\,\pi}\sum_{n\le y}\frac{\cos(4\pi\sqrt{nx}-3\pi/4)}{n^{3/4}}\Delta d_2(n;a_1,q_1,a_2,q_2), \tag{3.2}$$

$$R_{12}(x;y,H) = \frac{q_1q_2x^{1/4}}{4\sqrt{2}\,\pi} \times \sum_{y<n\le 2^{J+1}H^2}\frac{\cos(4\pi\sqrt{nx}-3\pi/4)}{n^{3/4}}\Delta d_{2,1}(n,H,J;a_1,q_1,a_2,q_2),$$

$$R_{21}(x;y,H) = \frac{q_1q_2x^{1/4}}{4\sqrt{2}\,\pi} \times \sum_{y<n\le 2^{J+1}H^2}\frac{\cos(4\pi\sqrt{nx}-3\pi/4)}{n^{3/4}}\Delta d_{2,2}(n,H,J;a_1,q_1,a_2,q_2),$$

$$G_{12}(x;H) = O\left(q_2\sum_{n_1\le q_1\sqrt{T}}\min\left(1, \frac{1}{H\|q_1x/n_1 - r_2/q_2\|}\right)\right),$$

$$G_{21}(x;H) = O\left(q_1\sum_{n_2\le q_2\sqrt{T}}\min\left(1, \frac{1}{H\|q_2x/n_2 - r_1/q_1\|}\right)\right).$$

4. Proof of Theorem 2.1. In this section, we prove Theorem 2.1 following the approach of [6].

Let n_0 be the smallest integer n such that $\Delta d_2(n;a_1,q_1,a_2,q_2) \ne 0$. By the definition of $\Delta d_2(n;a_1,q_1,a_2,q_2)$, it is easy to see that $\Delta d_2(n_0;a_1,q_1,a_2,q_2) = 1$ or -1, and $n_0 = \min\{a_1, q_1-a_1\}\times\min\{a_2, q_2-a_2\}$, which implies that $n_0 < \frac{1}{4}q_1q_2$.

Suppose $|f(t)| \le c_1 t^{1/4}$. Let

$$S^*(t) = 4\sqrt{2}\,\pi(q_1q_2)^{-1}t^{-1/2}\big(S(q_1q_2t^2; a_1/q_1, a_2/q_2) + f(q_1q_2t^2)\big) \quad \text{for } t \ge 1.$$

Define

$$K_\zeta(u) := (1-|u|)(1+\zeta\sin(4\pi\alpha\sqrt{n_0}\,u)) \quad \text{for } |u| \le 1,$$

with $\zeta = 1$ or -1, and $\alpha > n_0^{1/2}$ a large number.

Set $\zeta' = -\Delta d_2(n_0; a_1, q_1, a_2, q_2)\zeta$. Then it is easy to see that $\zeta' = 1$ or -1, and Theorem 2.1 follows from Lemma 4.1 below.

LEMMA 4.1. *Suppose $T \gg (q_1q_2)^\varepsilon$ is a large parameter. Then for each $\sqrt{T} \le t \le \sqrt{2T}$, we have*

$$\int_{-1}^{1} S^*(t+\alpha u)K_\zeta(u)\,du = \frac{\zeta'}{2n_0^{3/4}}\sin(4\pi t\sqrt{n_0} - 3\pi/4) + O\big(\alpha^{-2} + t^{-1/2}\mathcal{L}^3 + c_1(q_1q_2)^{-3/4}\big).$$

Proof. Let J and y be as prior to (3.1), with $H \ge 2$ to be determined. From (3.1), we have

$$\begin{aligned} S^*(t) = {} & R_0^*(t;y) + R_{12}^*(t;y,H) + R_{21}^*(t;y,H) \\ & + 4\sqrt{2}\,\pi(q_1q_2)^{-1}t^{-1/2}f(q_1q_2t^2) \\ & + O\big(t^{-1/2}(G_{12}^*(t;H) + G_{21}^*(t;H))\big) + O(t^{-1/2}\mathcal{L}^3), \end{aligned} \tag{4.1}$$

where

$$R_0^*(t;y) = \sum_{n\le y}\frac{\cos(4\pi t\sqrt{n} - 3\pi/4)}{n^{3/4}}\Delta d_2(n; a_1, q_1, a_2, q_2),$$

$$R_{12}^*(t;y,H) = \sum_{y<n\le 2^{J+1}H^2}\frac{\cos(4\pi t\sqrt{n} - 3\pi/4)}{n^{3/4}}\Delta d_{2,1}(n, H, J; a_1, q_1, a_2, q_2),$$

$$R_{21}^*(t;y,H) = \sum_{y<n\le 2^{J+1}H^2}\frac{\cos(4\pi t\sqrt{n} - 3\pi/4)}{n^{3/4}}\Delta d_{2,2}(n, H, J; a_1, q_1, a_2, q_2),$$

$$G_{12}^*(t;H) = O\bigg(\frac{1}{q_1}\sum_{n_1\le q_1\sqrt{T}}\min\bigg(1, \frac{1}{H\|q_1t^2/n_1 - r_2/q_2\|}\bigg)\bigg),$$

$$G_{21}^*(t;H) = O\bigg(\frac{1}{q_2}\sum_{n_2\le q_2\sqrt{T}}\min\bigg(1, \frac{1}{H\|q_2t^2/n_2 - r_1/q_1\|}\bigg)\bigg).$$

Denote

$$R^*(t) = R_0^*(t;y) + R_{12}^*(t;y,H) + R_{21}^*(t;y,H), \quad G^*(t) = G_{12}^*(t;H) + G_{21}^*(t;H).$$

Then

$$S^*(t) = R^*(t) + 4\sqrt{2}\,\pi(q_1q_2)^{-1}t^{-1/2}f(q_1q_2t^2) + O(t^{-1/2}G^*(t)) + O(t^{-1/2}\mathcal{L}^3). \tag{4.2}$$

We first consider $\int_{-1}^1 G^*(t+\alpha u)\,du$. Noting that

$$\min\Big(1, \frac{1}{H\|r\|}\Big) = \sum_{h=-\infty}^{\infty} a(h)e(hr)$$

with

$$a(0) \ll H^{-1}\log H, \qquad a(h) \ll \min(H^{-1}\log H, h^{-2}H),\ \ h \neq 0,$$

we have

$$\begin{aligned}\int_{-1}^{1} & G_{12}^*(t+\alpha u; H)\,du \\ &= \frac{1}{q_1}\sum_{h=-\infty}^{\infty} a(h) \sum_{n_1 \le q_1\sqrt{T}} e\Big(\frac{hq_1t^2}{n_1} - \frac{hr_2}{q_2}\Big)\int_{-1}^{1} e\Big(\frac{2hq_1t\alpha u + hq_1\alpha^2u^2}{n_1}\Big)\,du \\ &\ll |a(0)|\sqrt{T} + \frac{1}{q_1}\sum_{h=1}^{\infty}|a(h)| \sum_{n_1 \le q_1\sqrt{T}} \frac{n_1}{hq_1t\alpha} \\ &\ll H^{-1}T^{1/2}\log H + \sum_{h=1}^{H} H^{-1}(\log H)T(ht\alpha)^{-1} + \sum_{h=H}^{\infty} HT(t\alpha)^{-1}h^{-3} \\ &\ll H^{-1}T^{1/2}\log^2 H,\end{aligned}$$

where the first derivative test was used. This estimate remains valid with G_{12}^* replaced by G_{21}^*, which yields

$$\int_{-1}^{1} G^*(t+\alpha u)\,du \ll H^{-1}T^{1/2}\log^2 H. \tag{4.3}$$

Now we estimate $\int_{-1}^1 R^*(t+\alpha u)K_\zeta(u)\,du$. By the elementary formula

$$\begin{aligned}\cos(4\pi(t&+\alpha u)\sqrt{n} - 3\pi/4) \\ &= \cos(4\pi t\sqrt{n} - 3\pi/4)\cos(4\pi\alpha u\sqrt{n}) - \sin(4\pi t\sqrt{n} - 3\pi/4)\sin(4\pi\alpha u\sqrt{n}),\end{aligned}$$

we get

$$\int_{-1}^{1} \cos(4\pi(t+\alpha u)\sqrt{n} - 3\pi/4)(1-|u|)(1+\zeta\sin(4\pi\alpha\sqrt{n_0}\,u))\,du = I_1 + I_2$$

with

$$\begin{aligned}
I_1 &= \cos(4\pi t\sqrt{n}-3\pi/4)\int_{-1}^{1}\cos(4\pi\alpha u\sqrt{n})(1-|u|)(1+\zeta\sin(4\pi\alpha\sqrt{n_0}\,u))\,du\\
&= \cos(4\pi t\sqrt{n}-3\pi/4)\int_{-1}^{1}\cos(4\pi\alpha u\sqrt{n})(1-|u|)\,du,\\
I_2 &= \sin(4\pi t\sqrt{n}-3\pi/4)\int_{-1}^{1}\sin(4\pi\alpha u\sqrt{n})(1-|u|)(1+\zeta\sin(4\pi\alpha\sqrt{n_0}\,u))\,du\\
&= \zeta\sin(4\pi t\sqrt{n}-3\pi/4)\int_{-1}^{1}\sin(4\pi\alpha u\sqrt{n})(1-|u|)\sin(4\pi\alpha\sqrt{n_0}\,u)\,du\\
&= \frac{\zeta}{2}\sin(4\pi t\sqrt{n}-3\pi/4)\int_{-1}^{1}(1-|u|)\cos(4\pi\alpha u(\sqrt{n}-\sqrt{n_0}))\,du\\
&\quad -\frac{\zeta}{2}\sin(4\pi t\sqrt{n}-3\pi/4)\int_{-1}^{1}(1-|u|)\cos(4\pi\alpha u(\sqrt{n}+\sqrt{n_0}))\,du.
\end{aligned}$$

By using

$$\int_0^1 (1-u)\cos(Au)\,du \ll |A|^{-2}, \qquad A\neq 0,$$

we have

$$I_1 \ll \alpha^{-2}n^{-1}, \qquad I_2 = \begin{cases} (\zeta/2)\sin(4\pi t\sqrt{n_0}-3\pi/4)+O(\alpha^{-2}n_0^{-1}), & n=n_0,\\ O(\alpha^{-2}(\sqrt{n}-\sqrt{n_0})^{-2}), & n\neq n_0,\end{cases}$$

which yields

$$\int_{-1}^{1}\cos(4\pi(t+\alpha u)\sqrt{n}-3\pi/4)K_\zeta(u)\,du = \begin{cases} (\zeta/2)\sin(4\pi t\sqrt{n_0}-3\pi/4)+O(\alpha^{-2}n_0^{-1}), & n=n_0,\\ O(\alpha^{-2}(\sqrt{n}-\sqrt{n_0})^{-2}), & n\neq n_0.\end{cases}$$

Take $H=T$ and $y=T^{1/2}$. Then clearly $n_0<y$. Thus we get

$$\begin{aligned}
&\int_{-1}^{1} R^*(t+\alpha u)K_\zeta(u)\,du \qquad (4.4)\\
&= -\frac{\zeta}{2n_0^{3/4}}\sin(4\pi t\sqrt{n_0}-3\pi/4)\Delta d_2(n_0;a_1,q_1,a_2,q_2)+O\Big(\sum_{n>n_0}\frac{\alpha^{-2}n^{-3/4}d(n)}{(\sqrt{n}-\sqrt{n_0})^2}\Big)\\
&= -\frac{\zeta}{2n_0^{3/4}}\sin(4\pi t\sqrt{n_0}-3\pi/4)\Delta d_2(n_0;a_1,q_1,a_2,q_2)+O(\alpha^{-2}),
\end{aligned}$$

by using $\sum_{n>n_0}\frac{d(n)}{n^{3/4}(\sqrt{n}-\sqrt{n_0})^2}\ll 1$.

Note that $H = T$ and $t \asymp T^{1/2}$. From (4.2)–(4.4), we see that

$$\begin{aligned}
&\int_{-1}^{1} S^*(t+\alpha u)K_\zeta(u)\,du \\
&= -\frac{\zeta}{2n_0^{3/4}}\sin(4\pi t\sqrt{n_0} - 3\pi/4)\Delta d_2(n_0; a_1, q_1, a_2, q_2) + O(\alpha^{-2}) \\
&\quad + O\Big((q_1q_2)^{-1}t^{-1/2}\sup_{|u|\le 1} f(q_1q_2(t+\alpha u)^2)\Big) \\
&\quad + O(t^{-1/2}H^{-1}T^{1/2}\mathcal{L}^2) + O(t^{-1/2}\mathcal{L}^3) \\
&= \frac{\zeta'}{2n_0^{3/4}}\sin(4\pi t\sqrt{n_0} - 3\pi/4) + O(\alpha^{-2}) + O(c_1(q_1q_2)^{-3/4}) + O(t^{-1/2}\mathcal{L}^3).
\end{aligned}$$

This completes the proof of Lemma 4.1. ■

5. The mean value of $S(q_1q_2x; a_1/q_1, a_2/q_2)$ in short intervals. Suppose $T \gg (q_1q_2)^\varepsilon$ is a large parameter, and $1 \le h \le \frac{1}{2}\sqrt{T}$. Denote $S(q_1q_2x) = S(q_1q_2x; a_1/q_1, a_2/q_2)$. In this section we shall estimate the integral

$$I(T,h) = \int_1^T \big(S(q_1q_2(x+h)) - S(q_1q_2x)\big)^2\,dx,$$

which will play an important role in the proof of Theorem 2.2. This type of integral was studied for the error term in the mean square of $\zeta(1/2+it)$ by Good [4], for the error term in the Dirichlet divisor problem by Jutila [12] and for the error term in Weyl's law for a Heisenberg manifold by Tsang and Zhai [14]. Here we follow the approach of Tsang and Zhai [14] and prove

Lemma 5.1. *The estimate*

$$I(T,h) \ll (q_1q_2)^2 hT\log^3(\sqrt{T}/h) + (q_1q_2)^2 T\mathcal{L}^6$$

holds uniformly for $1 \le h \le \frac{1}{2}\sqrt{T}$.

Proof. Write

$$I(T,h) = \int_1 + \int_2, \tag{5.1}$$

where

$$\int_1 = \int_1^{100\max(h^2,T^{2/3})} \big(S(q_1q_2(x+h)) - S(q_1q_2x)\big)^2\,dx,$$

$$\int_2 = \int_{100\max(h^2,T^{2/3})}^{T} \big(S(q_1q_2(x+h)) - S(q_1q_2x)\big)^2\,dx.$$

From (1.5), we see that

$$\int_1 \ll (q_1q_2)^2(h^3+T) \ll (q_1q_2)^2 Th. \tag{5.2}$$

For $\int_2$, we first estimate the integral

$$J(U,h) = \int_U^{2U} \big(S(q_1q_2(x+h)) - S(q_1q_2x)\big)^2\,dx, \quad 100\max(h^2, T^{2/3}) \le U \le T. \tag{5.3}$$

Let $T = 2U$ in (3.1). Then

$$\begin{aligned} S(q_1q_2x) = R_0(x;y) &+ R_{12}(x;y,H) + R_{21}(x;y,H) \\ &+ G_{12}(x;H) + G_{21}(x;H) + O(q_1q_2\log^3 U). \end{aligned}$$

Take $H = U$ and $y = \min\big(\frac{1}{2}Uh^{-1}, U\log^{-6}U\big)$. From [11, Lemmas 6.2 and 6.5], we see that

$$\int_U^{2U} |G_{12}(x;H) + G_{21}(x;H)|^2\,dx \ll (q_1q_2)^2 U\log U,$$

$$\int_U^{2U} |R_{12}(x;y,H) + R_{21}(x;y,H)|^2\,dx \ll (q_1q_2)^2 U^{3/2}y^{-1/2}\log^3 U.$$

Thus we get

$$\begin{aligned} \int_U^{2U} (S(q_1q_2x) - R_0(x;y))^2\,dx \\ &\ll (q_1q_2)^2U^{3/2}y^{-1/2}\log^3 U + (q_1q_2)^2 U\log^6 U \\ &\ll (q_1q_2)^2 Uh^{1/2}\log^3 U + (q_1q_2)^2 U\log^6 U. \end{aligned} \tag{5.4}$$

We now estimate the integral $\int_U^{2U}(R_0(x+h;y)-R_0(x;y))^2\,dx$. From (3.2),

$$R_0(x+h;y) - R_0(x;y) = F_1(x) + F_2(x), \tag{5.5}$$

where

$$\begin{aligned} F_1(x) = \frac{q_1q_2}{4\sqrt{2}\pi}&((x+h)^{1/4} - x^{1/4}) \\ &\times \sum_{n\le y} \frac{\Delta d_2(n;a_1,q_1,a_2,q_2)}{n^{3/4}}\cos(4\pi\sqrt{n(x+h)} - 3\pi/4), \end{aligned}$$

$$\begin{aligned} F_2(x) = \frac{q_1q_2x^{1/4}}{4\sqrt{2}\,\pi}\sum_{n\le y}\frac{\Delta d_2(n;a_1,q_1,a_2,q_2)}{n^{3/4}}&\big(\cos(4\pi\sqrt{n(x+h)} - 3\pi/4) \\ &- \cos(4\pi\sqrt{nx} - 3\pi/4)\big). \end{aligned}$$

From [11, Lemma 6.3], we get

$$\int_U^{2U} F_1(x)^2\,dx \ll h^2U^{-2}\int_U^{2U} R_0(x+h)^2\,dx \ll (q_1q_2)^2h^2U^{-1/2}. \tag{5.6}$$

For the mean square of $F_2(x)$, we see

$$F_2^2 = F_{21}+F_{22}, \tag{5.7}$$

with

$$\begin{aligned}F_{21}(x) &= \frac{(q_1q_2)^2}{32\pi^2}x^{1/2}\sum_{n\le y}\frac{\Delta d_2(n;a_1,q_1,a_2,q_2)^2}{n^{3/2}}\\ &\quad\times\big(\cos(4\pi\sqrt{n(x+h)}-3\pi/4)-\cos(4\pi\sqrt{nx}-3\pi/4)\big)^2,\\ F_{22}(x) &= \frac{(q_1q_2)^2}{32\pi^2}x^{1/2}\sum_{\substack{m,n\le y\\ m\ne n}}\frac{\Delta d_2(m;a_1,q_1,a_2,q_2)\Delta d_2(n;a_1,q_1,a_2,q_2)}{(mn)^{3/4}}\\ &\quad\times\big(\cos(4\pi\sqrt{m(x+h)}-3\pi/4)-\cos(4\pi\sqrt{mx}-3\pi/4)\big)\\ &\quad\times\big(\cos(4\pi\sqrt{n(x+h)}-3\pi/4)-\cos(4\pi\sqrt{nx}-3\pi/4)\big).\end{aligned}$$

By writing

$$\begin{aligned}&\cos(4\pi\sqrt{n(x+h)}-3\pi/4)-\cos(4\pi\sqrt{nx}-3\pi/4)\\ &\qquad\qquad=\sum_{j=0}^{1}(-1)^{j+1}\cos(4\pi\sqrt{n(x+jh)}-3\pi/4),\end{aligned}$$

we get

$$\begin{aligned}F_{22}(x) &= \frac{(q_1q_2)^2}{32\pi^2}x^{1/2}\sum_{j_1=0}^{1}\sum_{j_2=0}^{1}(-1)^{j_1+j_2}\\ &\quad\times\sum_{\substack{m,n\le y\\ m\ne n}}\frac{\Delta d_2(m;a_1,q_1,a_2,q_2)\Delta d_2(n;a_1,q_1,a_2,q_2)}{(mn)^{3/4}}\\ &\quad\times\cos(4\pi\sqrt{m(x+j_1h)}-3\pi/4)\cos(4\pi\sqrt{n(x+j_2h)}-3\pi/4)\\ &=: F_{221}(x)+F_{222}(x),\end{aligned} \tag{5.8}$$

where

$$\begin{aligned}F_{221}(x) &= \frac{(q_1q_2)^2}{64\pi^2}x^{1/2}\sum_{j_1=0}^{1}\sum_{j_2=0}^{1}(-1)^{j_1+j_2}\sum_{\substack{m,n\le y\\ m\ne n}}\frac{1}{(mn)^{3/4}}\Delta d_2(m;a_1,q_1,a_2,q_2)\\ &\quad\times\Delta d_2(n;a_1,q_1,a_2,q_2)\cos\big(4\pi\sqrt{m(x+j_1h)}-4\pi\sqrt{n(x+j_2h)}\big),\end{aligned}$$

$$F_{222}(x) = \frac{(q_1q_2)^2}{64\pi^2}x^{1/2}\sum_{j_1=0}^{1}\sum_{j_2=0}^{1}(-1)^{j_1+j_2+1}\sum_{\substack{m,n\le y\\ m\ne n}}\frac{1}{(mn)^{3/4}}\Delta d_2(m;a_1,q_1,a_2,q_2)$$
$$\times \Delta d_2(n;a_1,q_1,a_2,q_2)\sin\big(4\pi\sqrt{m(x+j_1h)}+4\pi\sqrt{n(x+j_2h)}\big).$$

Let

$$g_\pm(x) = 4\pi\sqrt{m(x+j_1h)} \pm 4\pi\sqrt{n(x+j_2h)}.$$

Using

$$(1+t)^{1/2} = 1+\sum_{v=1}^{\infty} d_v t^v \quad (|t|\le 1/2)$$

with $|d_v|<1$, we see

$$g_\pm(x) = 4\pi\sqrt{x}(\sqrt{m}\pm\sqrt{n}) + 4\pi\sum_{v=1}^{\infty}\frac{d_v h^v}{x^{v-1/2}}(\sqrt{m}\,j_1^v \pm \sqrt{n}\,j_2^v).$$

Noting that $m,n\le y\le \frac{1}{2}Uh^{-1}$, we have

$$|g'_\pm(x)| \gg \frac{1}{\sqrt{x}}|\sqrt{m}\pm\sqrt{n}| \qquad (m\ne n).$$

Then by the first derivative test we get

$$\int_U^{2U} F_{221}(x)\,dx \ll (q_1q_2)^2U\sum_{\substack{m,n\le y\\ m\ne n}}\frac{\Delta d_2(m;a_1,q_1,a_2,q_2)\Delta d_2(n;a_1,q_1,a_2,q_2)}{(mn)^{3/4}|\sqrt{m}-\sqrt{n}|},$$
$$\int_U^{2U} F_{222}(x)\,dx \ll (q_1q_2)^2U\sum_{\substack{m,n\le y\\ m\ne n}}\frac{\Delta d_2(m;a_1,q_1,a_2,q_2)\Delta d_2(n;a_1,q_1,a_2,q_2)}{(mn)^{3/4}|\sqrt{m}+\sqrt{n}|}.$$

From (5.8), we obtain

$$\text{(5.9)}\qquad \int_U^{2U} F_{22}(x)\,dx$$
$$\ll (q_1q_2)^2U\sum_{\substack{m,n\le y\\ m\ne n}}\frac{\Delta d_2(m;a_1,q_1,a_2,q_2)\Delta d_2(n;a_1,q_1,a_2,q_2)}{(mn)^{3/4}|\sqrt{m}-\sqrt{n}|}$$
$$\ll (q_1q_2)^2U\log^4 y,$$

where we have used the estimate $\sum_{n\le N} d(n) \ll N\log N$.

By the elementary formulas

$$\cos u - \cos v = -2\sin\Big(\frac{u+v}{2}\Big)\sin\Big(\frac{u-v}{2}\Big), \qquad \sin^2 u = \frac{1}{2}(1-\cos 2u),$$

we have

$$\int_U^{2U} F_{21}(x)\,dx = \frac{(q_1q_2)^2}{8\pi^2}\sum_{n\le y}\frac{\Delta d_2(n;a_1,q_1,a_2,q_2)^2}{n^{3/2}} \tag{5.10}$$

$$\times \int_u^{2U} x^{1/2}\sin^2(2\pi\sqrt{n(x+h)}+2\pi\sqrt{nx}-3\pi/4)\sin^2(2\pi\sqrt{n(x+h)}-2\pi\sqrt{nx})\,dx$$

$$=: I_{211}+I_{212},$$

where

$$I_{211} = \frac{(q_1q_2)^2}{16\pi^2}\sum_{n\le y}\frac{\Delta d_2(n;a_1,q_1,a_2,q_2)^2}{n^{3/2}} \times \int_U^{2U} x^{1/2}\sin^2(2\pi\sqrt{n(x+h)}-2\pi\sqrt{nx})\,dx,$$

$$I_{212} = \frac{(q_1q_2)^2}{16\pi^2}\sum_{n\le y}\frac{\Delta d_2(n;a_1,q_1,a_2,q_2)^2}{n^{3/2}}\int_U^{2U} x^{1/2}\sin(4\pi\sqrt{n(x+h)}+4\pi\sqrt{nx}) \times \sin^2(2\pi\sqrt{n(x+h)}-2\pi\sqrt{nx})\,dx.$$

By the first derivative test, we have

$$L_n(t) := \int_U^t x^{1/2}\sin(4\pi\sqrt{n(x+h)}+4\pi\sqrt{nx})\,dx \ll Un^{-1/2}, \quad U\le t\le 2U.$$

Using integration by parts, we obtain

$$\int_U^{2U} x^{1/2}\sin(4\pi\sqrt{n(x+h)}+4\pi\sqrt{nx})\sin^2(2\pi\sqrt{n(x+h)}-2\pi\sqrt{nx})\,dx$$

$$= \int_U^{2U}\sin^2(2\pi\sqrt{n(x+h)}-2\pi\sqrt{nx})\,dL_n(x)$$

$$= L_n(2U)\sin^2(2\pi\sqrt{n(2U+h)}-2\pi\sqrt{2nU}) - 2\int_U^{2U} L_n(x)\Big(\frac{\pi\sqrt{n}}{\sqrt{x+h}}-\frac{\pi\sqrt{n}}{\sqrt{x}}\Big)$$

$$\times \sin(2\pi\sqrt{n(x+h)}-2\pi\sqrt{nx})\cos(2\pi\sqrt{n(x+h)}-2\pi\sqrt{nx})\,dx$$

$$\ll Un^{-1/2}+U^{1/2}h,$$

which yields

$$I_{212} \ll (q_1q_2)^2\sum_{n\le y}\frac{d(n)^2}{n^{3/2}}(Un^{-1/2}+U^{1/2}h) \tag{5.11}$$

$$\ll (q_1q_2)^2(U+U^{1/2}h) \ll (q_1q_2)^2U.$$

By using

$$\sqrt{x+h} = x^{1/2} + hx^{-1/2} + O(h^2x^{-3/2}), \quad x \ge 100h^2,$$

we get

$$\begin{aligned}\sin^2(2\pi\sqrt{n(x+h)} - 2\pi\sqrt{nx}) &= \sin^2(\pi hn^{1/2}x^{-1/2} + O(h^2n^{1/2}x^{-3/2})) \\ &= \sin^2(\pi hn^{1/2}x^{-1/2}) + O(h^2n^{1/2}x^{-3/2}).\end{aligned}$$

Noting that

$$\begin{aligned}\int_U^{2U} x^{1/2}\sin^2(\pi hn^{1/2}x^{-1/2})\,dx &\ll \int_U^{2U} x^{1/2}\min(1, h^2nx^{-1})\,dx \\ &\ll \begin{cases} U^{1/2}h^2n, & n \le Uh^{-2}, \\ U^{3/2}, & n > Uh^{-2},\end{cases}\end{aligned}$$

we have

$$\begin{aligned}(5.12)\quad I_{211} &\ll (q_1q_2)^2 \sum_{n\le y} \frac{d(n)^2}{n^{3/2}} \\ &\quad\times \int_U^{2U} x^{1/2}(\sin^2(\pi hn^{1/2}x^{-1/2}) + O(h^2n^{1/2}x^{-3/2}))\,dx \\ &\ll (q_1q_2)^2 \sum_{n\le y} \frac{d(n)^2}{n^{3/2}} \\ &\quad\times \int_U^{2U} x^{1/2}\sin^2(\pi hn^{1/2}x^{-1/2})\,dx + O\Big((q_1q_2h)^2 \sum_{n\le y}\frac{d(n)^2}{n}\Big) \\ &\ll (q_1q_2h)^2U^{1/2} \sum_{n\le Uh^{-2}} \frac{d(n)^2}{n^{1/2}} + (q_1q_2)^2U^{3/2} \\ &\quad\times \sum_{n>Uh^{-2}} \frac{d(n)^2}{n^{3/2}} + O((q_1q_2h)^2\log^4 y) \\ &\ll (q_1q_2)^2Uh\log^3(\sqrt{U}/h),\end{aligned}$$

where we have used the well-known estimate $\sum_{n\le N} d(n)^2 \ll N\log^3 N$.

From (5.10)–(5.12), we get

$$(5.13)\qquad \int_U^{2U} F_{21}(x)\,dx \ll (q_1q_2)^2Uh\log^3(\sqrt{U}/h).$$

Combining (5.7), (5.9) and (5.13), we obtain

$$\int_U^{2U} F_2(x)^2\,dx \ll (q_1q_2)^2Uh\log^3(\sqrt{U}/h) + (q_1q_2)^2U\log^4 y,$$

which together with (5.5), (5.6) yields

$$\int_U^{2U} (R_0(x+h;y) - R_0(x;y))^2\,dx \ll (q_1q_2)^2 Uh \log^3(\sqrt{U}/h) + (q_1q_2)^2 U \log^4 y. \tag{5.14}$$

From (5.3), (5.4), and (5.14), it follows that

$$J(U,h) \ll (q_1q_2)^2 Uh \log^3(\sqrt{U}/h) + (q_1q_2)^2 U \log^6 y,$$

which implies

$$\int_2 \ll (q_1q_2)^2 Th \log^3(\sqrt{T}/h) + (q_1q_2)^2 T\mathcal{L}^6, \tag{5.15}$$

via a splitting argument. Then Lemma 5.1 follows from (5.1), (5.2) and (5.15). ∎

6. Proof of Theorem 2.2. In this section, we prove Theorem 2.2 by following the approach of [14]. We still write $S(q_1q_2x) = S(q_1q_2x; a_1/q_1, a_2/q_2)$. Define

$$S_+(t) = \tfrac{1}{2}(|S(t)| + S(t)), \qquad S_-(t) = \tfrac{1}{2}(|S(t)| - S(t)).$$

LEMMA 6.1.

$$\int_T^{2T} S_\pm(q_1q_2t)^2\,dt \gg (q_1q_2)^2 T^{3/2}.$$

Proof. From (1.5) with $k = 2, 4$, by Hölder's inequality, we get

$$\begin{aligned}(q_1q_2)^2T^{3/2} \ll \int_T^{2T} S(q_1q_2t)^2\,dt &\ll \Big(\int_T^{2T} |S(q_1q_2t)|\,dt\Big)^{2/3}\Big(\int_T^{2T} S(q_1q_2t)^4\,dt\Big)^{1/3}\\ &\ll \Big(\int_T^{2T} |S(q_1q_2t)|\,dt\Big)^{2/3}(q_1q_2)^{4/3}T^{2/3},\end{aligned}$$

which yields

$$\int_T^{2T} |S(q_1q_2t)|\,dt \gg q_1q_2T^{5/4}. \tag{6.1}$$

From (1.4), we see that

$$\int_T^{2T} S(q_1q_2t)\,dt \ll q_1q_2T^{3/4}.$$

Thus, from the definition of $S_{\pm}(q_1q_2t)$, we have

$$\int_T^{2T} S_{\pm}(q_1q_2t)\,dt \gg q_1q_2T^{5/4}.$$

Then by Cauchy–Schwarz's inequality, we get

$$q_1q_2T^{5/4} \ll \Big(\int_T^{2T} dt\Big)^{1/2}\Big(\int_T^{2T} S_{\pm}(q_1q_2t)^2\,dt\Big)^{1/2} \ll T^{1/2}\Big(\int_T^{2T} S_{\pm}(q_1q_2t)^2\,dt\Big)^{1/2},$$

which immediately implies Lemma 6.1. ∎

LEMMA 6.2. *Suppose* $2 \le H_0 \le \sqrt{T}$. *Then*

$$\int_T^{2T} \max_{h\le H_0} \big(S_{\pm}(q_1q_2(t+h)) - S_{\pm}(q_1q_2t)\big)^2\,dt \ll (q_1q_2)^2H_0T\mathcal{L}^7.$$

Proof. Since

$$|S_{\pm}(q_1q_2(t+h)) - S_{\pm}(q_1q_2t)| \le |S(q_1q_2(t+h)) - S(q_1q_2t)|,$$

it is sufficient to prove that

$$I = \int_T^{2T} \max_{h\le H_0} \big(S(q_1q_2(t+h)) - S(q_1q_2t)\big)^2\,dt \ll (q_1q_2)^2H_0T\mathcal{L}^7.$$

Write $H_0 = 2^{\lambda}b$ with $\lambda \in \mathbb{N}$ and $1 \le b < 2$. By Lemma 5.1, we get

$$\begin{aligned} I &\ll \lambda\sum_{\mu\le\lambda}\sum_{0\le\nu\le 2^{\mu}} \int_{T+\nu2^{\lambda-\mu}b}^{2T+\nu2^{\lambda-\mu}b} \big(S(q_1q_2(t+2^{\lambda-\mu}b)) - S(q_1q_2t)\big)^2\,dt + (q_1q_2)^2T\mathcal{L}^2 \\ &\ll \lambda\sum_{\mu\le\lambda}\sum_{0\le\nu\le 2^{\mu}} \big((q_1q_2)^2 2^{\lambda-\mu}bT\mathcal{L}^3 + (q_1q_2)^2T\mathcal{L}^6\big) \\ &\ll \lambda\sum_{\mu\le\lambda} \big((q_1q_2)^2 2^{\lambda}bT\mathcal{L}^3 + (q_1q_2)^2 2^{\mu}T\mathcal{L}^6\big) \\ &\ll \lambda^2(q_1q_2)^2H_0T\mathcal{L}^3 + \lambda(q_1q_2)^2H_0T\mathcal{L}^6 \ll (q_1q_2)^2H_0T\mathcal{L}^7, \end{aligned}$$

where we have used the well-known estimate

$$\sum_{x<n\le x+y} d(n) \ll y\log x, \quad x^{\varepsilon} < y < x. \ \blacksquare$$

Now we finish the proof of Theorem 2.2. For any function $P(t)$ and $Q(t)$ such that

$$\omega(t) = P(t)^2 - 4\max_{h\le H_0}\big(P(t+h) - P(t)\big)^2 - Q(t)^2 > 0,$$

we see that $P(t+h)$ has the same sign as $P(t)$, and $|P(t+h)| > \frac{1}{2}|Q(t)|$ for any $0 \le h \le H_0$. Take $P(t) = S_{\pm}(q_1q_2t)$ and $Q(t) = \delta q_1q_2t^{1/4}$ for a sufficiently small $\delta > 0$. By Lemmas 6.1 and 6.2, we get

$$\int_T^{2T} \omega(t)\,dt \gg (q_1q_2)^2T^{3/2} - O\big((q_1q_2)^2(H_0T\mathcal{L}^7 + \delta^2T^{3/2})\big) \gg (q_1q_2)^2T^{3/2}, \tag{6.2}$$

by taking $H_0 = \delta T^{1/2}\mathcal{L}^{-7}$. Let

$$\mathscr{S} = \{t \in [T, 2T] : \omega(t) > 0\}.$$

From (1.5) and (6.2), using Cauchy–Schwarz's inequality, we have

$$\begin{aligned}(q_1q_2)^2T^{3/2} &\ll \int_T^{2T} \omega(t)\,dt \le \int_{\mathscr{S}} \omega(t)\,dt \le \int_{\mathscr{S}} S_{\pm}(q_1q_2t)^2\,dt \\ &\le |\mathscr{S}|^{1/2}\Big(\int_T^{2T} S(q_1q_2t)^4\,dt\Big)^{1/2} \ll |\mathscr{S}|^{1/2}(q_1q_2)^2T,\end{aligned}$$

which implies

$$|\mathscr{S}| \gg T.$$

This completes the proof of Theorem 2.2. ∎

7. Proof of Theorem 2.3. Suppose $k \ge 3$ is a fixed odd integer and $T \ge (q_1q_2)^{\varepsilon}$ is a large parameter. Set

$$\delta = \begin{cases} -1 & \text{if } C_k \ge 0, \\ 1 & \text{if } C_k < 0, \end{cases}$$

where C_k is defined in (1.5).

By Theorem 2.2, there exists $t \in [T, 2T]$ such that $\delta S(q_1q_2u; a_1/q_1, a_2/q_2) > c_5q_1q_2t^{1/4}$ for any $u \in [t, t + H_0]$, with $H_0 = c_4\sqrt{T}\,\mathcal{L}^{-7}$. Thus

$$\begin{aligned}c_5^kH_0t^{k/4} &< (q_1q_2)^{-k} \int_t^{t+H_0} \delta^kS(q_1q_2u; a_1/q_1, a_2/q_2)^k\,du \\ &= \delta^kC_k((t + H_0)^{1+k/4} - t^{1+k/4}) \\ &\quad + \delta^k\big(\mathcal{F}_k(q_1q_2(t + H_0); a_1/q_1, a_2/q_2) - \mathcal{F}_k(q_1q_2t; a_1/q_1, a_2/q_2)\big) \\ &= \delta^kC_k(1 + k/4)t^{k/4}H_0 + O(H_0^2t^{k/4-1}) \\ &\quad + \delta^k\big(\mathcal{F}_k(q_1q_2(t + H_0); a_1/q_1, a_2/q_2) - \mathcal{F}_k(q_1q_2t; a_1/q_1, a_2/q_2)\big),\end{aligned}$$

which yields

$$\delta^k\big(\mathcal{F}_k(q_1q_2(t + H_0); a_1/q_1, a_2/q_2) - \mathcal{F}_k(q_1q_2t; a_1/q_1, a_2/q_2)\big) > C_k^*H_0t^{k/4}\big(1 + O(H_0T^{-1})\big)$$

with

$$C_k^* = c_5^k - \delta^kC_k(1 + k/4) > 0.$$

Thus we get

$$|\mathcal{F}_k(q_1q_2(t + H_0); a_1/q_1, a_2/q_2) - \mathcal{F}_k(q_1q_2t; a_1/q_1, a_2/q_2)| \gg H_0T^{k/4},$$

which immediatly implies Theorem 2.3. ∎

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