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GENERATORS OF SIMPLE GRADED LIE ALGEBRAS OF FINITE GROWTH

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Abstract. Let X be a simple graded Lie algebra of finite growth over an algebraically closed field of characteristic zero. We prove that X can be generated by two elements.

1. Introduction. A $\mathbb{Z}$-*graded Lie algebra* is a Lie algebra L endowed with a decomposition $L = \bigoplus_{n\in\mathbb{Z}} L_n$ such that $[L_n, L_m] \subset L_{n+m}$ for every $n, m \in \mathbb{Z}$. A *simple* graded Lie algebra L is a $\mathbb{Z}$-graded Lie algebra L which is not abelian and does not contain any nontrivial graded ideal. A $\mathbb{Z}$-graded Lie algebra L has *finite growth* provided that the homogenous component L_n is finite-dimensional and the function $n \mapsto \dim L_n$ is bounded by some polynomial. Kac [6, 7, 8] posed a conjecture about the classification of simple graded Lie algebras of finite growth over an algebraically closed field of characteristic zero, which was proved by Mathieu [10]. All the simple graded Lie algebras of finite growth consist of the following series:

- finite-dimensional simple Lie algebras,
- loop algebras,
- Cartan algebras,
- Virasoro algebra.

The aim of the paper is to determine the minimal number of generators of any simple graded Lie algebra of finite growth over an algebraically closed field of characteristic zero. Generators play an important role in the structure and representation theory of an algebra, in questions regarding derivations, automorphisms, maximal subalgebras and so on. In 2009, Bois [1] proved that any simple Lie algebra in characteristics $p \neq 2, 3$ can be generated by two elements, and moreover, the classical Lie algebras and Zassenhaus algebras can be generated by 1.5 elements, that is, any given nonzero element can be paired with a suitable element such that these two elements generate

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the whole algebra. In 2010, Bois [2] proved that the simple graded Cartan type Lie algebras S, H or K are never generated by 1.5 elements. The results in [1, 2] cover certain classical results: in 1976, Ionescu [5] proved that a simple Lie algebra L over $\mathbb{C}$ can be generated by 1.5 elements, and in 1951, Kuranashi [9] showed that a semisimple Lie algebra in characteristic 0 can be generated by two elements.

The methods used in [1] to investigate the 2-generation property were geometrical. The approach used in the present paper involves properties of simple graded Lie algebras. The ideas come also from [1]. In order to study the generators of a simple graded Lie algebra, a general strategy is to compute weights of some appropriate gradations as modules over the $\mathbb{Z}$-null part. We start from some weight vectors corresponding to different weights by virtue of the weight space decomposition.

Throughout the paper, unless otherwise specified, all vector spaces and algebras are over an algebraically closed field of characteristic zero. For an algebra $\mathfrak{A}$ and $x, y \in \mathfrak{A}$, we write $\langle x, y\rangle$ for the subalgebra generated by x and y.

The main result of this paper is:

THEOREM 1.1. *If $\mathbb{F}$ is an algebraically closed field of characteristic zero, then any simple $\mathbb{Z}$-graded Lie algebra of finite growth is generated by two elements.*

2. Finite-dimensional simple Lie algebras. We first briefly recall some notions and conclusions about generators of simple Lie algebras.

Let $\mathfrak{g}$ be a simple Lie algebra. The natural $\mathbb{Z}$-grading of $\mathfrak{g}$ is defined by $\mathfrak{g} = \mathfrak{g}_0$. Consider the root decomposition relative to a Cartan subalgebra $\mathfrak{h}$: $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}^\alpha$. For $x \in \mathfrak{g}$ we write

$$x = x_{\mathfrak{h}} + \sum_{\alpha\in\Phi} x^\alpha$$

for the corresponding decomposition. It is well-known that [4]

$$\dim \mathfrak{g}^\alpha = 1 \quad \text{for all } \alpha \in \Phi, \tag{2.1}$$

$$\mathfrak{h} = \sum_{\alpha\in\Phi} [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}], \tag{2.2}$$

$$[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = \mathfrak{g}^{\alpha+\beta} \quad \text{whenever } \alpha, \beta, \alpha+\beta \in \Phi. \tag{2.3}$$

Let V be a vector space and $\mathfrak{F} := \{f_1, \dots, f_n\}$ a finite set of nonzero linear functions on V. Write

$$\Omega_{\mathfrak{F}} := \Big\{ v \in V \;\Big|\; \prod_{1\le i\ne j\le n} (f_i - f_j)(v) \ne 0 \Big\}.$$

LEMMA 2.1. *Let $\mathfrak{F}$ be a finite set of nonzero functions in V^*. Then $\Omega_{\mathfrak{F}} \neq \emptyset$. If $\mathfrak{G} \subset \mathfrak{F}$ then $\Omega_{\mathfrak{F}} \subset \Omega_{\mathfrak{G}}$.*

Proof. The first statement is proved in [1, Lemma 2.2.1] and the second is straightforward. ∎

This lemma will be usually applied in the special situation when V is a Cartan subalgebra of a simple Lie algebra.

LEMMA 2.2. *Let $\mathfrak{A}$ be an algebra. For $a \in \mathfrak{A}$ write L_a for the left-multiplication operator given by a. Suppose $x = x_1 + \cdots + x_n$ is a sum of eigenvectors of L_a associated with mutually distinct eigenvalues. Then all x_i's lie in the subalgebra $\langle a, x\rangle$.*

Denote by $\Pi := \{\alpha_1, \ldots, \alpha_n\}$ the system of simple roots of a simple Lie algebra $\mathfrak{g}$ relative to a Cartan subalgebra $\mathfrak{h}$. The element $x \in \mathfrak{g}$ is called *Π-balanced* if x is the sum of all the simple-root vectors, that is, $x = \sum_{\alpha\in\Pi} x^\alpha$, where x^α is a root vector of α. Recall that $\Omega_\Pi \neq \emptyset$ by Lemma 2.1.

PROPOSITION 2.3. *A simple Lie algebra $\mathfrak{g}$ is generated by a Π-balanced element and an element in Ω_Π.*

Proof. This is a consequence of Lemma 2.2 and the basic facts (2.1)–(2.3). ∎

3. Loop algebras. We refer to [10, Section 0.1] and [3, Section 3.2] for the definition and natural grading of a loop algebra. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra and $\mathbb{F}[t, t^{-1}]$ the Laurent polynomial ring with parameter t. Set $L(\mathfrak{g}) := \mathfrak{g} \otimes \mathbb{F}[t, t^{-1}]$. Let ω be an automorphism of $\mathfrak{g}$ of finite order l, and η a primitive l-root of unity. Then one can define an automorphism $\widetilde{\omega}$ of $L(\mathfrak{g})$ by setting

$$\widetilde{\omega}(x \times t^n) := \eta^n \omega(x) \otimes t^n.$$

The *loop algebra* $L(\mathfrak{g}, \omega, \eta)$ is defined to be the set of fixed points of $\widetilde{\omega}$. Let

$$\mathfrak{g}_{\bar\epsilon} := \{x \in \mathfrak{g} \mid \omega(x) = \eta^\epsilon x\}$$

for $\bar\epsilon = \bar 0, \ldots, \overline{l-1} \in \mathbb{Z}_l$. Then $\mathfrak{g} = \bigoplus_{\bar\epsilon\in\mathbb{Z}_l} \mathfrak{g}_{\bar\epsilon}$ is a $\mathbb{Z}_l$-grading of $\mathfrak{g}$. Furthermore, $\mathfrak{g}_{\bar 0}$ is a simple Lie algebra and $\mathfrak{g}_{\bar\epsilon}$ is an irreducible $\mathfrak{g}_{\bar 0}$-module for all $0 \le \bar\epsilon \le l-1$. Also,

$$L(\mathfrak{g}, \omega, \eta) \cong \bigoplus_{\bar\epsilon\in\mathbb{Z}_l} (\mathfrak{g}_{\bar\epsilon} \otimes t^{l-\epsilon}\mathbb{F}[t^l, t^{-l}]) = \bigoplus_{n\in\mathbb{Z}} (\mathfrak{g}_{\overline{-n}} \otimes t^n).$$

The natural grading of $L(\mathfrak{g}, \omega, \eta)$ is defined by

$$L(\mathfrak{g}, \omega, \eta)_n = \mathfrak{g}_{\overline{-n}} \otimes t^n.$$

Suppose $\{x_i^{\bar\epsilon} \mid i = 1, \ldots, n_{\bar\epsilon}\}$ is a basis of $\mathfrak{g}_{\bar\epsilon}$. Then $\bigcup_{\bar\epsilon\in\mathbb{Z}_l} \{x_1^{\bar\epsilon}, \ldots, x_{n_{\bar\epsilon}}^{\bar\epsilon}\}$ is a basis of $\mathfrak{g}$. Multiplication in $L(\mathfrak{g}, \omega, \eta)$ is determined by

$$[x_i^{\bar\epsilon} \otimes t^{-\epsilon}, x_j^{\bar\rho} \otimes t^{-\rho}] = [x_i^{\bar\epsilon}, x_j^{\bar\rho}] \otimes t^{-\epsilon-\rho}.$$

LEMMA 3.1. *Let* $L(\mathfrak{g},\omega,\eta)=\bigoplus_{n\in\mathbb{Z}} L(\mathfrak{g},\omega,\eta)_n$ *be a Loop algebra. Then:*

(1) $L(\mathfrak{g},\omega,\eta)_{-1}$ *is irreducible as an* $L(\mathfrak{g},\omega,\eta)_0$*-module.*
(2) $L(\mathfrak{g},\omega,\eta)_1$ *is irreducible as an* $L(\mathfrak{g},\omega,\eta)_0$*-module.*
(3) $L(\mathfrak{g},\omega,\eta)$ *is generated by* $L(\mathfrak{g},\omega,\eta)_{-1}\oplus L(\mathfrak{g},\omega,\eta)_0\oplus L(\mathfrak{g},\omega,\eta)_1$.

Proof. (1) and (3) follow at once from Kac's recognition theorem [6, 10].

(2) Suppose $\{x_i^{\overline{-1}} \mid i=1,\dots,n_{\overline{-1}}\}$ is a basis of $\mathfrak{g}_{\overline{-1}}$. Then

$$\{x_i^{\overline{-1}}\otimes t \mid i=1,\dots,n_{\overline{-1}}\}$$

is a basis of $L(\mathfrak{g},\omega,\eta)_1$. For any nonzero $L(\mathfrak{g},\omega,\eta)_0$-submodule M of $L(\mathfrak{g},\omega,\eta)_1$, pick a nonzero element

$$x=\sum_{i=1}^{n_{\overline{-1}}} a_i x_i^{\overline{-1}}\otimes t\in M,\qquad a_i\in\mathbb{F}.$$

Since $U(L(\mathfrak{g},\omega,\eta)_0)=U(\mathfrak{g}_{\bar 0})\otimes\mathbb{F}$ and $\mathfrak{g}_{\overline{-1}}$ is an irreducible $\mathfrak{g}_{\bar 0}$-module, we have

$$U(L(\mathfrak{g},\omega,\eta)_0)x=(U(\mathfrak{g}_{\bar 0})\otimes\mathbb{F})\Big(\sum_{i=1}^{n_{\overline{-1}}} a_i x_i^{\overline{-1}}\otimes t\Big)=U(\mathfrak{g}_{\bar 0})x\otimes t=\mathfrak{g}_{\overline{-1}}\otimes t.$$

Consequently, $M=L(\mathfrak{g},\omega,\eta)_1$. ∎

Let $L(\mathfrak{g},\omega,\eta)=\bigoplus_{n\in\mathbb{Z}} L(\mathfrak{g},\omega,\eta)_n$ be a loop algebra and H be the standard Cartan subalgebra of $L(\mathfrak{g},\omega,\eta)_0$. For any n we have a weight decomposition

$$L(\mathfrak{g},\omega,\eta)_n=\bigoplus_{\alpha\in H^*} L(\mathfrak{g},\omega,\eta)_n^\alpha.$$

Write

$$\widetilde{\Delta}=\{(\alpha,n)\in H^*\times\mathbb{Z}\mid L(\mathfrak{g},\omega,\eta)_n^\alpha\neq 0\}$$

and let $\widetilde{Q}$ be the subgroup of $H^*\times\mathbb{Z}$ generated by $\widetilde{\Delta}$. For $\widetilde{\alpha}=(\alpha,n)\in\widetilde{Q}$, we set $L(\mathfrak{g},\omega,\eta)_{\widetilde{\alpha}}:=L(\mathfrak{g},\omega,\eta)_n^\alpha$. Moreover for $h\in H$, let $\widetilde{\alpha}(h):=\alpha(h)$. From (2.1) every root space for a simple Lie algebra $\mathfrak{g}$ is 1-dimensional. Recall that $\mathfrak{g}_{\bar 0}$ is a simple Lie algebra. Write Π for the simple root system for $\mathfrak{g}_{\bar 0}$. Then the following lemma holds:

LEMMA 3.2. *For a loop algebra* $L(\mathfrak{g},\omega,\eta)=\bigoplus_{\widetilde{\alpha}\in\widetilde{Q}} L(\mathfrak{g},\omega,\eta)_{\widetilde{\alpha}}$*, there exist* $\widetilde{\alpha'}=(\alpha',1)$, $\widetilde{\alpha''}=(\alpha'',-1)\in\widetilde{Q}$ *such that* $\alpha',\alpha''\notin\Pi$ *and* $\alpha'\neq\alpha''$.

THEOREM 3.3. *A loop algebra can be generated by two elements.*

Proof. Let $L(\mathfrak{g},\omega,\eta)=\bigoplus_{\widetilde{\alpha}\in\widetilde{Q}} L(\mathfrak{g},\omega,\eta)_{\widetilde{\alpha}}$ be a loop algebra. In the notation of Lemma 3.2, set

$$y:=x'\otimes t+x''\otimes t^{-1}+\sum_{\alpha\in\Pi} x_\alpha\otimes 1,$$

where

$$x'\otimes t\in L(\mathfrak{g},\omega,\eta)_{\widetilde{\alpha'}},\quad x''\otimes t^{-1}\in L(\mathfrak{g},\omega,\eta)_{\widetilde{\alpha''}},\quad x_\alpha\otimes 1\in L(\mathfrak{g},\omega,\eta)_{\widetilde{\alpha}},\ \forall\alpha\in\Pi,$$

and $\widetilde{\alpha'}=(\alpha',1)$, $\widetilde{\alpha''}=(\alpha'',-1)$ and $\widetilde{\alpha}=(\alpha,0)$, $\alpha\in\Pi$, where α', α'' and all α are pairwise distinct. By Lemma 2.1 we choose

$$h\otimes 1\in\Omega_{\{\widetilde{\alpha'},\widetilde{\alpha''}\}\cup\{\widetilde{\alpha}\mid\alpha\in\Pi\}}.$$

Then

$$\langle h\otimes 1,y\rangle=L(\mathfrak{g},\omega,\eta).$$

In fact, we have

$$\begin{aligned}(\operatorname{ad}h\otimes 1)^k\Big(x'\otimes t+x''\otimes t^{-1}+\sum_{\alpha\in\Pi}x_\alpha\otimes 1\Big)\\ =\alpha'^k(h)x'\otimes t+\alpha''^k(h)x''\otimes t^{-1}+\sum_{\alpha\in\Pi}\alpha^k(h)x_\alpha\otimes 1.\end{aligned}$$

According to Lemma 2.2, we have

$$x'\otimes t,x''\otimes t^{-1},x_\alpha\otimes 1\in\langle h\otimes 1,y\rangle,\qquad\forall\alpha\in\Pi.$$

Furthermore, by Proposition 2.3 we have $L(\mathfrak{g},\omega,\eta)_0\subset\langle h\otimes 1,y\rangle$. By Lemma 3.1(1),(2) the irreducibility of $L(\mathfrak{g},\omega,\eta)_{-1}$ and $L(\mathfrak{g},\omega,\eta)_1$ as $L(\mathfrak{g},\omega,\eta)_0$-modules ensures that $L(\mathfrak{g},\omega,\eta)_{-1}\subset\langle h\otimes 1,y\rangle$ and $L(\mathfrak{g},\omega,\eta)_1\subset\langle h\otimes 1,y\rangle$. By Lemma 3.1(3), we have $L(\mathfrak{g},\omega,\eta)=\langle h\otimes 1,y\rangle$. ■

4. Cartan algebras. All the Cartan algebras are listed below [10, 6]:

$$W_n,\ \ S_n,\ \ H_{2m},\ \ K_{2m+1}.$$

Let us briefly describe these algebras.

• W_n: Let $n\geq 2$ and W_n be the algebra of derivations of the polynomial algebra $\mathbb{F}[x_1,\dots,x_n]$. The standard basis of the algebra is

$$\{x_1^{a_1}\cdots x_n^{a_n}\partial/\partial x_i\mid a_1,\dots,a_n\in\mathbb{N},\ i=1,\dots,n\}.$$

Write $D_i:=\partial/\partial x_i$, $x^a:=x_1^{a_1}\dots x_n^{a_n}$ and $x^{(a)}:=(1/a!)x^a$, where $a!=a_1!\dots a_n!$. We have $D_i(x^{(\alpha)})=x^{(a-\varepsilon_i)}$ and define the *degree* of $x^{(a)}$ to be $|a|:=a_1+\dots+a_n$. Multiplication is given by

$$[x^{(a)}D_i,x^{(b)}D_j]=x^{(a)}D_i(x^{(b)})D_j-x^{(b)}D_j(x^{(a)})D_i.$$

The grading is determined by

$$(W_n)_i=\operatorname{span}_{\mathbb{F}}\{x^{(a)}D_i\mid |a|=i+1\}.$$

• S_n: Let $n \geq 2$ and

$$S_n = \Big\{\sum_{j=1}^{n} f_j D_j \in W_n \;\Big|\; \sum_{j=1}^{n} D_j(f_j) = 0\Big\}$$
$$= \mathrm{span}_{\mathbb{F}}\{D_{ij}(f) \mid f \in \mathbb{F}[x_1, \ldots, x_n],\, 1 \leq i < j \leq n\},$$

where $D_{ij}(f) := D_j(f)D_i - D_i(f)D_j$. The standard basis of S_n is

$$\{D_{ij}(x^{(a)}) \mid 1 \leq i < j \leq n\}.$$

The grading is determined by $(S_n)_i = \{D_{ij}(x^{(a)}) \mid |a| = i + 2\}$.

• H_{2m}: Let $m \geq 2$ and

$$H_{2m} = \Big\{\sum_{j=1}^{2m} f_j D_j \in W_n \;\Big|\; \sigma(j')D_i(f_{j'}) = \sigma(i')D_j(f_{i'}), 1 \leq i, j \leq 2m\Big\}$$
$$= \mathrm{span}_{\mathbb{F}}\{D_H(f) \mid f \in \mathbb{F}[x_1, \ldots, x_{2m}]\},$$

where $'$ is the involution of the index set $\{1, \ldots, 2m\}$ satisfying $j' = j+m$ and $\sigma(j) = 1$ for $j \leq m$, $\sigma(j) = -1$ for $j > m$ and $D_H(f) := \sum_{j=1}^{2m} \sigma(j)D_j(f)D_{j'}$. Multiplication is given by

$$[D_H(x^{(a)}), D_H(x^{(b)})] = D_H(D_H(x^{(a)})(x^{(b)})).$$

The grading is determined by $(H_{2m})_i = \{D_H(x^{(a)}) \mid |a| = i + 2\}$.

• K_{2m+1}: Let $m \geq 2$. As a Lie superalgebra,

$$(K_{2m+1}, [\,,]) \cong (\mathbb{F}[x_1, \ldots, x_{2m+1}], [\,,]'),$$

where multiplication is given by

$$[x^{(a)}, x^{(b)}]' = D_K(x^{(a)})(x^{(b)}) - 2D_{2m+1}(x^{(a)})(x^{(b)})$$

for $x^{(a)}, x^{(b)} \in \mathbb{F}[x_1, \ldots, x_{2m+1}]$. Here

$$D_K : \mathbb{F}[x_1, \ldots, x_{2m+1}] \to W_{2m+1}$$

is an even linear operator with

$$D_K(x^{(a)}) := \sum_{i=1}^{2m} \big(x_i D_{2m+1}(x^{(a)}) + \sigma(i')D_{i'}(x^{(a)})\big) D_i$$
$$+ \Big(2x^{(a)} - \sum_{i=1}^{2m} x_i D_i(x^{(a)})\Big) D_{2m+1},$$

where $'$ is the involution of $\{1, \ldots, 2m\}$ satisfying $j' = j + m$ and $\sigma(j) = 1$ for $j \leq m$, $\sigma(j) = -1$ for $j > m$. Grading $\mathbb{F}[x_1, \ldots, x_{2m+1}]$ by declaring $x_1, \ldots, x_{2m}$ to be homogeneous of degree 1 and x_{2m+1} to be homogeneous of degree 2 induces a grading on K_{2m+1}.

Throughout this section, L denotes one of the Cartan algebras W_n, S_n, H_{2m} or K_{2m+1}. Consider its decomposition into subspaces:

$$L = L_{-l} \oplus \cdots \oplus L_s. \tag{4.1}$$

For $L = W_n, S_n, H_{2m}$, $l = 1$ or K_{2m+1}, $l = 2$, the null space L_0 is isomorphic to

$$\mathfrak{gl}(n), \mathfrak{sl}(n), \mathfrak{so}(2m), \text{ or } \mathfrak{so}(2m) \oplus \mathbb{F}I,$$

respectively.

By Kac's recognition theorem, the following lemma holds.

LEMMA 4.1.

(1) *L_{-1} is irreducible as an L_0-module.*
(2) *L_1 is completely reducible as an L_0-module.*
(3) *L can be generated by $L_{-1} \oplus L_0 \oplus L_1$.*

We list the standard bases of the standard Cartan subalgebras $\mathfrak{h}_{L_0}$:

Table 4.1

L	The standard basis of $\mathfrak{h}_{L_0}$
W_n	x_iD_i, $1 \le i \le n$
S_n	$x_1D_1 - x_jD_j$, $2 \le j \le n$
H_{2m}	$x_iD_i - x_{i'}D_{i'}$, $1 \le i \le m$
K_{2m+1}	x_ix_{i+m}, x_{2m+1}, $1 \le i \le m$

The weight space decomposition of the component L_k relative to $\mathfrak{h}_{L_0}$ is

$$L_k = \delta_{k,0}\mathfrak{h}_{L_0} \oplus \bigoplus_{\alpha \in \Delta_k} L_k^{\alpha}, \quad \text{where} \quad -l \le k \le s.$$

Write Π for the set of simple roots of L_0 relative to the Cartan subalgebra $\mathfrak{h}_{L_0}$.

For $L = S$ or H, L_1 is an irreducible L_0-module. For $L = W$ or K, L_1 is a direct sum of two irreducible L_0-submodules,

$$L_1 = L_1^1 \oplus L_1^2.$$

Let Δ_1^i be the weight set of L_1^i, $i = 1, 2$. We have:

LEMMA 4.2.

(1) *If $L = W_n$ then $\Pi \cap \Delta_{-1} = \Pi \cap \Delta_1 = \Delta_{-1} \cap \Delta_1 = \emptyset$ and there exist nonzero weights $\alpha_1^i \in \Delta_1^i$ such that $\alpha_1^1 \neq \alpha_1^2$.*
(2) *If $L = S_n$ then $\Pi \cap \Delta_{-1} = \Pi \cap \Delta_1 = \Delta_{-1} \cap \Delta_1 = \emptyset$.*
(3) *If $L = H_{2m}$ then $\Pi \cap \Delta_{-1} = \Pi \cap \Delta_1 = \emptyset$ and $\Delta_{-1} \neq \Delta_1$.*
(4) *If $L = K_{2m+1}$ then $0 \in \Delta_{-1}$, $\Pi \neq \Delta_1$ and $\Delta_{-1} \neq \Delta_1$.*
(5) *If $L = W_n$ or K_{2m+1} then there exist nonzero weights $\alpha_1^i \in \Delta_1^i$ such that $\alpha_1^1 \neq \alpha_1^2$.*

Proof. For W_n, define a linear function ε_i^w on $\mathfrak{h}_{W_0}$ by

$$\varepsilon_i^w(x_j D_j) = \delta_{ij}, \quad 1 \le i, j \le n.$$

We have

$$\begin{aligned} \Delta_{-1} &= \{-\varepsilon_j^w \mid 1 \le j \le n\}, \\ \Delta_0 &= \{\varepsilon_i^w - \varepsilon_j^w \mid 1 \le i \ne j \le n\}, \\ \Pi &= \{\varepsilon_i^w - \varepsilon_{i+1}^w \mid 1 \le i \le n-1\}, \\ \Delta_1 &= \{\varepsilon_k^w + \varepsilon_l^w - \varepsilon_j^w \mid 1 \le k \ne l, j \le n\}. \end{aligned} \tag{4.2}$$

For S_n, define a linear function ε_i^s on $\mathfrak{h}_{S_0}$ by

$$\varepsilon_i^s(x_1 D_1 - x_j D_j) = \delta_{ij}, \quad 2 \le i, j \le n,$$

and write $\varepsilon_1^s := -\sum_{l=2}^n \varepsilon_l^s$. We have

$$\begin{aligned} \Delta_{-1} &= \{\varepsilon_j^s \mid 1 \le j \le n\}, \\ \Delta_0 &= \{\varepsilon_i^s - \varepsilon_j^s \mid 1 \le i \ne j \le n\}, \\ \Pi &= \{\varepsilon_i^s - \varepsilon_{i+1}^s \mid 1 \le i \le n-1\}, \\ \Delta_1 &= \{\varepsilon_j^s - \varepsilon_k^s - \varepsilon_l^s \mid 1 \le k \ne l, j \le n\}. \end{aligned}$$

For H_{2m}, define a linear function ε_i^h on $\mathfrak{h}_{H_0}$ by

$$\varepsilon_i^h(x_j D_j - x_{j'} D_{j'}) = \delta_{ij}, \quad 1 \le i, j \le m.$$

We have

$$\begin{aligned} \Delta_{-1} &= \{\ \pm\varepsilon_j^h \mid 1 \le j \le m\}, \\ \Delta_0 &= \{\pm(\varepsilon_i^h + \varepsilon_j^h), \pm(\varepsilon_i^h - \varepsilon_j^h) \mid 1 \le i, j \le m\}, \\ \Pi &= \{\varepsilon_i^h - \varepsilon_{i+1}^h, \varepsilon_{m-1}^h + \varepsilon_m^h \mid 1 \le i < m\}, \\ \Delta_1 &= \{\pm(\varepsilon_i^h + \varepsilon_j^h) \pm \varepsilon_k^h, \pm(\varepsilon_i^h - \varepsilon_j^h) \pm \varepsilon_k^h \mid 1 \le i, j, k \le m\} \\ &\quad \cup \{\pm\varepsilon_l^h \mid 1 \le l \le m\}. \end{aligned}$$

For K_{2m+1}, define a linear function ε_i^k on $\mathfrak{h}_{K_0}$ by

$$\varepsilon_i^k(x_j x_{j+m}) = \delta_{ij}, \quad 1 \le j \le m, \quad \varepsilon_{2m+1}^k(x_{2m+1}) = 1.$$

We have

$$\begin{aligned} \Delta_{-1} &= \{0\} \cup \{\pm\varepsilon_i^k \mid 1 \le i \le m\}, \\ \Delta_0 &= \{\pm\varepsilon_{2m+1}^k, \pm(\varepsilon_i^k + \varepsilon_j^k), \pm(\varepsilon_i^k - \varepsilon_j^k) \mid 1 \le l \le m,\ 1 \le i, j \le m\}, \\ \Pi &= \{\varepsilon_i^k - \varepsilon_{i+1}^k, \varepsilon_{2m+1}^k \mid 1 \le i < m\}, \\ \Delta_1 &= \{0\} \cup \{\pm\varepsilon_l^k, \pm(\varepsilon_i^k + \varepsilon_j^k), \pm(\varepsilon_i^k - \varepsilon_j^k) \mid 1 \le l \le m,\ 1 \le i, j \le m\} \\ &\quad \cup \{\pm(\varepsilon_i^k + \varepsilon_j^k) \pm \varepsilon_l^k, \pm(\varepsilon_i^k - \varepsilon_j^k) \pm \varepsilon_l^k) \mid 1 \le i, j, l \le m\}. \end{aligned} \tag{4.3}$$

All the statements follow directly except (5) for $L = W_n$ or K_{2m+1}. In these cases, from (4.2) and (4.3) one sees that $|\Delta_1| > 1$, respectively. Consequently, (5) holds. ∎

Recall that an element of $\mathfrak{g}$ is Π-balanced if it is a sum of all the simple-root vectors.

THEOREM 4.3. *A Cartan algebra L can be generated by two elements.*

Proof. Recall the null space L_0 is isomorphic to $\mathfrak{gl}(n), \mathfrak{sl}(n), \mathfrak{so}(2m)$ or $\mathfrak{so}(2m) \oplus \mathbb{F}I$. From Proposition 2.3, for a Π-balanced element $x_0 \in L_0$ and $h_0 \in \Omega_\Pi \subset \mathfrak{h}_{L_0}$ we have $L_0 = \langle x_0 + \delta_{L,W_n} z_1 + \delta_{L,K_{2m+1}} z_2, h_0\rangle$, where z_1 and z_2 are central elements in $\mathfrak{gl}(n)$ and $\mathfrak{so}(2m) \oplus \mathbb{F}I$, respectively.

Suppose $L = S_n$ or H_{2m}. According to Lemma 4.2(2),(3), we may choose nonzero weights $\alpha_{-1} \in \Delta_{-1}$ and $\alpha_1 \in \Delta_1$ such that $\alpha_{-1} \neq \alpha_1$ and $\alpha_{-1}, \alpha_1 \notin \Pi$. Set $x := x_{-1} + x_0 + x_1$ for some weight vectors $x_{-1} \in L_{-1}^{\alpha_{-1}}$ and $x_1 \in L_1^{\alpha_1}$. Now write $\Phi := \Pi \cup \{\alpha_{-1}\} \cup \{\alpha_1\} \subset \mathfrak{h}_{L_0}^*$ and choose $h_0 \in \Omega_\Phi$. We assert $\langle x, h_0\rangle = L$. Lemma 2.2 implies that all components x_{-1}, x_0, and x_1 belong to $\langle x, h_0\rangle$. As $h_0 \in \Omega_\Phi \subset \Omega_\Pi$, we obtain $L_0 = \langle x_0, h_0\rangle \subset \langle x, h_0\rangle$. By Lemma 4.1(1),(2), since L_{-1} and L_1 are irreducible L_0-modules, we have $L_{-1}, L_1 \subset \langle x, h_0\rangle$. From Lemma 4.1(3), we thus get $L = \langle x, h_0\rangle$.

If $L = W_n$, by Lemma 4.2(1),(4),(5), choose $\alpha_{-1} \in \Delta_{-1}$, $\alpha_1^1 \in \Delta_1^1$ and $\alpha_1^2 \in \Delta_1^2$ such that $\alpha_{-1}, \alpha_1^1, \alpha_1^2$ are pairwise distinct and $\alpha_{-1}, \alpha_1^1, \alpha_1^2 \notin \Pi$. Set $x := x_{-1} + x_0 + x_1^1 + x_1^2$ for some weight vectors $x_{-1} \in L_{-1}^{\alpha_{-1}}$ and $x_1^i \in L_1^{\alpha_1^i}$, $i = 1, 2$. Write $\Phi := \Pi \cup \{\alpha_{-1}\} \cup \{\alpha_1^1\} \cup \{\alpha_1^2\}$. For $h_0 \in \Omega_\Phi \subset \Omega_\Pi$, as above, one may show that $L = \langle x, h_0\rangle$.

If $L = K_{2m+1}$, by Lemma 4.2(4), choose $\alpha_{-1} \in \Delta_{-1}$ and $\alpha_1 \in \Delta_1$ such that $\alpha_{-1} = 0$ and $\alpha_1 \notin \Pi$. Set $x := x_{-1} + x_0 + x_1$ for some weight vectors $x_{-1} \in L_{-1}^{\alpha_{-1}}$ and $x_1 \in L_1^{\alpha_1}$. Now write $\Phi := \Pi \cup \{\alpha_{-1}\} \cup \{\alpha_1\} \subset \mathfrak{h}_{L_0}^*$. Let $h_0 \in \Omega_\Phi \subset \Omega_\Pi$. We claim that $L = \langle x, h_0\rangle$. By Lemma 2.2, $x_0, x_{-1}, x_1 \in \langle x, h_0\rangle$. Then $L_0 \subset L$, and hence the irreducibility of L_{-1} and L_1 ensures $L_{-1}, L_1 \subset \langle x, h_0\rangle$. By Lemma 4.1(3), $L = \langle x, h_0\rangle$. ∎

5. Virasoro algebra. Let $L := \operatorname{Der} \mathbb{F}[t, t^{-1}]$. The standard basis of the algebra is $\{t^n d/dt \mid n \in \mathbb{Z}\}$. Multiplication is given by

$$[t^n d/dt, t^m d/dt] = (m - n)t^{m+n-1} d/dt.$$

The grading is defined by $L_i = \operatorname{span}_{\mathbb{F}}\{t^{i+1} d/dt\}$.

LEMMA 5.1. *L is generated by $L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_2$.*

Proof. Since $L_1 = [L_{-1}, L_2]$, $L_{i+1} = [L_1, L_i]$ and $L_{-(i+1)} = [L_{-1}, L_{-i}]$, $1 \neq i \in \mathbb{Z}^+$, by induction, the conclusion follows. ∎

THEOREM 5.2. *L is generated by td/dt and $d/dt + t^{-1}d/dt + t^3 d/dt$.*

Proof. We have

$$(\operatorname{ad} td/dt)^k (d/dt + t^{-1}d/dt + t^3 d/dt) = (-1)^k d/dt + (-2)^k t^{-1} d/dt + 2^k t^3 d/dt, \qquad k \in \mathbb{Z}^+.$$

Then by Lemma 2.2 we have

$$d/dt, t^{-1}d/dt, t^3d/dt \in \langle td/dt, d/dt + t^{-1}d/dt + t^3d/dt\rangle.$$

So $L_{-1}, L_{-2}, L_2, L_0 \subset \langle td/dt, d/dt + t^{-1}d/dt + t^3d/dt\rangle$. According to Lemma 5.1 the elements td/dt and $d/dt + t^{-1}d/dt + t^3d/dt$ generate L. ∎

Proposition 2.3 and Theorems 3.3, 4.3, 5.2 combine to give the main Theorem 1.1 of this paper.

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