APPLICATIONES MATHEMATICAE
44,1 (2017), pp. 123–133

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EXPANDING THE APPLICABILITY OF INEXACT NEWTON METHODS USING RESTRICTED CONVERGENCE DOMAINS

Abstract. We provide a new semilocal convergence analysis for the inexact Newton method in order to approximate a solution of a nonlinear equation in a Banach space setting. Using a new idea of restricted convergence domains we present a convergence analysis with the following advantages over earlier studies: larger convergence domain, tighter error bounds on the distances involved and an at least as precise information on the location of the solution. This way we expand the applicability of the inexact Newton method. Special cases and numerical examples are also provided.

1. Introduction. Let X, Y be Banach spaces and D be a convex subset of X. In this study we are concerned with the problem of approximating a solution x^* of the equation

$$F(x) = 0, \tag{1.1}$$

where $F : D \subseteq X \to Y$ has a continuous Fréchet derivative $F' : D \subseteq X \to L(X, Y)$. Here $L(X, Y)$ denotes the space of bounded linear operators from X into Y.

Many problems can be formulated like equation (1.1) using mathematical modeling [2, 5, 13]. Solutions of these equations can be found in explicit form only in special cases. That is why most solution methods for these equations are usually iterative.

2010 *Mathematics Subject Classification*: 65G99, 65H10, 47H17, 49M15.
Key words and phrases: Newton's method, inexact Newton method, semilocal convergence, weak Lipschitz condition, Fréchet derivative.
Received 22 January 2016.
Published online 7 April 2017.

DOI: 10.4064/am2292-3-2016

The most popular methods for solving equation (1.1) is undoubtedly Newton's method defined for each $n = 0, 1, 2, \dots$ by

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n),$$

where $x_0 \in D$ is an initial point. There are many results on the local as well as semilocal convergence analysis of Newton's method under various Lipschitz-type conditions. We refer the reader to [2] and references therein for a survey of such results. However, the implementation of Newton's method has two drawbacks. Firstly, one must evaluate F' at each step and secondly compute the exact solution of the Newton equations

$$F'(x_n)(x_{n+1} - x_n) = -F(x_n) \quad \text{for each } n = 0, 1, 2, \dots.$$

If the number of unknowns is large, e.g. in Euclidean spaces, the application of the Gaussian elimination may be very expensive. That is why inexact Newton methods defined for each $n = 0, 1, 2, \dots$ by

$$x_{n+1} = x_n + s_n, \qquad F'(x_n)s_n = -F(x_n) + r_n \tag{1.2}$$

have been used instead of Newton's method, where $\{r_n\}$ is a sequence in Y chosen in such a way that the sequence $\{x_n\}$ converges to x^*. Several conditions on the residual controls $\{r_n\}$ have been suggested [2, 3, 5, 12, 13].

In the present study, we are motivated by the work [12] and optimization considerations. In particular, we assume that

$$\|F'(x_0)^{-1}r_n\| \le \eta_n \|F'(x_0)^{-1}F(x_n)\|^2 \quad \text{for each } n = 0, 1, 2, \dots \tag{1.3}$$

where $\{\eta_n\}$ is a nonnegative sequence. Let $x_0 \in D$ and $R_0 > 0$. Define

$$R := \sup\{t \in [0, R_0] : \bar{U}(x_0, t) \subseteq D\}.$$

Suppose that F has a continuous Fréchet derivative in a closed ball $\bar{U}(x_0, r)$, $0 < r \le R_0$, $F'(x_0)^{-1}F'$ exists and $F'(x_0)^{-1}F'$ satisfies the weak Lipschitz condition:

$$\|F'(x_0)^{-1}(F'(x) - F'(y))\| \le \int_{s(x)}^{s(\overline{xy})} L(u)\,du \tag{1.4}$$

for each $x \in U(x_0, r)$ and

$$y \in \bar{U}(x, r - s(x)), \qquad s(x) = \|x - x_0\|, \qquad s(\overline{xy}) = s(x) + \|y - x\| \le r$$

and L is a positive, integrable nondecreasing function on $[0, r]$. Using (1.3) and (1.4) a semilocal convergence result was given in [12] for the inexact Newton method (1.2).

Next, we show how to improve on the convergence domain, the error bounds on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$ and the information on the location of the solution x^*. In view of (1.4) there exists a positive, integrable nondecreasing function L_0 on $[0,r]$ such that

$$\|F'(x_0)^{-1}(F'(x)-F'(x_0))\| \le \int_0^{s(x)} L_0(u)\,du \tag{1.5}$$

for each $x \in \bar{U}(x_0,r)$. Notice that

$$L_0(u) \le L(u) \quad \text{for each } u \in [0,r] \tag{1.6}$$

in general and L/L_0 can be arbitrarily large [1, 2, 4]. Define

$$r_0 := \sup\Big\{t \in [0,r] : \int_0^{s(x)} L_0(u)\,du < 1\Big\}.$$

Then there exists a positive, integrable nondecreasing function L_1 on $[0,r_0]$ such that

$$\|F'(x_0)^{-1}(F'(x)-F'(y))\| \le \int_{s(x)}^{s(\overline{xy})} L_1(u)\,du \tag{1.7}$$

for each $x \in U(x_0,r_0)$ and $y \in \bar{U}(x, r_0 - s(x))$. Notice that by (1.4) and (1.7),

$$L_1(u) \le L(u) \quad \text{for each } u \in [0,r_0]. \tag{1.8}$$

It turns out that the function L_1 can replace L. Moreover, L_0 can also replace L for the computation of the upper bounds on the norms of $\|F'(x)^{-1}F'(x_0)\|$ in the semilocal convergence analysis of the inexact Newton method (1.2). L_0 and L_1 are at least as small as L and the iterates x_n lie in a more precise location $U(x_0,r_0)$ than $U(x_0,r)$ since $U(x_0,r_0) \subset U(x_0,r)$. The resulting semilocal convergence analysis has the following advantages denoted by $(\mathcal{A})$ over [12]:

(a) Larger convergence domain.
(b) Tighter error bounds on the distances $\|x_{n+1}-x_n\|$, $\|x_n - x^*\|$.
(c) At least as precise information on the location of the solution x^*.

Advantages $(\mathcal{A})$ are obtained under the same computational cost, since in practice the computation of L requires the computation of L_0 or L_1 as special cases.

The rest of the paper is organized as follows: Section 2 contains the semi-local convergence analysis of an inexact Newton-like method. Special cases and applications appear in the concluding Section 3.

2. Semilocal convergence. As in [12], we first introduce some functions and parameters. Let $\beta > 0$, $0 \le \lambda < 1$, $\omega \ge 1$ and $\sigma \ge 0$. Define functions φ and ψ on the interval $[0, R]$ by

$$\varphi(t) := \beta - (1-\lambda)t + \sigma t^2 + \omega \int_0^t L(u)(t-u)\,du,$$

$$\psi(t) := \beta - t + \omega \int_0^t L(u)(t-u)\,du.$$

Define parameters r_λ, b_λ by

$$r_\lambda := \sup \Big\{ t \in (0, R) : \omega \int_0^t L(u)\,du + 2\sigma t \le 1 - \lambda \Big\},$$

$$b_\lambda := (1-\lambda) r_\lambda - \sigma r_\lambda^2 - \omega \int_0^{r_\lambda} L(u)(r_\lambda - u)\,du.$$

Set

$$\delta := \omega \int_0^R L(u)\,du + 2\sigma R.$$

Clearly,

$$r_\lambda = \begin{cases} R & \text{if } \delta < 1-\lambda, \\ r'_\lambda & \text{if } \delta \ge 1-\lambda, \end{cases}$$

$$b_\lambda \ge \omega \int_0^{r_\lambda} L(u)u\,du + \sigma (r'_\lambda)^2 \quad \text{if } \delta < 1-\lambda,$$

$$b_\lambda = \omega \int_0^{r_\lambda} L(u)u\,du + \sigma (r'_\lambda)^2 \quad \text{if } \delta \ge 1-\lambda,$$

where $r'_\lambda \in [0, R]$ satisfies

$$\omega \int_0^{r'_\lambda} L(u)\,du + 2\sigma r'_\lambda = 1 - \lambda.$$

Under condition (1.3) further suppose that

$$\eta := \sup_{\eta \ge 0} \eta_n < 1. \tag{2.1}$$

Then by (1.3) and (2.1), we have

$$\|F'(x_0)^{-1} r_n\| \le \eta_n \|F'(x_0)^{-1} F(x_n)\|^2 \le \eta \|F'(x_0)^{-1} F(x_n)\|^2. \tag{2.2}$$

Moreover, set

$$\alpha := \|F'(x_0)^{-1} F(x_0)\|, \quad \beta := (1+\sqrt{\eta})\alpha,$$

$$\omega := 1 + \sqrt{\eta}, \quad \sigma := \frac{\eta(1+\sqrt{\eta})(1+\int_0^R L(u)\,du)}{(1-\sqrt{u})^2}.$$

Using the preceding notation the following semilocal convergence result for the inexact Newton method (1.2) was shown in [12, p. 9].

THEOREM 2.1. *Suppose*

$$\beta \le \min\{1/\sqrt{\eta}, b_\lambda\}, \qquad \bar{U}(x_0, t^*) \subseteq U(x_0, R),$$

and $F'(x_0)^{-1}F'$ *satisfies the weak Lipschitz condition* (1.4) *on* $U(x_0, t^*)$, *where*

$$t^* = \lim_{n\to\infty} t_n,$$

and $\{t_n\}$ *is defined by*

$$t_0 = 0, \qquad t_{n+1} = t_n - \frac{\varphi(t_n)}{\psi'(t_n)} \quad \text{for each } n = 0, 1, 2, \ldots.$$

Then the sequence $\{x_n\}$ *generated by the inexact Newton method* (1.2) *is well defined in* $\bar{U}(x_0, t^*)$, *remains in* $\bar{U}(x_0, t^*)$ *for each* $n = 0, 1, 2, \ldots$ *and converges to a solution* $x^* \in \bar{U}(x_0, t^*)$ *of the equation* $F(x) = 0$. *Moreover,*

$$\|x_{n+1} - x_n\| \le t_{n+1} - t_n, \qquad \|x_n - x^*\| \le t^* - t_n.$$

Next, we show how to improve Theorem 2.1. Define functions φ_0 and ψ_0 on the interval $[0, R]$ by

$$\varphi_0(t) := \beta - (1-\lambda)t + \sigma_0 t^2 + \omega \int_0^t L(u)(t-u)\,du,$$

$$\psi_0(t) := \beta - t + \omega \int_0^t L_0(u)(t-u)\,du,$$

where

$$\sigma_0 := \frac{\eta(1+\sqrt{\eta})(1+\int_0^R L_0(u)\,du)}{(1-\sqrt{u})^2}.$$

Let

$$r_{\lambda,0} := \sup\Big\{t \in (0, R) : \omega \int_0^t L(u)u\,du + 2\sigma_0 t \le 1-\lambda\Big\},$$

$$b_{\lambda,0} := (1-\lambda)r_{\lambda,0} - \sigma_0 r_{\lambda,0}^2 - \omega \int_0^{r_{\lambda,0}} L(u)(r_{\lambda,0}-u)\,du,$$

$$\delta_0 := \omega \int_0^R L(u)\,du + 2\sigma_0 R.$$

Then

$$r_{\lambda,0} = \begin{cases} R & \text{if } \delta_0 < 1-\lambda, \\ r'_{\lambda,0} & \text{if } \delta_0 \ge 1-\lambda, \end{cases}$$

where $r'_{\lambda,0} \in [0, R]$ satisfies

$$\omega \int_0^{r'_{\lambda,0}} L(u)\,du + 2\sigma_0 r'_{\lambda,0} = 1 - \lambda.$$

We also have

$$b_{\lambda,0} \geq \omega \int_0^{r_{\lambda,0}} L(u)u\,du + \sigma_0 (r'_{\lambda,0})^2 \quad \text{if } \delta_0 < 1 - \lambda,$$

$$b_{\lambda,0} = \omega \int_0^{r_{\lambda,0}} L(u)u\,du + \sigma_0 (r'_{\lambda,0})^2 \quad \text{if } \delta_0 \geq 1 - \lambda.$$

Notice that in view of the preceding definitions and (1.6), we have

$$\varphi_0(t) \leq \varphi(t), \quad \psi_0(t) \leq \psi(t), \quad \varphi_0'(t) \leq \varphi'(t), \quad \varphi_0''(t) \leq \varphi''(t),$$

and

$$\psi_0'(t) \leq \psi'(t), \quad \psi_0''(t) \leq \psi''(t), \tag{2.3}$$

$$\sigma_0 \leq \sigma, \quad r_\lambda \leq r_{\lambda,0}, \quad \delta_0 \leq \delta, \quad r'_\lambda \leq r'_{\lambda,0}, \quad b_\lambda \leq b_{\lambda,0}. \tag{2.4}$$

Define a majorizing sequence $\{\bar{t}_n\}$ by

$$\bar{t}_0 = 0, \quad \bar{t}_{n+1} = \bar{t}_n - \frac{\varphi_0(\bar{t}_n)}{\psi_0'(\bar{t}_n)} \quad \text{for each } n = 0, 1, 2, \ldots.$$

Then, using the preceding notation, we can show the following improved semilocal convergence result for the inexact Newton method (1.2).

Theorem 2.2. *Suppose*

$$\beta \leq \min\{1/\sqrt{\eta}, b_{\lambda,0}\}, \quad \bar{U}(x_0, \bar{t}^*) \subseteq U(x_0, R)$$

and $F'(x_0)^{-1}F'$ *satisfies the weak Lipschitz condition* (1.4) *on* $U(x_0, \bar{t}^*)$, *where*

$$\bar{t}^* = \lim_{n\to\infty} \bar{t}_n. \tag{2.5}$$

Then the sequence $\{x_n\}$ *generated by the inexact Newton method* (1.2) *is well defined in* $\bar{U}(x_0, \bar{t}^*)$, *remains in* $\bar{U}(x_0, \bar{t}^*)$ *for each* $n = 0, 1, 2, \ldots$ *and converges to a solution* $x^* \in \bar{U}(x_0, \bar{t}^*)$ *of* $F(x) = 0$. *Moreover,*

$$\|x_{n+1} - x_n\| \leq \bar{t}_{n+1} - \bar{t}_n, \tag{2.6}$$

$$\|x_n - x^*\| \leq \bar{t}^* - \bar{t}_n. \tag{2.7}$$

Proof. Condition (1.4) implies condition (1.5). By applying the Banach Lemma on invertible operators as in [2], we get

$$\|F'(x_k)^{-1}F'(x_0)\| \leq \frac{1}{1 - \int_0^{s(x_k)} L_0(u)\,du} \tag{2.8}$$

instead of the less precise estimate

$$\|F'(x_k)^{-1}F'(x_0)\| \le \frac{1}{1-\int_0^{s(x_k)} L(u)\,du}$$

obtained in [12]. Replacing σ, φ, ψ in the proof in [12] by $\sigma_0, \varphi_0, \psi_0$ respectively we obtain

$$\begin{aligned}
&(1+\sqrt{\eta})\|F'(x_0)^{-1}F(x_k)\| \qquad (2.9)\\
&\le (1+\sqrt{\eta}) \int_0^{\|x_k-x_{k-1}\|} (\|x_k-x_{k-1}\|-u)L(\|x_k-x_0\|+u)\,du\\
&\quad + \frac{\eta(1+\sqrt{\eta})(1+\int_0^R L_0(u)\,du)}{(1-\sqrt{\eta})^2}\|x_k-x_{k-1}\|^2\\
&= \omega \int_0^{\|x_k-x_{k-1}\|} (\|x_k-x_{k-1}\|-u)L(\|x_{k-1}-x_0\|+u)\,du + \sigma_0\|x_k-x_{k-1}\|^2\\
&\le \left(\frac{\omega}{(\bar{t}_k-\bar{t}_{k-1})^2}\int_0^{\bar{t}_k-\bar{t}_{k-1}} (\bar{t}_k-\bar{t}_{k-1}-u)L(\bar{t}_{k-1}+u)\,du + \sigma_0\right)(\bar{t}_k-\bar{t}_{k-1})^2\\
&= \omega \int_0^{\bar{t}_k-\bar{t}_{k-1}} (\bar{t}_k-\bar{t}_{k-1}-u)L(\bar{t}_{k-1}+u)\,du + \sigma_0[\bar{t}_k^2-\bar{t}_{k-1}^2-2\bar{t}_{k-1}(\bar{t}_k-\bar{t}_{k-1})]\\
&= \varphi_0(\bar{t}_k)-\varphi_0(\bar{t}_{k-1})-\varphi_0'(\bar{t}_{k-1})(\bar{t}_k-\bar{t}_{k-1})\\
&\le \varphi_0(\bar{t}_k)-(\lambda+2\sigma_0\bar{t}_{k-1})(\bar{t}_k-\bar{t}_{k-1}) \le \varphi_0(t_k) \le \varphi(\bar{t}_0)=\beta,
\end{aligned}$$

leading together with (2.8) to $\sqrt{\eta}\|F'(x_0)^{-1}F(x_{k+1})\| \le 1$ and

$$\|x_k-x_{k-1}\| \le \bar{t}_k-\bar{t}_{k-1}, \qquad (2.10)$$

where we have also used the estimates

$$\begin{aligned}
\|F'(x_0)^{-1}F'(x_{k-1})\| &= \|I+F'(x_0)^{-1}(F'(x_{k-1})-F'(x_0))\|\\
&\le 1+\int_0^{s(x_{k-1})} L_0(u)\,du \le 1+\int_0^R L_0(u)\,du
\end{aligned}$$

and

$$\begin{aligned}
\|F'(x_0)^{-1}F(x_{k-1})\| &\le \frac{\|F'(x_0)^{-1}F'(x_{k-1})\|\,\|x_k-x_{k-1}\|}{1-\sqrt{\eta}}\\
&\le \frac{1+\int_0^R L_0(u)\,du}{1-\sqrt{\eta}}\|x_k-x_{k-1}\|.
\end{aligned}$$

It follows from (2.10) that $\{x_k\}$ is a Cauchy sequence (since $\{\bar{t}_k\}$ is convergent) in a Banach space X and as such it converges to some $x^* \in \bar{U}(x_0, \bar{t}^*)$

(since $\bar{U}(x_0, \bar{t}^*)$ is a closed set). By letting $k \to \infty$ in (2.9) we get $F(x^*) = 0$. Estimate (2.6) follows from (2.5) by using standard majorization techniques [2, 6, 7, 8, 9, 10]. ∎

REMARK 2.3. If $L = L_0$, then Theorem 2.2 reduces to Theorem 2.1. However, if $L_0 < L$, the advantages as stated in the abstract are obtained.

Next, we present another refinement of Theorem 2.1 using conditions (1.5) and (1.7). Suppose that

$$L_0(u) \leq L_1(u) \quad \text{for each } u \in [0, R_0]. \tag{2.11}$$

Define a function φ_1 and parameters $r_{\lambda,1}, b_{\lambda,1}, \delta_1, r'_{\lambda,1}$ just as $\varphi_0, r_{\lambda,0}, b_{\lambda,0}, \delta_0$, $r'_{\lambda,0}$, respectively but with L_1 replacing L. Moreover, define majorizing sequences $\{t_n^2\}, \{t_n^3\}$ by

$$t_0^2 = 0, \quad t_{n+1}^2 = t_n^2 - \frac{\varphi_1(t_n^2)}{\varphi_0'(t_n^2)},$$
$$t_0^3 = 0, \quad t_{n+1}^3 = t_n^3 - \frac{\varphi_1(t_n^3)}{\varphi_1'(t_n^3)}.$$

Then we arrive at:

THEOREM 2.4. *Suppose*

$$\beta \leq \min\{1/\sqrt{\eta}, b_{\lambda,1}\}, \quad \bar{U}(x_0, t^{2*}) \subseteq U(x_0, R),$$

(2.11) *holds and* $F'(x_0)^{-1}F'$ *satisfies the weak Lipschitz condition* (1.5) *and* (1.7), *where* $\bar{t}^{2*} = \lim_{n\to\infty} t_n^2$. *Then the sequence* $\{x_n\}$ *generated by the inexact Newton method* (1.2) *is well defined in* $\bar{U}(x_0, t^{2*})$, *remains in* $\bar{U}(x_0, t^{2*})$ *for each* $n = 0, 1, 2, \dots$ *and converges to a solution* $x^* \in \bar{U}(x_0, t^{2*})$ *of* $F(x) = 0$. *Moreover,*

$$\|x_{n+1} - x_n\| \leq t_{n+1}^2 - t_n^2 \leq t_{n+1}^3 - t_n^3,$$
$$\|x_n - x^*\| \leq t^{2*} - t_n^2, \quad \|x_n - x^*\| \leq t^{3*} - t_n^{3*},$$

where

$$t^{3*} = t^{2*}.$$

Proof. Simply notice that the iterates $\{x_n\}$ lie in $U(x_0, r_0)$ which is a more precise location than $U(x_0, r)$, and then follow the proof of Theorem 2.2. ∎

REMARK 2.5. Theorem 2.4 improves Theorems 2.1 and 2.2 if $L_1 < L$. Estimate (1.3) is used in the proof of Theorem 2.1 and holds for $x_n \in U(x_0, r)$. However, in the rest of the results, we can replace (1.3) by (2.1) and (2.2), that is, use

$$\|F'(x_0)^{-1}r_n\| \leq \bar{\eta}_n \|F'(x_0)^{-1}F(x_n)\|^2 \leq \bar{\eta}\|F'(x_0)^{-1}F(x_n)\|^2$$

for each $x_n \in U(x_0, r_0)$, where

$$\bar{\eta} = \sup_{n \geq 0} \bar{\eta}_n < 1.$$

Then we have

$$\bar{\eta}_n \leq \eta_n \quad \text{and} \quad \bar{\eta} \leq \eta$$

since $U(x_0, r_0) \subset U(x_0, r)$. Concerning the uniqueness of the solution x^* not studied in [12] we have:

PROPOSITION 2.6. *Suppose that the hypotheses of Theorem 2.2 hold. Then for $T \in [\bar{t}^*, r_0]$ the point x^* is the only solution of the equation $F(x) = 0$ in $D_0 = U(x_0, r_0) \cap U(x_0, R)$.*

Proof. The existence of the solution x^* inside $\bar{U}(x_0, \bar{t}^*)$ has been established in Theorem 2.2. Let $y^* \in D_0$ be such that $F(y^*) = 0$. Let $M = \int_0^1 F'(x^* + \theta(y^* - x^*))\, d\theta$. Then, in view of (1.5) and the definition of r_0, we have

$$\|F'(x_0)^{-1}(M - F'(x_0))\| \leq \int_0^{s(\overline{x^* y^*})} L_0(u)\, du \leq \int_0^{r_0} L_0(u)\, du < 1.$$

It follows that $M^{-1} \in L(Y, X)$. Then, in view of the identity

$$0 = F(y^*) - F(x^*) = M(y^* - x^*),$$

we conclude that $y^* = x^*$. ∎

3. Special cases and applications

3.1. Special case. If $\bar{\eta}_n = \eta_n = \bar{\eta} = \eta = 0$, $\omega = 1$, $\sigma_0 = 0$ and $\beta = \|F'(x_0)^{-1}F(x_0)\|$, our theorems improve the corresponding ones in [12, Corollaries 3.3, 3.4 and Theorem 3.1].

3.2. Application: Kantorovich-type condition. Let L_0, L, L_1 be constant functions. Then the convergence criteria corresponding to Theorems 2.1, 2.2 and 2.4 are respectively

$$\|F'(x_0)^{-1}F(x_0)\| \leq \frac{(1-\lambda)^2 \omega L}{2(\omega L + \sigma)}, \tag{3.1}$$

$$\|F'(x_0)^{-1}F(x_0)\| \leq \frac{(1-\lambda)^2 \omega L}{2(\omega L + \sigma_0)}, \tag{3.2}$$

$$\|F'(x_0)^{-1}F(x_0)\| \leq \frac{(1-\lambda)^2 \omega L_1}{2(\omega L_1 + \sigma_0)}. \tag{3.3}$$

Notice that (3.1)⇒(3.2) and (3.1)⇒(3.3) by (1.8) and (2.3).

3.3. Application: γ-condition [2, 11, 12]. Suppose that

$$\|F'(x_0)^{-1}F''(x)\| \leq \frac{2\gamma}{(1-\gamma\|x-x_0\|)^3}$$

for each $x \in U(x_0, 1/\gamma)$. Define

$$L(u) = \frac{2\gamma}{(1-\gamma u)^3}$$

for each $u \in [0, 1/\gamma)$. Then the corresponding convergence criteria for Theorems 2.1 and 2.2 are respectively

$$\|F'(x_0)^{-1}F(x_0)\| \leq \frac{2r_\lambda^2}{(1-\gamma r_\lambda)^2}, \tag{3.4}$$

$$\|F'(x_0)^{-1}F(x_0)\| \leq \frac{2r_{\lambda,0}^2}{(1-\gamma r_{\lambda,0})^2)}. \tag{3.5}$$

Then (3.4)$\Rightarrow$(3.5) by (2.4). Examples where (1.6) or (2.11) hold as strict inequalities and therefore the advantages $(\mathcal{A})$ hold can be found in [1, 2, 3, 4].

All the above justifies the advantages stated in the abstract.

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