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MULTI-BOUNDED SETS AND AMENABILITY FOR A CLASS OF BANACH ALGEBRAS

BY

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Abstract. Let $\mathcal{A}$ be a Lau algebra. We obtain some equivalent conditions for left amenability of $\mathcal{A}$ in terms of multi-bounded sets. We then apply these results to certain Lau algebras on a locally compact group G to give characterizations for amenability of G.

1. Introduction. The theory of multi-normed spaces was first introduced and studied by Dales and Polyakov [DP]. This study was continued in [DD$^+$2] to resolve a long-standing question of Barry Johnson on the injectivity of Banach left $L^1(G)$-modules (where G is a locally compact group and $L^1(G)$ is the group algebra based on G); indeed, Dales, Daws, Pham and Ramsden characterized amenability of a locally compact group G in terms of multi-norms. The theory of multi-norms has been further developed by several authors; see for example [BDP], [DD$^+$1] and [DLOT].

A Banach algebra $\mathcal{A}$ is called a *Lau algebra* if the dual space $\mathcal{A}'$ of $\mathcal{A}$ is a W^*-algebra and the identity element of $\mathcal{A}'$ is a multiplicative linear functional on $\mathcal{A}$. This class of Banach algebras originated with a paper published in 1983 by Lau [L1] who referred to them as F-algebras. Later on, in his useful monograph Pier [PI] introduced the name of Lau algebra (which includes the group algebra and the Fourier algebra of a locally compact group and quantum group algebras, or more generally the predual algebra of a Hopf von Neumann algebra).

The notion of left amenability of Lau algebras was introduced by Lau [L1]. In the same paper he obtained several characterizations of left amenable Lau algebras. See also Lau [L2] and Lau and Wong [LW].

In this paper, we characterize left amenability of Lau algebras in the language of multi-norms. We need the notions of a (p,q)-multi-norm and a multi-bounded set. First, we recall the background material on multi-

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normed spaces. Then we present some characterizations of left amenability and left contractibility for Lau algebras using multi-norms. Finally, we give applications of our results to group algebras.

2. Survey of some properties of multi-normed spaces. The purpose of this section is to sketch the background material on multi-normed spaces. For a more detailed and in-depth treatment spaces see [DP]. Let E be a Banach space and $n \in \mathbb{N}$. We denote by E^n the vector space Cartesian product of n copies of E, and by $\mathfrak{G}_n$ the group of all permutations of the set $\{1, \ldots, n\}$.

A *multi-norm* based on $\{E^n : n \in \mathbb{N}\}$ is a sequence $(\|\cdot\|_n) = (\|\cdot\|_n : n \in \mathbb{N})$ such that $\|\cdot\|_n$ is a norm on E^n such that $\|x\|_1 = \|x\|$ for each $x \in E$ and the following axioms are satisfied for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in E$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$:

(A1) $$\|(x_{\sigma(1)}, \ldots, x_{\sigma(n)})\|_n = \|(x_1, \ldots, x_n)\|_n \quad (\sigma \in \mathfrak{G}_{\mathfrak{n}}),$$

(A2) $$\|(\alpha_1 x_1, \ldots, \alpha_n x_n)\|_n \leq \Big(\max_{1\leq i\leq n} |\alpha_i|\Big) \|(x_1, \ldots, x_n)\|_n,$$

(A3) $$\|(x_1, \ldots, x_{n-1}, 0)\|_n = \|(x_1, \ldots, x_{n-1})\|_{n-1},$$

(A4) $$\|(x_1, \ldots, x_{n-1}, x_{n-1})\|_n = \|(x_1, \ldots, x_{n-1})\|_{n-1}.$$

In this case, we say that $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is a *multi-normed space*. The following properties are almost immediate consequences of the axioms:

(C1) $$\|(x, \ldots, x)\|_n = \|x\| \quad (n \in \mathbb{N},\ x \in E),$$

(C2) $$\max_{1\leq i\leq n} \|x_i\| \leq \|(x_1, \ldots, x_n)\|_n \leq \sum_{i=1}^{n} \|x_i\| \quad (n \in \mathbb{N},\ x_1, \ldots, x_n \in E).$$

Now, we recall two important examples of multi-norms for an arbitrary normed space E.

EXAMPLE 2.1. (a) The sequence $(\|\cdot\|_n : n \in \mathbb{N})$ for each $x_1, \ldots, x_n \in E$ defined by

$$\|(x_1, \ldots, x_n)\|_n := \max_{1\leq i\leq n} \|x_i\|$$

is a multi-norm called the *minimum multi-norm*. The "minimum" here is justified by property (C2).

(b) Let $\{(\|\cdot\|_n^\alpha : n \in \mathbb{N}) : \alpha \in A\}$ be the (non-empty) family of all multi-norms on $\{E^n : n \in \mathbb{N}\}$ for each α in a non-empty index set A (perhaps finite). For $n \in \mathbb{N}$, set

$$|||(x_1, \ldots, x_n)|||_n := \sup_{\alpha\in A} \|(x_1, \ldots, x_n)\|_n^\alpha \quad (x_1, \ldots, x_n \in E).$$

Then $(|||\cdot||| : n \in \mathbb{N})$ is a multi-norm on $\{E^n : n \in \mathbb{N}\}$, called the *maximum multi-norm*.

Let $(E, \|\cdot\|)$ be a normed space. A *dual multi-norm* based on $\{E^n : n \in \mathbb{N}\}$ is a sequence $(\|\cdot\|_n : n \in \mathbb{N})$ such that $\|\cdot\|_n$ is a norm on E^n for each $n \in \mathbb{N}$, such that $\|x\|_1 = \|x\|$ for each $x \in E$, and Axioms (A1)–(A3) and the following modified form of (A4) are satisfied for all $n \in \mathbb{N}$ with $n \geq 2$:

(B4) $$\|(x_1, \dots, x_{n-2}, x_{n-1}, x_{n-1})\|_n = \|(x_1, \dots, x_{n-2}, 2x_{n-1})\|_{n-1}.$$

In this case, we say that $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is a *dual multi-normed space.*

If $(\|\cdot\|_n : n \in \mathbb{N})$ is a multi-norm or a dual multi-norm based on E, and $\|\cdot\|'_n$ is the dual norm to $\|\cdot\|_n$ for each $n \in \mathbb{N}$, then $(\|\cdot\|'_n : n \in \mathbb{N})$ is a dual multi-norm or multi-norm, respectively, based on E' (the dual space of E).

Now, we recall the definition of the weak p-summing norm on a normed space; following [DD$^+$2], [DP] and [J] we denote the weak p-summing norm (for $1 \leq p < \infty$) on E^n by

$$\mu_{p,n}(x) = \sup\Big\{ \Big(\sum_{i=1}^{n} |\langle x_i, \lambda \rangle|^p \Big)^{1/p} : \lambda \in E'_{[1]} \Big\},$$

where $x = (x_1, \dots, x_n) \in E^n$, the closed unit ball of E is denoted by $E_{[1]}$, and the action of $\lambda \in E'$ on $x \in E$ is written as $\langle x, \lambda \rangle$. See also [DJT] and [RY].

The dual of $\|\cdot\|_n^{\max}$ is the weak 1-summing norm $\mu_{1,n}$ [DP, Proposition 3.33], and hence for each $n \in \mathbb{N}$ and $x = (x_1, \dots, x_n) \in E^n$ we have

$$\|x\|_n^{\max} = \sup\Big\{ \Big| \sum_{i=1}^{n} \langle x_i, \lambda_i \rangle \Big| : \lambda_1, \dots, \lambda_n \in E',\ \mu_{1,n}(\lambda_1, \dots, \lambda_n) \leq 1 \Big\},$$

where the supremum is taken over all $\lambda_1, \dots, \lambda_n \in (E')^n$.

Now we define another class of multi-norms. The following definition was first given in [DP, §4.1]. Let E be a normed space, and take p, q with $1 \leq p, q < \infty$. For each $n \in \mathbb{N}$ and each $x = (x_1, \dots, x_n) \in E^n$, we set

$$\|x\|_n^{(p,q)} := \sup\Big\{ \Big(\sum_{i=1}^{n} |\langle x_i, \lambda_i \rangle|^q \Big)^{1/q} : \mu_{p,n}(\lambda_1, \dots, \lambda_n) \leq 1 \Big\},$$

where the supremum is taken over all $\lambda_1, \dots, \lambda_n \in (E')^n$.

It is clear that $\|\cdot\|_n^{(p,q)}$ is a norm on E^n. As noted in [DP, Theorem 4.1], for $1 \leq p \leq q < \infty$, the sequence $(\|\cdot\|_n^{(p,q)} : n \in \mathbb{N})$ is a multi-norm based on E; it is called the (p,q)-*multi-norm.*

As a consequence of the above definition and the Principal of Local Reflexivity [RY, Theorem 5.54] we have

$$\|\lambda\|_n^{(p,q)} = \sup\Big\{ \Big(\sum_{i=1}^{n} |\langle x_i, \lambda_i \rangle|^q \Big)^{1/q} : \mu_{p,n}(x_1, \dots, x_n) \leq 1 \Big\}$$

for all $n \in \mathbb{N}$ and $\lambda = (\lambda_1, \dots, \lambda_n) \in (E')^n$. This implies that, for each normed space E, the (p,q)-multi-norm based on E is the same as the one induced from the (p,q)-multi-norm based on E''.

Let E be a normed space. Suppose that multi-norms $(\|\cdot\|_n^1 : n \in \mathbb{N})$ and $(\|\cdot\|_n^2 : n \in \mathbb{N})$ belong to $\mathcal{E}_E$, the family of all multi-norms based on E. Then $(\|\cdot\|_n^1) \leq (\|\cdot\|_n^2)$ if with respect to the ordering $\leq$,

$$\|(x_1,\dots,x_n)\|_n^1 \leq \|(x_1,\dots,x_n)\|_n^2$$

for all $(x_1,\dots,x_n) \in E^n$ and $n \in \mathbb{N}$.

Consider the above (p,q)-multi-norms based on E, where $1 \leq p \leq q < \infty$. It is clear that for each fixed $p \geq 1$ we have

$$p \leq q_2 \leq q_1 \;\Rightarrow\; (\|\cdot\|_n^{(p,q_1)}) \leq (\|\cdot\|_n^{(p,q_2)}),$$

and for each fixed $q \geq 1$ we have

$$p_1 \leq p_2 \leq q \;\Rightarrow\; (\|\cdot\|_n^{(p_1,q)}) \leq (\|\cdot\|_n^{(p_2,q)}).$$

Some elementary relations were given in [DP]; for example, it is proved in Theorem 4.5 there that

$$1 \leq p \leq q < \infty \;\Rightarrow\; \|\cdot\|_n^{(p,p)} \geq \|\cdot\|_n^{(q,q)}.$$

The next consequence was obtained exactly from [R, Proposition 3.10].

COROLLARY 2.2. *Let E be a Banach space and let $1 < p < q < \infty$. For each $n \in \mathbb{N}$ the following inequalities hold on E^n:*

(1) $\|\cdot\|_n^{(1,q)} \leq \|\cdot\|_n^{(p,q)} \leq \|\cdot\|_n^{(q,q)}$,
(2) $\|\cdot\|_n^{(q,q)} \leq \|\cdot\|_n^{(p,p)} \leq \|\cdot\|_n^{(1,1)}$.

In general, $\|\cdot\|_n^{(1,1)}$ is maximal among these multi-norms.

Let $((E^n,\|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space. A subset $F \subset E$ is *multi-bounded* if

$$\mathbf{mb}(F) := \sup\{\|(x_1,\dots,x_n)\|_n : x_1,\dots,x_n \in F,\, n \in \mathbb{N}\} < \infty;$$

the constant $\mathbf{mb}(F)$ is the *multi-bound* of F.

For example, a subset of E is called (p,q)-*multi-bounded* if it is multi-bounded with respect to the (p,q)-multi-norm. The (p,q)-multi-bound of such a subset F is denoted by

$$\mathbf{mb}_{(p,q)}(F) := \sup\{\|(x_1,\dots,x_n)\|_n^{(p,q)} : x_1,\dots,x_n \in F,\, n \in \mathbb{N}\}.$$

3. Left amenability of Lau algebras in terms of multi-norms. In this section we characterize left amenability of Lau algebras in terms of multi-bounded sets.

Let $\mathcal{A}$ be a Lau algebra and let u be the identity element of $\mathcal{A}'$. Then the second dual $\mathcal{A}''$ of $\mathcal{A}$ equipped with the first Arens multiplication $\odot$ is a Lau algebra [L1, Proposition 3.2], where $\odot$ is defined by the equalities

$$\langle M \odot N, f\rangle = \langle M, Nf\rangle, \quad \langle Nf, a\rangle = \langle N, fa\rangle, \quad \langle fa, b\rangle = \langle f, ab\rangle,$$

for all $M, N \in \mathcal{A}''$, $f \in \mathcal{A}'$, and $a, b \in \mathcal{A}$.

Let $P(\mathcal{A})$ be the set of all elements a in $\mathcal{A}$ that induce positive functionals on the W^*-algebra $\mathcal{A}'$; note that

$$P(\mathcal{A}) = \{a \in \mathcal{A} : \|a\| = \langle u, a\rangle\},$$

and hence span $\mathcal{A}$ (see [S, Propositions 1.5.1 and 1.5.2]). An element a in $P(\mathcal{A})$ is called a *mean* in $\mathcal{A}$ if $\langle u, a\rangle = 1$. We denote by $P_1(\mathcal{A})$ the set of all means in $\mathcal{A}$.

A mean M of $P_1(\mathcal{A}'')$ is said to be a *topological left invariant mean* if $a \odot M = M$ (for all $a \in P_1(\mathcal{A})$), or equivalently

$$\langle M, fa\rangle = \langle M, f\rangle \quad (a \in P_1(\mathcal{A}),\ f \in \mathcal{A}').$$

The Lau algebra $\mathcal{A}$ is called *left amenable* if there exists a topological left invariant mean in $\mathcal{A}''$.

For each $M \in P_1(\mathcal{A}'')$, we define the subset $\mathcal{A}M$ of $\mathcal{A}''$ by

$$\mathcal{A}M := \{a \odot M : a \in P_1(\mathcal{A})\}.$$

We are now ready to give a characterization of left amenability of Lau algebras which is inspired by the results of Dales, Daws, Pham and Ramsden for a locally compact group G.

THEOREM 3.1. *Let $\mathcal{A}$ be a Lau algebra and $1 \le p < \infty$. Then the following statements are equivalent:*

(a) *$\mathcal{A}$ is left amenable.*
(b) *There exists a mean $M \in \mathcal{A}''$ such that $\mathbf{mb}_{(p,p)}(\mathcal{A}M) < \infty$.*

Proof. (a)⇒(b). Let $\mathcal{A}$ be left amenable with $M \in P_1(\mathcal{A}'')$ a topological left invariant mean. Then for every $a \in P_1(\mathcal{A})$ we have $a \odot M = M$. Hence, immediately from property (C1), we obtain

$$\begin{aligned}\mathbf{mb}_{(p,p)}(\mathcal{A}M) &= \sup_{n\in\mathbb{N}}\{\|(a_1 \odot M, \ldots, a_n \odot M)\|_n^{(p,p)} : a_1, \ldots, a_n \in P_1(\mathcal{A})\}\\ &= \sup_{n\in\mathbb{N}}\{\|(M, \ldots, M)\|_n^{(p,p)}\} = \|M\| = 1.\end{aligned}$$

This shows that (b) holds.

(b)⇒(a). Suppose that there is a mean $M \in P_1(\mathcal{A}'')$ such that $\mathcal{A}M$ is (p,p)-multi-bounded. Then by [DD$^+$2, Theorem 5.7], $\mathcal{A}M$ is relatively weakly compact, and also, by the Krein–Šmulian theorem [M], the following set is weakly compact:

$$K := \overline{\mathrm{co}(\mathcal{A}M)} = \overline{\Big\{\sum_{i=1}^{n} t_i a_i \odot M : 0 \le t_i \le 1,\ \sum_{i=1}^{n} t_i = 1,\ a_i \in P_1(\mathcal{A})\Big\}}.$$

Now, let $\Sigma = \{L_b : b \in P_1(\mathcal{A})\}$ be a semigroup of affine maps from the weakly compact convex set K into itself defined by

$$L_b(\Lambda) = b \odot \Lambda \quad (b \in P_1(\mathcal{A}),\ \Lambda \in K).$$

Then L_b is an isometric map since, for every $b \in P_1(\mathcal{A})$,

$$\|L_b(\Lambda)\| = \|b \odot \Lambda\| = \|\Lambda\|\cdot$$

Hence, by the Ryll-Nardzewski fixed point theorem [P, G], there exists $M_0 \in K$ which is a common fixed point for Σ such that $L_b(M_0) = M_0$. Therefore, for all $a \in P_1(\mathcal{A})$ and $f \in \mathcal{A}'$ we have

$$\langle M_0, fa\rangle = \langle a \odot M_0, f\rangle = \langle M_0, f\rangle, \qquad \|M_0\| = \langle M_0, u\rangle = 1.$$

This implies that M_0 is a topological left invariant mean on $\mathcal{A}'$. ■

As a consequence of Theorem 3.1 and Corollary 2.2, we have the following result.

Corollary 3.2. *Let $\mathcal{A}$ be a Lau algebra and $1 \le p < \infty$. Consider the following statements:*

(a) *$\mathcal{A}$ is left amenable.*
(b) *There is a mean $M \in \mathcal{A}''$ such that $\mathbf{mb}_{(p,p)}(\mathcal{A}M) < \infty$.*
(c) *There is a mean $M \in \mathcal{A}''$ such that $\mathbf{mb}_{(q,q)}(\mathcal{A}M) < \infty$ for all $q \ge p$.*
(d) *There is a mean $M \in \mathcal{A}''$ such that $\mathbf{mb}_{(p,q)}(\mathcal{A}M) < \infty$ for all $q > p$.*
(e) *There is a mean $M \in \mathcal{A}''$ such that $\mathbf{mb}_{(1,q)}(\mathcal{A}M) < \infty$ for all $q \ge 1$.*

Then (a)⇔(b)⇒(c)⇒(d)⇒(e).

A Lau algebra $\mathcal{A}$ is called an *FC-algebra* if the W^*-algebra $\mathcal{A}'$ is commutative. In this case, $\mathcal{A} \cong L^1(\Omega, \mu)$ (isometrc isomorphism), and $\mathcal{A}'$ may be regarded as the W^*-algebra $L^\infty(\Omega, \mu)$ for some measure space (Ω, μ).

The following result shows that statements (a) and (e) in Corollary 3.2 are equivalent for certain Lau algebras.

Proposition 3.3. *Let $\mathcal{A}$ be an FC-algebra and $1 \le p < \infty$. Then the following statements are equivalent:*

(a) *$\mathcal{A}$ is left amenable.*
(b) *There is a mean $M \in \mathcal{A}''$ such that $\mathbf{mb}_{(p,q)}(\mathcal{A}M) < \infty$ for all $q \ge p$.*
(c) *There is a mean $M \in \mathcal{A}''$ such that $\mathbf{mb}_{(1,q)}(\mathcal{A}M) < \infty$ for all $q \ge 1$.*

Proof. (b)⇔(c). Since $\mathcal{A}$ is an FC-algebra, its dual space is an AM-space as a (real) Banach lattice. Thus $\mathcal{A}''$ can be isometrically identified as a Banach lattice with $L^1(\Omega)$ for some measure space. For these results on Banach lattices, see [AB, §12], for example. Then by [DD$^+$2, Theorem 5.6] statements (b) and (c) are equivalent.

(b)⇒(a). Suppose that there is a mean $M \in P_1(\mathcal{A}'')$ such that the set $\mathcal{A}M$ is (p,q)-multi-bounded. Since $\mathcal{A}$ is an FC-algebra, it follows that $\mathcal{A}M$ is relatively weakly compact [DD+2, Corollary 5.8]. So by a similar proof to that of Theorem 3.1, there is a topological left invariant mean in $\mathcal{A}''$. Therefore, $\mathcal{A}$ is left amenable.

(a)⇒(b). This is trivial by property (C1). ∎

Let $\mathcal{A}$ be a Lau algebra. We recall [NS] that $\mathcal{A}$ is *left u-contractible* if there exists a topological left invariant mean in $\mathcal{A}$, that is, a mean $m \in \mathcal{A}$ such that

$$a \odot m = am = m \quad (a \in P_1(\mathcal{A})).$$

Now, we characterize left u-contractibility of Lau algebras in the language of multi-norms.

PROPOSITION 3.4. *Let $\mathcal{A}$ be a Lau algebra and $1 \le p < \infty$. Then the following statements are equivalent:*

(a) *$\mathcal{A}$ is left u-contractible.*
(b) *There is a mean $m \in \mathcal{A}$ such that $\mathbf{mb}_{(p,p)}(\mathcal{A}m) < \infty$.*

Proof. This is obtained immediately from replacing $\mathcal{A}$ by $\mathcal{A}''$ in the proof of Theorem 3.1. ∎

We end this section with the following result.

COROLLARY 3.5. *Let $\mathcal{A}$ be an FC-algebra and $1 \le p < \infty$. Then the following statements are equivalent:*

(a) *$\mathcal{A}$ is left u-contractible.*
(b) *There is a mean $m \in \mathcal{A}$ such that $\mathbf{mb}_{(p,q)}(\mathcal{A}m) < \infty$ for all $q \ge p$.*
(c) *There is a mean $m \in \mathcal{A}$ such that $\mathbf{mb}_{(p,q)}(\mathcal{A}m) < \infty$ for some $q \ge p$.*
(d) *There is a mean $m \in \mathcal{A}$ such that $\mathbf{mb}_{(1,q)}(\mathcal{A}m) < \infty$ for all $q \ge 1$.*
(e) *There is a mean $m \in \mathcal{A}$ such that $\mathbf{mb}_{(1,q)}(\mathcal{A}m) < \infty$ for some $q \ge 1$.*

4. Applications to group algebras. In this final section we give applications of the results obtained in the previous section to group algebras.

Let G be a locally compact group with left Haar measure λ_G and let $L^1(G) = L^1(G, \lambda_G)$ be the group algebra of G as defined in [HR] endowed with the norm $\|\cdot\|_1$ and the convolution product $*$ given by

$$(\varphi * \psi)(s) = \int_G \varphi(t)\psi(t^{-1}s)\, d\lambda_G(t) \quad (s \in G),$$

where $\varphi, \psi \in L^1(G)$ and the integral is defined for almost all $s \in G$.

Also suppose that $L^\infty(G)$ is the usual Lebesgue space with the essential supremum norm $\|\cdot\|_\infty$.

We now consider $L^1(G)$ as an FC-algebra: for $\varphi, \psi \in L^1(G)$ and $s \in G$, we define $s \cdot \varphi, \psi \cdot s \in L^1(G)$ by

$$(s \cdot \varphi)(t) = \varphi(s^{-1}t), \quad (\psi \cdot s)(t) = \psi(ts^{-1})\Delta(s^{-1}) \quad (t \in G),$$

thus defining an action of G on the space $L^1(G)$.

We can extend this action by duality to the space $L^\infty(G)' = L^1(G)''$. An element $M \in L^\infty(G)'$ is a mean on $L^\infty(G)$ if

$$\langle 1, M\rangle = \|M\| = 1,$$

and M is left invariant if $\{s \cdot M : s \in G\} = \{M\}$. The group G is amenable if there exists a left invariant mean on $L^\infty(G)$. It is well-known that all locally abelian groups and compact groups are amenable, but the free group $\mathbb{F}_2$ on two generators is not amenable; see [P], for example, for a full account.

There is a famous theorem of Lau [L1] that the algebra $L^1(G)$ is left amenable as an FC-algebra if and only if the locally compact group G is amenable.

Note that Dales, Daws, Pham and Ramsden [DD$^+$2, Theorem 8.4] characterized amenability of the locally compact group G in terms of multi-norms; indeed, they proved that G is amenable if and only if there is a mean $M \in L^1(G)''$ such that

$$\mathbf{mb}_{(p,q)}(GM) := \mathbf{mb}_{(p,q)}(\{s \cdot M : s \in G\}) < \infty.$$

Related to this result, we have the following consequence of Proposition 3.3.

THEOREM 4.1. *Let G be a locally compact group and $1 \le p < \infty$. Then for all $q \ge p$ the following statements are equivalent:*

(a) *G is amenable.*
(b) *There is a mean $M \in L^1(G)''$ such that $\mathbf{mb}_{(p,q)}(L^1(G)M) < \infty$.*
(c) *There is a mean $M \in L^1(G)''$ such that $\mathbf{mb}_{(p,q)}(GM) < \infty$.*

Proof. (a)⇔(b). This follows immediately from Proposition 3.3 and the fact that G is amenable if and only if $L^1(G)$ is left amenable.

(b)⇒(c). Suppose that $M \in P_1(L^1(G)'')$ is a topological left invariant mean. Choose $\varphi_0 \in P_1(L^1(G))$, and set $M_{\varphi_0} = \varphi_0 \odot M$. Then

$$\{s \cdot M_{\varphi_0} : s \in G\} = \Big\{ \underbrace{s \cdot \varphi_0}_{\in P_1(L^1(G))} \odot M : s \in G\Big\}$$
$$\subseteq \{\varphi \odot M : \varphi \in P_1(L^1(G))\} \subseteq L^1(G)M.$$

Therefore, the set GM_{φ_0} is (p, q)-multi-bounded, so (c) holds.

(c)⇒(b). Suppose that there is $M \in P_1(L^1(G)'')$ such that $\mathbf{mb}_{(p,q)}(GM) < \infty$ for all $q \ge p$. Define $\theta : L^1(G) \to L^1(G)$ by

$$\theta(\varphi)(s) = \varphi(s^{-1})\Delta(s^{-1})$$

for $\varphi \in L^1(G)$ and $s \in G$. Therefore, the adjoint operator $\theta'' : L^1(G)'' \to L^1(G)''$ is automatically (p,q)-multi-bounded and takes the set $GM = \{s \cdot M : s \in G\}$ to $\theta''(M)G = \{\theta''(M) \cdot s : s \in G\}$, and $\theta'(1) = 1$; thus the set $\theta''(M)G$ is (p,q)-multi-bounded.

Now, we show that the set

$$L^1(G)M = \{\varphi \odot M : \varphi \in P_1(L^1(G))\} = \{\theta(\varphi) \odot M : \varphi \in P_1(L^1(G))\}$$

is (p,q)-multi-bounded. To that end, we first show that

$$\mathbf{mb}_{(p,q)}(\{\theta''(M) \cdot s \cdot \varphi : s \in G,\, \varphi \in P_1(L^1(G))\}) \le \mathbf{mb}_{(p,q)}(\theta''(M)G). \tag{1}$$

Indeed, by the definition of the (p,q)-multi-norm, we have

$$\begin{aligned}
\mathbf{mb}_{(p,q)}(\theta''(M)G) &= \sup_{n\in\mathbb{N}} \sup_{\mu_{p,n}(f_i)\le 1} \Big\{ \Big(\sum_{i=1}^{n} |\langle \theta''(M) \cdot s_i, f_i\rangle|^q \Big)^{1/q} \Big\} \\
&\ge \sup_{n\in\mathbb{N}} \sup_{\mu_{p,n}(f_i)\le 1,\, \varphi_i\in P_1(L^1(G))} \Big\{ \Big(\sum_{i=1}^{n} |\langle \theta''(M) \cdot s_i, \varphi_i f_i\rangle|^q \Big)^{1/q} \Big\} \\
&= \sup_{n\in\mathbb{N}} \sup_{\mu_{p,n}(f_i)\le 1,\, \varphi_i\in P_1(L^1(G))} \Big\{ \Big(\sum_{i=1}^{n} |\langle \theta''(M) \cdot s_i \cdot \varphi_i, f_i\rangle|^q \Big)^{1/q} \Big\} \\
&= \mathbf{mb}_{(p,q)}\big(\big\{\theta''(M) \cdot s \cdot \varphi : s \in G, \varphi \in P_1(L^1(G))\big\}\big).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\mathbf{mb}_{(p,q)}(\theta''(M)L^1(G)) \\
&\qquad = \mathbf{mb}_{(p,q)}\big(\big\{\theta''(M) \cdot s \cdot \varphi : s \in G,\, \varphi \in P_1(L^1(G))\big\}\big).
\end{aligned} \tag{2}$$

Moreover,

$$\theta''(M)L^1(G) = \theta''\big(\big\{\theta(\varphi) \odot M : \varphi \in P_1(L^1(G))\big\}\big); \tag{3}$$

in fact, since

$$\begin{aligned}
(\theta(\psi) * \varphi)(s) &= \int_G \theta(\psi)(t)\varphi(t^{-1}s)\, dt = \int_G \psi(t^{-1})\Delta(t^{-1})\varphi(t^{-1}s)\, dt \\
&= \int_G \psi(t^{-1}s^{-1})\theta(\varphi)(t)\Delta(s^{-1})\, dt = (\theta(\varphi) * \psi)(s^{-1})\Delta(s^{-1}) \\
&= \theta(\theta(\varphi) * \psi)(s)
\end{aligned}$$

for all $\varphi, \psi \in L^1(G)$ and $s \in G$, we conclude that

$$\begin{aligned}
\theta'(\varphi f)(\psi) &= f(\theta(\psi) * \varphi) = f(\theta(\theta(\varphi) * \psi)) = \theta'(f)(\theta(\varphi) * \psi) \\
&= (\theta'(f)\theta(\varphi))(\psi)
\end{aligned}$$

for all $f \in L^\infty(G)$. Thus we have

$$\begin{aligned}
(\theta''(M) \odot \varphi)(f) &= \theta''(M)(\varphi f) = M(\theta'(\varphi f)) = M(\theta'(f)\theta(\varphi)) \\
&= (\theta(\varphi) \odot M)(\theta'(f)) = \theta''(\theta(\varphi) \odot M)(f).
\end{aligned}$$

Therefore (1)–(3) show that the set

$$\theta''\big(\{\theta(\varphi)\odot M : \varphi\in P_1(L^1(G))\}\big) = \theta''(L^1(G)M)$$

is (p,q)-multi-bounded. This shows that $L^1(G)M$ is also (p,q)-multi-bounded. ∎

Our last result shows that compactness of a locally compact group G can be characterized by a (p,q)-multi-bounded subset of $L^1(G)$.

COROLLARY 4.2. *Let G be a locally compact group and $1\le p<\infty$. Then the following statements are equivalent:*

(a) *G is compact.*
(b) *There is a mean $m\in L^1(G)$ such that $\mathbf{mb}_{(p,q)}(L^1(G)m)<\infty$ for all $q\ge p$.*
(c) *There is a mean $m\in L^1(G)$ such that $\mathbf{mb}_{(p,q)}(L^1(G)m)<\infty$ for some $q\ge p$.*
(d) *There is a mean $m\in L^1(G)$ such that $\mathbf{mb}_{(1,q)}(L^1(G)m)<\infty$ for all $q\ge 1$.*
(e) *There is a mean $m\in L^1(G)$ such that $\mathbf{mb}_{(1,q)}(L^1(G)m)<\infty$ for some $q\ge 1$.*

Proof. The result follows immediately from Corollary 3.5 and the fact that $L^1(G)$ is left u-contractible if and only if G is compact [NS, Theorem 6.1]. ∎

This result was also obtained by Ramsden [R, Proposition 4.2].

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