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Global properties of Dirichlet forms on discrete spaces

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Contents

Abstract

We provide an introduction to Dirichlet forms on discrete spaces and study their global properties including recurrence, stochastic completeness and the regularity of the Neumann form. In this setting we compare the notion of a recurrent Dirichlet form and a recurrent discrete-time Markov chain of a given graph. We prove several known and several new characterizations of recurrence by using functional-analytic Dirichlet form methods only. Finally, we compare all the above mentioned global properties and discuss their relation to spectral theory.

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Introduction

The study of the long-time behavior of sample paths of discrete-time random walks on countable sets is a very well established area of mathematical research. Probably the most fundamental question is whether a given random walk returns to each point infinitely often or not, i.e., whether the random walk is recurrent or transient. When dealing with the special case of Markov chains, it is now well-known that this question is intimately linked to the study of certain linear operators on function spaces over the state space (see e.g. the textbooks [2, 3, 23, 25]). If, additionally, the Markov chains are reversible, these operators satisfy a further symmetry condition making them accessible to the theory of self-adjoint operators. In this case they are sometimes referred to as discrete Laplacians, which, independently, gained a lot of attention outside of probability theory in the realm of spectral geometry and spectral theory.

Starting with the work of Yamasaki [27, 28] it was realized that potential theory also provides useful tools for studying the long-time behavior of discrete-time Markov chains. In the simplest case the potential theory is induced by a certain quadratic form coming from a weighted graph on the state space. The weights of the graph and the random walk correspond through transition probabilities. It can be seen that this approach is indeed a special case of the potential theory of Dirichlet spaces as presented in [4]. However, there the authors are interested in the study of symmetric Markov processes in continuous time. On countable state spaces this means that additional holding times at each site are introduced. As a consequence, the local behavior, i.e., the holding times and the transition probabilities, does not necessarily determine the process uniquely and the process might have a finite lifetime (be stochastically incomplete).

One of the advantages of the approach of [4] is that these so-called global properties (uniqueness of the process, stochastic completeness, recurrence) can be formulated in purely functional-analytic terms. In [15] Keller and Lenz introduced the full power of the theory of regular Dirichlet forms of [4] to the study of discrete Laplacians. They were inspired by the need of a convenient framework for the investigation of unbounded Laplacians on discrete spaces, which recently gained considerable attention starting with the works [13, 24, 26]. Keller and Lenz study the relation of spectral theory to the uniqueness of associated processes and their lifetime in quite some detail. However, a treatment of recurrence and transience is missing in their exposition.

This is where the present text comes into play. It is intended to provide a convenient and self-contained exposition of recurrence and transience within the theory of Dirichlet forms on discrete spaces. In particular, it contains an introduction to the theory of such forms.

As discussed previously, recurrence and transience of random walks are usually investigated in discrete time and, when passing to continuous time, other global properties enter the game. Thus, the study of these properties via Dirichlet forms raises three main questions:

(a) How is the notion of recurrence/transience for discrete-time random walks related to the one for regular Dirichlet forms?
(b) How can one use the theory of regular Dirichlet forms to prove known and new criteria for recurrence/transience?
(c) How is recurrence related to the uniqueness of the processes and their lifetime?

The present article provides answers to all three questions.

Question (a) is answered by Theorem 3.4. It establishes an explicit formula relating Green's function of a discrete-time random walk on a given graph and Green's function of the regular Dirichlet form associated with this graph. In particular, it shows that both notions of recurrence agree. While the last result is certainly known to experts, a reference proving it in the full generality of the present framework seems to be missing.

Given the previously discussed answer to question (a), question (b) makes sense. For an answer the known criteria of recurrence and transience which deal with the existence of monopoles of finite energy (Theorem 4.2), the existence of superharmonic functions of finite energy (Theorem 4.5) and the capacity of points (Theorem 4.3) are proven within the framework provided by [4]. Two new criteria which are inspired by the recent works [6, 11] and deal with the vanishing of a boundary term in the discrete version of Green's formula are obtained (Theorems 4.7 and 4.8). Finally, the connection between recurrence and the spectral theory of the discrete Laplacian on the space of functions of finite energy is discussed in Theorem 6.6.

Question (c) is answered at the end of the paper. Uniqueness of processes is addressed in analytic terms by asking when the Neumann form associated with a graph is regular, while the lifetime problem is studied analytically in terms of stochastic completeness of the regular Dirichlet form (see Chapter 5). It turns out that regularity of the Neumann form and stochastic completeness are also related to the validity of Green's formula (Theorems 5.3 and 5.6) on certain function spaces. Furthermore, it is shown that recurrence always implies the regularity of the Neumann form and stochastic completeness (Theorem 6.3), and that all these global properties are equivalent when the underlying measure is finite (Theorem 6.5).

The paper is organized as follows. Chapter 1 introduces the basic objects of investigation, namely weighted graphs and an ensemble of associated forms, operators and spaces. In this presentation we basically follow [9, 15]. Chapter 2 outlines the theory of recurrence and transience of Dirichlet forms as presented in [4, Section 1.5]. The contents of this chapter are certainly well-known. However, as the proofs simplify substantially in the discrete setting, we include them for the convenience of the reader. Chapter 3 compares the notion of recurrence of discrete-time Markov chains and Dirichlet forms associated with graphs. In Chapter 4 we discuss the consequences of the theory developed in Chapter 2 when applied to a regular Dirichlet form on a discrete space. We recover known characterizations of recurrence purely by Dirichlet form methods and even obtain new ones with the same techniques. Chapter 5 introduces two more global properties, namely stochastic completeness and regularity of the Neumann form. We prove several characterizations. Chapter 6 deals with the connection between recurrence and the other global properties, and with the connection between recurrence and the spectral theory of the discrete Laplacian.

Remark on the history of this paper. A preliminary version of this paper [22] was published on the arXiv. It was not sent to a refereed journal since it is rather long and, besides new theory, contains some known results. However, this exposition turned out to close the gap between the textbooks [4] and [23, 25] and seems to be quite useful. In fact, the way the theory is presented here is used in the publications [5, 8, 10, 14, 17, 18] which cite [22].

1. Forms and spaces associated with graphs

In this chapter we introduce the objects of our studies. Following [9] we specify what we will call a weighted graph (b, c) over a vertex set V, define the ensemble $(\widetilde{D}, \widetilde{F}, \widetilde{L}, \widetilde{Q})$ of associated objects and show their basic connections. Afterwards we equip the space $\widetilde{D}$ with an inner product which turns it into a Hilbert space. Since spaces of this sort were introduced in [27], we call it the Yamasaki space $\mathbf{D}$. We prove some results about its structure. At the end of this chapter we introduce the notion of a Dirichlet form associated with a graph. Most results of this chapter are well-known; we include their proofs for the convenience of the reader.

1.1. Basic definitions. Assume that V is an infinite, countable set. Let $b : V \times V \to [0, \infty)$ be such that

(b0) $b(x, x) = 0$ for all $x \in V$,
(b1) $b(x, y) = b(y, x)$ for all $x, y \in V$,
(b2) $\sum_{y \in V} b(x, y) < \infty$ for all $x \in V$,

and let $c : V \to [0, \infty)$. We call the pair (b, c) a *weighted graph* over the *vertex set* V. We say $x, y \in V$ are *connected by an edge* whenever $b(x, y) > 0$. In this case we write $x \sim y$ and call $b(x, y)$ the *weight* of the edge connecting x and y. Vertices $x \in V$ with $c(x) > 0$ might be thought of as being connected to a point ∞ which is not contained in V. We call a finite sequence of vertices $x_0, \ldots, x_n \in V$ a *path* connecting x_0 and x_n if $x_j \sim x_{j+1}$ for $j = 0, \ldots, n-1$. A subset $W \subseteq V$ is said to be *connected* if for any $x, y \in W$ there is a path in W connecting x and y. The quantity

$$\deg(x) := \sum_{y \in V} b(x, y) + c(x)$$

is said to be the *generalized vertex degree* of x. For a graph $(b, 0)$ with $b \in \{0, 1\}$ the number $\deg(x)$ coincides with the number of edges emerging from x. A graph (b, c) over V

is called *locally finite* if the sets

$$\{y \in V \mid b(x,y) > 0\}$$

are finite for every $x \in V$, i.e., each vertex is only connected to finitely many other vertices.

Let $C(V)$ be the set of all real-valued functions on V and let $C_c(V)$ be the subset of all real-valued functions of finite support. To a graph (b,c) over V we associate the quadratic form

$$\widetilde{Q} := \widetilde{Q}_{b,c} : C(V) \to [0,\infty]$$

defined by

$$\widetilde{Q}(u) := \frac{1}{2}\sum_{x,y\in V} b(x,y)(u(x)-u(y))^2 + \sum_{x\in V} u(x)^2 c(x).$$

The next lemma shows that $\widetilde{Q}$ enjoys certain cut-off properties.

Lemma 1.1. *Let $F : \mathbb{R} \to \mathbb{R}$ be a* normal contraction *(i.e. $F(0) = 0$ and $|F(x) - F(y)| \leq |x-y|$ for all $x, y \in \mathbb{R}$). For $u \in C(V)$,*

$$\widetilde{Q}(F \circ u) \leq \widetilde{Q}(u).$$

Proof. Since $F(0) = 0$, we have $|F(x)| \leq |x|$ for all $x \in \mathbb{R}$. With this observation a direct calculation yields the statement. ∎

Remark 1.2. • Typical examples of normal contractions are given by $F(x) = |x|$ and $F(x) = (0 \vee x) \wedge 1$ (here $a \vee b = \max\{a,b\}$ and $a \wedge b = \min\{a,b\}$).

• The previous lemma is important for showing that certain restrictions of $\widetilde{Q}$ are Dirichlet forms.

We will be interested in the space of all *functions of finite energy*, which is defined by

$$\widetilde{D} := \{u \in C(V) \mid \widetilde{Q}(u) < \infty\}.$$

For $x \in V$ let δ_x be the function on V which vanishes everywhere except at x, where it takes value 1. It satisfies

$$\widetilde{Q}(\delta_x) = \deg(x).$$

Therefore, assumption (b2) implies that $C_c(V) \subseteq \widetilde{D}$.

Abusing notation we extend $\widetilde{Q}$ to a bilinear map $\widetilde{Q} : \widetilde{D} \times \widetilde{D} \to \mathbb{R}$ given by

$$\widetilde{Q}(u,v) := \frac{1}{2}\sum_{x,y\in V} b(x,y)(u(x)-u(y))(v(x)-v(y)) + \sum_{x\in V} u(x)v(x)c(x).$$

Note that the above sum is absolutely convergent by the definition of $\widetilde{D}$.

Remark 1.3. Just as for $\widetilde{Q}$, we will usually write $B(u)$ instead of $B(u,u)$ when dealing with a bilinear form B.

We come to the *formal operator* $\widetilde{L}$ which is associated with $\widetilde{Q}$. Let

$$\widetilde{F} := \left\{u : V \to \mathbb{R} \;\middle|\; \sum_{y\in V} b(x,y)|u(y)| < \infty \text{ for all } x \in V\right\}$$

and $m : V \to (0, \infty)$. We define

$$\widetilde{L} := \widetilde{L}_{b,c,m} : \widetilde{F} \to C(V)$$

via

$$\widetilde{L}u(x) := \frac{1}{m(x)} \sum_{y \in V} b(x,y)(u(x) - u(y)) + \frac{c(x)}{m(x)} u(x).$$

The definition of $\widetilde{F}$ and (b2) ensure that the sum is absolutely convergent. The following lemma is the crucial link between $\widetilde{L}$ and $\widetilde{Q}$. For various versions of this statement, see e.g. [7, 9, 15].

Lemma 1.4 (Green's formula). *The space $\widetilde{D}$ is contained in $\widetilde{F}$. Furthermore, for all $u \in \widetilde{D}$ and $v \in C_c(V)$,*

$$\widetilde{Q}(u,v) = \sum_{x \in V} (\widetilde{L}u)(x)v(x)m(x) = \sum_{x \in V} u(x)(\widetilde{L}v)(x)m(x).$$

Proof. The proof will be in two steps. First, we show $\widetilde{D} \subseteq \widetilde{F}$ following [9, Proposition 2.8]. As a second step we prove the desired equality as in [7, Lemma 4.7].

Step 1. Let $u \in \widetilde{D}$. Using the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}
\sum_{y \in V} b(x,y)|u(y)| &\leq \sum_{y \in V} b(x,y)|u(x) - u(y)| + \sum_{y \in V} b(x,y)|u(x)| \\
&\leq \Big(\sum_{y \in V} b(x,y)\Big)^{1/2} \Big(\sum_{y \in V} b(x,y)|u(x) - u(y)|^2\Big)^{1/2} + \deg(x)|u(x)| \\
&\leq \deg(x)^{1/2} \widetilde{Q}(u)^{1/2} + \deg(x)|u(x)| < \infty.
\end{aligned}$$

This shows $u \in \widetilde{F}$.

Step 2. Let $u \in \widetilde{D}$ and $v \in C_c(V)$. Step 1 and (b1) yield

$$\sum_{x,y \in V} b(x,y)|u(x)v(y)| = \sum_{y \in V} |v(y)| \sum_{x \in V} b(x,y)|u(x)| < \infty.$$

Moreover, by condition (b2) we obtain

$$\sum_{x,y \in V} b(x,y)|u(x)v(x)| = \sum_{x \in V} |v(x)|\,|u(x)| \sum_{y \in V} b(x,y) < \infty.$$

This allows us to rearrange the summation of

$$\sum_{x \in V} \Big(\sum_{y \in V} b(x,y)(u(x) - u(y))v(x) + c(x)u(x)v(x)\Big),$$

and the statement follows by a simple computation. ∎

The next lemma is standard (see e.g. [11, Lemma 2.5]). It shows the continuity of differences of function values with respect to $\widetilde{Q}$.

Lemma 1.5. *Let (b,c) be connected and let $x, y \in V$. There exists a constant $K_{x,y} > 0$ such that for every $u \in \widetilde{D}$,*

$$|u(x) - u(y)| \leq K_{x,y}\, \widetilde{Q}(u)^{1/2}.$$

Proof. Let $x = x_0, \dots, x_n = y$ be a path of pairwise different vertices connecting x and y. Set

$$K_{x,y} = \left(\sum_{j=1}^{n} \frac{1}{b(x_{j-1}, x_j)} \right)^{1/2}.$$

By the Cauchy–Schwarz inequality,

$$|u(x) - u(y)| \leq \sum_{j=1}^{n} |u(x_j) - u(x_{j-1})| \leq K_{x,y} \left(\sum_{j=1}^{n} b(x_{j-1}, x_j) |u(x_j) - u(x_{j-1})|^2 \right)^{1/2}.$$

Since the x_j were assumed to be pairwise different, the last sum can be estimated by $\widetilde{Q}$. This yields the statement. ■

1.2. The Yamasaki space. Following [23], we introduce a Hilbert space associated with a connected graph (b, c). Fix some reference point $o \in V$. For $u, v \in \widetilde{D}$ we define the inner product

$$\langle u, v \rangle_o := \widetilde{Q}(u, v) + u(o)v(o).$$

Let $\|\cdot\|_o$ be the corresponding norm, which is non-degenerate due to the connectedness of (b, c). The following proposition is part of [23, Lemma 3.14 and Theorem 3.15].

PROPOSITION 1.6 (Properties of $(\widetilde{D}, \langle \cdot, \cdot \rangle_o)$). *Let (b, c) be a connected graph over V.*

(a) *For every $x \in V$ the functional $F_x : \widetilde{D} \to \mathbb{R}$, $F_x[u] = u(x)$ is continuous with respect to $\|\cdot\|_o$. In particular, for $o, o' \in V$ the norms $\|\cdot\|_o$ and $\|\cdot\|_{o'}$ are equivalent.*
(b) *The space $\widetilde{D}$ equipped with $\langle \cdot, \cdot \rangle_o$ is a Hilbert space.*

Proof. (a) We only need to prove the first statement. The 'in particular' part then follows, since the first statement shows that $u \mapsto u(o)$ is continuous with respect to $\|\cdot\|_{o'}$.

Let $x \in V$. Choose a path $o = x_0, \dots, x_n = x$ of pairwise different points from o to x and let K_{x_{i-1}, x_i} be constants for these vertices as in Lemma 1.5. We obtain

$$|u(x)| \leq \sum_{i=1}^{n} |u(x_{i-1}) - u(x_i)| + |u(o)| \leq \widetilde{Q}(u)^{1/2} \sum_{i=1}^{n} K_{x_{i-1}, x_i} + \|u\|_o.$$

(b) It suffices to show completeness. Let (u_n) be a Cauchy sequence in $(\widetilde{D}, \langle \cdot, \cdot \rangle_o)$. By (a) the sequence (u_n) converges pointwise to a function u. Fatou's lemma yields

$$\widetilde{Q}(u - u_n) \leq \liminf_{l \to \infty} \widetilde{Q}(u_l - u_n),$$

which can be made small by choosing n large enough. This shows $u \in \widetilde{D}$ and $u_n \to u$ with respect to $\|\cdot\|_o$. ■

DEFINITION 1.7 (Yamasaki space). The pair $(\widetilde{D}, \langle \cdot, \cdot \rangle_o)$ is called the *Yamasaki space* associated with (b, c). We write $\mathbf{D}$ for short whenever we refer to $\widetilde{D}$ endowed with the topology generated by $\|\cdot\|_o$. The set $\mathbf{D}_0$ is the closed subspace of $\mathbf{D}$ given by

$$\mathbf{D}_0 := \overline{C_c(V)}^{\|\cdot\|_o}.$$

REMARK 1.8. Proposition 1.6 shows that the topology on $\mathbf{D}$ and, in particular, the space $\mathbf{D}_0$ do not depend on the choice of the reference point $o \in V$.

For our further discussion we need some approximation results in $\mathbf{D}$ which are based on the following theorem. The proof we present was suggested by Daniel Lenz [20].

THEOREM 1.9 (Convergence in $\mathbf{D}$). *Let $u_n, u \in \mathbf{D}$. The following assertions are equivalent:*

(i) $\|u_n - u\|_o \to 0$ *as* $n \to \infty$.
(ii) $\limsup_{n\to\infty} \widetilde{Q}(u_n) \le \widetilde{Q}(u)$, *and* $u_n \to u$ *pointwise as* $n \to \infty$.

Proof. The implication (i)⇒(ii) follows from Proposition 1.6(a).

To prove (ii)⇒(i) we let $u_n, u \in \widetilde{D}$ with $u_n \to u$ pointwise and $\limsup \widetilde{Q}(u_n) \le \widetilde{Q}(u)$. Then (u_n) is a bounded sequence in $\mathbf{D}$. Since $\mathbf{D}$ is a Hilbert space, every ball is weakly compact, so every subsequence of (u_n) has a weakly convergent subsequence. As (u_n) converges pointwise to u, all the occurring limits must coincide. We infer $u_n \to u$ weakly in $\mathbf{D}$. With this observation the desired statement follows from the inequality

$$0 \le \widetilde{Q}(u - u_n) + (u(o) - u_n(o))^2 = \widetilde{Q}(u) + u(o)^2 + \widetilde{Q}(u_n) + u_n(o)^2 - 2\langle u, u_n\rangle_o,$$

after taking lim sup. ■

COROLLARY 1.10. *Let (b, c) be connected and $u \in \mathbf{D}$. For any natural number N the function $u_N := ((-N) \vee u) \wedge N$ belongs to $\mathbf{D}$, and $\|u_N - u\|_o \to 0$ as $N \to \infty$.*

Proof. Lemma 1.1 shows $\widetilde{Q}(u_N) \le \widetilde{Q}(u)$. Now apply the previous theorem. ■

COROLLARY 1.11. *Let (b, c) be connected and assume $c \equiv 0$. Let (e_n) be a sequence in $\mathbf{D}$ such that*

$$\|e_n - 1\|_o \to 0 \qquad \text{as } n \to \infty.$$

Furthermore, let $u \in \mathbf{D}$ with $0 \le u \le 1$. Then

$$\|e_n \wedge u - u\|_o \to 0 \qquad \text{as } n \to \infty.$$

Proof. Proposition 1.6 shows that convergence with respect to $\|\cdot\|_o$ implies pointwise convergence, hence $e_n \to 1$ pointwise. Since $0 \le u \le 1$, this also yields $e_n \wedge u \to u$ pointwise. Using Theorem 1.9 it suffices to show $\limsup \widetilde{Q}(e_n \wedge u) \le \widetilde{Q}(u)$. By the triangle inequality for $\widetilde{Q}^{1/2}$ and Lemma 1.1 we obtain

$$\begin{aligned}\widetilde{Q}(e_n \wedge u)^{1/2} &= \tfrac{1}{2}\widetilde{Q}(u + e_n - |u - e_n|)^{1/2} \\ &\le \tfrac{1}{2}[\widetilde{Q}(u)^{1/2} + \widetilde{Q}(e_n)^{1/2} + \widetilde{Q}(|u - e_n|)^{1/2}] \le \widetilde{Q}(u)^{1/2} + \widetilde{Q}(e_n)^{1/2}.\end{aligned}$$

Since $c \equiv 0$, the constant function 1 satisfies $\widetilde{Q}(1) = 0$. Hence, $\|e_n - 1\|_o \to 0$ implies $\widetilde{Q}(e_n) \to 0$, as $n \to \infty$. Now the above calculation yields the statement after taking lim sup. ■

1.3. Dirichlet forms. In this section we introduce Dirichlet forms associated with graphs and prove some of their properties. For basic definitions from Dirichlet form theory we refer to Appendix A.1. Let m be a *measure of full support* on V, i.e., a function $m : V \to (0, \infty)$ which induces a measure by setting

$$m(A) := \sum_{x\in A} m(x).$$

The sets

$$\ell^p(V,m) := \Big\{u : V \to \mathbb{R} \;\Big|\; \sum_{x\in V} |u(x)|^p m(x) < \infty\Big\}$$

endowed with the norm

$$\|u\|_p := \Big(\sum_{x\in V} |u(x)|^p m(x)\Big)^{1/p}$$

are Banach spaces. Moreover, the case $p = 2$ provides a Hilbert space with inner product given by

$$\langle u, v\rangle := \sum_{x\in V} u(x)v(x)m(x).$$

For non-negative $u, v \in C(V)$ we abuse notation and let

$$\langle u, v\rangle := \sum_{x\in V} u(x)v(x)m(x) \in [0,\infty].$$

As usual, $\ell^\infty(V)$ denotes the set of all bounded functions on V, equipped with the norm

$$\|u\|_\infty := \sup_{x\in V} |u(x)|.$$

In the subsequent chapters we are concerned with restrictions of $\widetilde{Q}$ to certain ℓ^2-domains such that the emerging forms are Dirichlet forms. There is a maximal and a minimal choice for such domains. The maximal ℓ^2-domain is given by

$$D(Q^{(N)}) := \widetilde{D} \cap \ell^2(V,m).$$

We call $\widetilde{Q}|_{D(Q^{(N)})}$ the *Neumann form associated to* (b,c) and write $Q^{(N)}$ for short. The following summarizes properties of $Q^{(N)}$ and its associated operator.

Proposition 1.12. *$Q^{(N)}$ is a Dirichlet form. Its associated self-adjoint operator $L^{(N)}$ is a restriction of $\widetilde{L}$ with domain $D(L^{(N)})$ satisfying*

$$D(L^{(N)}) \subseteq \{u \in \widetilde{D} \cap \ell^2(V,m) \mid \widetilde{L}u \in \ell^2(V,m)\}.$$

Proof. We first show the closedness of $Q^{(N)}$. Let (u_n) be a Cauchy sequence in $D(Q^{(N)})$ with respect to the inner product $Q^{(N)}(\cdot,\cdot)+\langle\cdot,\cdot\rangle$. It has a limit $u \in \ell^2(V,m)$ with respect to $\|\cdot\|_2$. Since ℓ^2-convergence implies pointwise convergence, Fatou's lemma yields

$$\widetilde{Q}(u - u_n) \le \liminf_{m\to\infty} \widetilde{Q}(u_m - u_n) = \liminf_{m\to\infty} Q^{(N)}(u_m - u_n).$$

Hence $u \in D(Q^{(N)})$ and $u_n \to u$ with respect to $Q^{(N)}(\cdot,\cdot) + \langle\cdot,\cdot\rangle$, i.e., $Q^{(N)}$ is closed.

Next we turn to the Markov property of $Q^{(N)}$. Let $F : \mathbb{R} \to \mathbb{R}$ be a normal contraction. Lemma 1.1 shows that for $u \in \widetilde{D}$,

$$\widetilde{Q}(F \circ u) \le \widetilde{Q}(u).$$

If furthermore $u \in \ell^2(V,m)$ then also $F \circ u \in \ell^2(V,m)$. This proves the Markov property of $Q^{(N)}$.

For the statement on the associated operator let $u \in D(L^{(N)})$. Using the notation $\hat{\delta}_x = m(x)^{-1}\delta_x$ and Lemma 1.4 we obtain

$$(L^{(N)}u)(x) = \langle L^{(N)}u, \hat{\delta}_x\rangle = Q^{(N)}(u, \hat{\delta}_x) = \langle \widetilde{L}u, \hat{\delta}_x\rangle = (\widetilde{L}u)(x).$$

Consequently, $L^{(N)}$ is a restriction of $\widetilde{L}$ and

$$D(L^{(N)}) \subseteq \{u \in \widetilde{D} \cap \ell^2(V,m) \mid \widetilde{L}u \in \ell^2(V,m)\}. \blacksquare$$

The second important choice for a domain is given by

$$D(Q^{(D)}) = \overline{C_c(V)}^{\|\cdot\|_Q},$$

where the closure is taken with respect to the form norm

$$\|\cdot\|_Q := \sqrt{\widetilde{Q}(\cdot) + \|\cdot\|_2^2}$$

in $\ell^2(V,m) \cap \widetilde{D}$. We call $\widetilde{Q}|_{D(Q^{(D)})}$ the *regular Dirichlet form associated to* (b,c) and denote it by $Q^{(D)}$. It might be thought of as being the minimal closed ℓ^2-restriction of $\widetilde{Q}$ containing $C_c(V)$.

PROPOSITION 1.13. *$Q^{(D)}$ is a regular Dirichlet form. Its associated operator $L^{(D)}$ is a restriction of $\widetilde{L}$ with domain*

$$D(L^{(D)}) = \{u \in D(Q^{(D)}) \mid \widetilde{L}u \in \ell^2(V,m)\}.$$

Proof. The closedness of $Q^{(D)}$ follows from the closedness of $Q^{(N)}$ and the fact that $D(Q^{(D)})$ is a closed subspace of $D(Q^{(N)})$ with respect to $\|\cdot\|_Q$.

To prove the Markov property of $Q^{(D)}$ we let F be a normal contraction and $u \in D(Q^{(D)})$. It suffices to show $F \circ u \in D(Q^{(D)})$. By the definition of $D(Q^{(D)})$ there exists a sequence (u_n) in $C_c(V)$ converging to u with respect to $\|\cdot\|_Q$. Obviously,

$$\|F \circ u_n - F \circ u\|_2 \to 0 \quad \text{as } n \to \infty.$$

Furthermore,

$$\limsup_{n\to\infty} \widetilde{Q}(F \circ u_n) \leq \limsup_{n\to\infty} \widetilde{Q}(u_n) = \widetilde{Q}(u).$$

Therefore, the sequence $(F \circ u_n)$ is bounded in the Hilbert space $(D(Q^{(N)}), \|\cdot\|_Q)$. By the weak compactness of balls in Hilbert spaces we conclude that any of its subsequences has a weakly convergent subsequence. Because of the ℓ^2-convergence of $(F\circ u_n)$ all the occurring limits must coincide with $F \circ u$, and we infer $F \circ u_n \to F \circ u$ weakly in $(D(Q^{(N)}), \|\cdot\|_Q)$. This shows that $F \circ u$ belongs to the weak closure of $C_c(V)$ in $(D(Q^{(N)}), \|\cdot\|_Q)$. Since $C_c(V)$ is convex, this weak closure coincides with the closure of $C_c(V)$ with respect to $\|\cdot\|_Q$, and hence $F \circ u \in D(Q^{(D)})$.

We now turn to the statement on the operator $L^{(D)}$. With the same arguments as in the proof of Proposition 1.12 we can show that $L^{(D)}$ is a restriction of $\widetilde{L}$ and

$$D(L^{(D)}) \subseteq \{u \in D(Q^{(D)}) \mid \widetilde{L}u \in \ell^2(V,m)\}.$$

It remains to prove the opposite inclusion. Let $v \in \{u \in D(Q^{(D)}) \mid \widetilde{L}u \in \ell^2(V,m)\}$. By the correspondence of $Q^{(D)}$ and $L^{(D)}$ (see Appendix A.1) we need to prove

$$Q^{(D)}(v,w) = \langle \widetilde{L}v, w\rangle$$

for all $w \in D(Q^{(D)})$. Lemma 1.4 shows that the above equality holds for all $w \in C_c(V)$. Since $C_c(V)$ is dense in $D(Q^{(D)})$ with respect to $\|\cdot\|_Q$, it extends to all $w \in D(Q^{(D)})$. $\blacksquare$

Remark 1.14. • The previous proof and the one of Theorem 1.9 use a similar argument. They show that a bounded sequence in a Hilbert space that is convergent in some 'weak' sense (pointwise, in $\ell^2, \ldots,$) is already weakly convergent in that Hilbert space.

• There seems to be no explicit proof of $Q^{(D)}$ being a Dirichlet form in the literature. Usually this property is deduced from [4, Theorem 3.1.1], which uses general principles.

• The given characterization of the domain of $L^{(D)}$ seems to be new. For further properties we refer the reader to [9, 15].

We now come to the main objects of our studies.

Definition 1.15 (Dirichlet form associated with (b,c)). Let (b,c) be a graph over V. A Dirichlet form Q is said to be *associated to* (b,c) if it is a restriction of $\widetilde{Q}_{b,c}$ and its domain $D(Q)$ satisfies

$$D(Q^{(D)}) \subseteq D(Q) \subseteq D(Q^{(N)}).$$

Remark 1.16. Dirichlet forms associated with graphs seem to be a special class of examples of Dirichlet forms on $\ell^2(V,m)$. However, it turns out that every regular Dirichlet form on a discrete space coincides with a regular Dirichlet form associated to a graph (b,c) (see the discussion in Appendix A.2). In this sense the forms $Q^{(D)}$ are exactly the regular Dirichlet forms on discrete spaces.

2. General theory

In this chapter we study recurrence and transience of Dirichlet forms which are associated to a graph (b,c) by using the theory presented in [4, Section 1.5]. The discrete structure of the underlying ℓ^2-space allows us to simplify many technical details. Therefore, some definitions and statements differ slightly from the ones found in [4].

Let Q be a Dirichlet form associated to a graph (b,c) and let e^{-tL} be its associated semigroup. Recall that e^{-tL} is positivity preserving, i.e., it maps non-negative functions to non-negative functions (see Appendix A.1). Therefore, its *Green's function* $G : V \times V \to [0,\infty]$ with

$$G(x,y) := \int_0^\infty e^{-tL}\delta_x(y)\,dt$$

is well-defined.

Definition 2.1 (Recurrent/transient semigroup). The semigroup e^{-tL} is called *transient* if its Green's function satisfies $G(x,y) < \infty$ for all $x,y \in V$. It is called *recurrent* if $G(x,y) = \infty$ for all $x,y \in V$.

The next proposition shows the dichotomy of recurrence and transience of the semigroup whenever the graph (b,c) is connected.

Proposition 2.2. *Let (b,c) be connected. The semigroup e^{-tL} is transient if and only if there exist some $x,y \in V$ such that $G(x,y) < \infty$. In particular, e^{-tL} is either recurrent or transient.*

Proof. Let $x, y \in V$ be such that $G(x,y) < \infty$ and let $w, z \in V$ be arbitrary. We need to show $G(w,z) < \infty$. The functions $\widetilde{\delta}_v = m(v)^{-1/2}\delta_v$ form an orthonormal basis in $\ell^2(V,m)$. Using the semigroup property and the fact that e^{-tL} is positivity preserving (see Appendix A.1) for $t > 1$ we then obtain

$$\begin{aligned}(e^{-tL}\delta_x)(y) &= (e^{-(t-1)L}e^{-L}\delta_x)(y) = \Big[e^{-(t-1)L}\Big(\sum_{v\in V}\langle e^{-L}\delta_x, \widetilde{\delta}_v\rangle\widetilde{\delta}_v\Big)\Big](y)\\ &= \frac{1}{m(y)}\sum_{v\in V}\langle e^{-L}\delta_x, \widetilde{\delta}_v\rangle\langle e^{-(t-1)L}\widetilde{\delta}_v, \delta_y\rangle\\ &\geq \frac{1}{m(y)}\langle e^{-L}\delta_x, \widetilde{\delta}_w\rangle\langle e^{-(t-1)L}\widetilde{\delta}_w, \delta_y\rangle. \end{aligned}\tag{2.1}$$

Since (b,c) is connected, Theorem A.7 shows that e^{-tL} is positivity improving (see Definition A.6), hence

$$\langle e^{-L}\delta_x, \widetilde{\delta}_w\rangle > 0.$$

We can now deduce the finiteness of $G(w,y)$ by integrating both sides of (2.1) from 1 to ∞. A similar computation shows the finiteness of $G(w,z)$. ∎

The next proposition shows how transience of the semigroup is related to the resolvent $(L+\alpha)^{-1}$ of Q.

PROPOSITION 2.3. *For all $x, y \in V$,*

$$G(x,y) = \int_0^\infty e^{-tL}\delta_x(y)\,dt = \lim_{\alpha\to 0+}(L+\alpha)^{-1}\delta_x(y).$$

Proof. By the monotone convergence theorem we infer that

$$\int_0^\infty e^{-tL}\delta_x(y)\,dt = \lim_{\alpha\to 0+}\int_0^\infty e^{-t\alpha}e^{-tL}\delta_x(y)\,dt.$$

Let $\mu_{x,y}$ be the spectral measure associated with L such that

$$\frac{1}{m(y)}\langle e^{-tL}\delta_x, \delta_y\rangle = \int_0^\infty e^{-t\lambda}\,d\mu_{x,y}(\lambda).$$

Since $e^{-t(\alpha+\lambda)}$ is integrable in t on $[0,\infty)$ and $\mu_{x,y}$ is of bounded total variation, we can use Fubini's theorem to obtain

$$\begin{aligned}\int_0^\infty e^{-t\alpha}e^{-tL}\delta_x(y)\,dt &= \int_0^\infty\int_0^\infty e^{-t\alpha}e^{-t\lambda}\,d\mu_{x,y}(\lambda)\,dt = \int_0^\infty\int_0^\infty e^{-t(\alpha+\lambda)}\,dt\,d\mu_{x,y}(\lambda)\\ &= \int_0^\infty \frac{1}{\lambda+\alpha}\,d\mu_{x,y}(\lambda) = (L+\alpha)^{-1}\delta_x(y). \ ∎\end{aligned}$$

There is also a notion of recurrence and transience for Dirichlet forms. We call a function $g : V \to [0,\infty)$ a *reference function for Q* if

$$\sum_{x\in V}|u(x)|g(x)m(x) \leq \sqrt{Q(u)} \quad \text{for all } u \in D(Q).$$

DEFINITION 2.4 (Recurrent/transient Dirichlet form). A Dirichlet form Q is called *transient* if it has a strictly positive reference function $g \in \ell^1(V,m)\cap\ell^\infty(V)$. It is called *recurrent* if the constant function 0 is its only reference function.

Just as for semigroups, recurrence and transience of Dirichlet forms are mutually exclusive whenever the underlying graph is connected.

Lemma 2.5. *Let (b,c) be connected. If Q has a non-trivial reference function, it also has a strictly positive reference function in $\ell^1(V,m)\cap\ell^\infty(V)$. In particular, Q is either recurrent or transient.*

Proof. Let g be a non-trivial reference function and assume $g(w)>0$. We obtain

$$|u(w)| \leq \frac{1}{g(w)}\sum_{x\in V} g(x)|u(x)| \leq \sqrt{Q(u)} \quad \text{for all } u\in D(Q).$$

Furthermore, Lemma 1.5 implies that for each $x\in V$ there exists a constant $K_x>0$ such that

$$|u(x)-u(w)| \leq K_x\sqrt{Q(u)} \quad \text{for all } u\in D(Q).$$

Combining both inequalities yields the existence of $C_x>0$ with

$$|u(x)| \leq C_x\sqrt{Q(u)} \quad \text{for all } u\in D(Q).$$

Thus, for a suitable choice of coefficients (a_x) the function

$$\widetilde{g}(x) := \frac{a_x}{C_x m(x)}$$

is a strictly positive reference function in $\ell^1(V,m)\cap\ell^\infty(V)$. ∎

We will show next that a Dirichlet form Q is recurrent/transient if and only if its associated semigroup is recurrent/transient.

The semigroup (e^{-tL}) is a strongly continuous family of bounded operators on $\ell^2(V,m)$. Thus, the real function $t\mapsto e^{-tL}f(x)$ is continuous for any $f\in\ell^2(V,m)$ and any $x\in V$. We abuse notation and define the 0*th order resolvent operator*

$$G:\ell^2_+(V,m)\to\{u:V\to[0,\infty]\} \quad \text{by} \quad (Gf)(x)=\int_0^\infty e^{-tL}f(x)\,dt.$$

Here $\ell^2_+(V,m)$ denotes the set of all non-negative ℓ^2-functions. Note that $(Gf)(x)$ may take the value ∞.

Remark 2.6. It is immediate from the definition that Green's function of Q satisfies $G(x,y)=(G\delta_x)(y)$. In this sense it is the integral kernel of the 0th order resolvent operator with respect to the counting measure.

The following lemma will allow us to prove the equivalence of Definitions 2.1 and 2.4; it is a variant of [4, Lemma 1.5.3].

Lemma 2.7. *For a non-negative $g\in\ell^1(V,m)\cap\ell^2(V,m)$,*

$$\sup_{u\in D(Q)}\frac{\langle |u|,g\rangle}{\sqrt{Q(u)}} = \sqrt{\langle g,Gg\rangle}.$$

In particular,

$$\sup_{u\in D(Q)}\frac{\langle |u|,g\rangle}{\sqrt{Q(u)}} < \infty \quad \textit{if and only if} \quad \langle g,Gg\rangle<\infty.$$

Proof. For $f \in \ell^2(V,m)$ and $x \in V$ we use the notation

$$(S_t f)(x) := \int_0^t e^{-sL} f(x)\, ds.$$

With the help of Theorem A.8 we conclude that

$$\|S_t f\|_2 \le \int_0^t \|e^{-sL} f\|_2\, ds \le t\|f\|_2.$$

Therefore, $S_t f$ is a bounded linear operator on $\ell^2(V,m)$. The proof is in three steps.

STEP 1. For $f \in \ell^2(V,m)$ and $h \in D(Q)$ we prove the identity

$$Q(S_t f, h) = \langle f - e^{-tL} f, h\rangle \tag{2.2}$$

by showing that $S_t f \in D(L)$ and $LS_t f = f - e^{-tL} f$. Using the correspondence of L and e^{-tL} we need to compute the derivative of the function $[0,\infty) \to \ell^2(V,m)$, $s \mapsto -e^{-sL} S_t f$, at 0 (see Appendix A.1). Theorem A.9 yields

$$\frac{S_t f - e^{-sL} S_t f}{s} = \frac{S_t f - S_t e^{-sL} f}{s} = \frac{1}{s}\Big(\int_0^s e^{-uL} f\, du - \int_t^{s+t} e^{-uL} f\, du\Big).$$

With the help of Theorem A.8 and the mean value theorem for Riemann integrals we compute

$$\begin{aligned}\Big\|\frac{1}{s}\int_0^s e^{-uL} f\, du - f\Big\|_2 &= \Big\|\frac{1}{s}\int_0^s e^{-uL} f - f\, du\Big\|_2 \\ &\le \frac{1}{s}\int_0^s \|e^{-uL} f - f\|_2\, du = \|e^{-\theta L} f - f\|_2\end{aligned}$$

where $\theta \in (0,s)$. Taking the limit as $s \to 0$ yields

$$\frac{1}{s}\int_0^s e^{-uL} f\, du \to f.$$

An analogous computation shows

$$\frac{1}{s}\int_t^{s+t} e^{-uL} f\, du \to e^{-tL} f \qquad \text{as } s \to 0.$$

This implies (2.2).

STEP 2. We show that

$$c := \sup_{u \in D(Q)} \frac{\langle |u|, g\rangle}{\sqrt{Q(u)}} < \infty$$

implies $\sqrt{\langle g, Gg\rangle} \le c$. Since $e^{-tL} g \ge 0$ and $S_t g \ge 0$ we obtain

$$\langle g, S_t g\rangle \le c\sqrt{Q(S_t g)} \overset{(2.2)}{=} c\sqrt{\langle g - e^{-tL} g, S_t g\rangle} \le c\sqrt{\langle g, S_t g\rangle}.$$

We conclude $\sqrt{\langle S_t g, g\rangle} \le c$. Letting $t \to \infty$ we obtain $\sqrt{\langle g, Gg\rangle} \le c$ by the monotone convergence theorem.

STEP 3. Suppose that $\langle g, Gg\rangle < \infty$. By monotone convergence and semigroup properties, we get

$$\langle g, Gg\rangle = \int_0^\infty \langle e^{-tL} g, g\rangle\, dt = \int_0^\infty \langle e^{-t/2L} g, e^{-t/2L} g\rangle\, dt.$$

The function $t \mapsto \langle e^{-t/2L}g, e^{-t/2L}g\rangle$ is non-increasing in t (use the semigroup property and $\|e^{-sL}\| \le 1$). Thus, the finiteness of $\langle g, Gg\rangle$ yields

$$\lim_{t\to\infty} \langle e^{-tL}g, g\rangle = \lim_{t\to\infty} \langle e^{-t/2L}g, e^{-t/2L}g\rangle = 0.$$

On account of (2.2) for $u \in D(Q)$ we obtain

$$\begin{aligned}
\langle |u|, g\rangle &\overset{(2.2)}{=} Q(|u|, S_t g) + \langle |u|, e^{-tL}g\rangle \\
&\le \sqrt{Q(S_t g)}\sqrt{Q(|u|)} + \sqrt{\langle e^{-tL}g, e^{-tL}g\rangle}\sqrt{\langle u, u\rangle} \\
&\overset{(2.2)}{=} \sqrt{\langle g - e^{-tL}g, S_t g\rangle}\sqrt{Q(|u|)} + \sqrt{\langle e^{-tL}g, e^{-tL}g\rangle}\sqrt{\langle u, u\rangle} \\
&\le \sqrt{\langle S_t g, g\rangle}\sqrt{Q(u)} + \sqrt{\langle e^{-2tL}g, g\rangle}\sqrt{\langle u, u\rangle}.
\end{aligned}$$

The inequality $\langle |u|, g\rangle \le \sqrt{\langle g, Gg\rangle}\sqrt{Q(u)}$ follows by letting $t \to \infty$. ■

The next theorem shows that transience of a Dirichlet form Q and of its associated semigroup coincide. It is a variant of [4, Theorem 1.5.1].

Theorem 2.8. *A Dirichlet form Q is transient if and only if its associated semigroup e^{-tL} is transient.*

Proof. Assume Q is transient and let $g \in \ell^1(V, m) \cap \ell^\infty(V) \subseteq \ell^2(V, m)$ be a strictly positive reference function of Q. Since by definition of reference function, we have

$$\sum_{x\in V} |u(x)|g(x)m(x) \le \sqrt{Q(u)}$$

for all $u \in D(Q)$, Lemma 2.7 shows $\langle g, Gg\rangle \le 1$. As g is strictly positive, this implies that $Gg(x) < \infty$ for all $x \in V$. Therefore, the non-negativity of $e^{-tL}\delta_x$ and the self-adjointness of e^{-tL} yield

$$\begin{aligned}
\int_0^\infty e^{-tL}\delta_x(y)\,dt &= \frac{1}{g(y)m(y)}\int_0^\infty \langle e^{-tL}\delta_x, g(y)\delta_y\rangle\,dt \\
&\le \frac{1}{g(y)m(y)}\int_0^\infty \langle e^{-tL}\delta_x, g\rangle\,dt = \frac{m(x)}{g(y)m(y)}\int_0^\infty e^{-tL}g(x)\,dt < \infty.
\end{aligned}$$

Conversely, assume that e^{-tL} is transient, i.e., its Green's function satisfies $G(x, y) < \infty$ for all $x, y \in V$. By Remark 2.6,

$$G(x, y) = G\delta_x(y)$$

for all $x, y \in V$. Hence, an application of Lemma 2.7 to δ_x yields

$$\sup_{u\in D(Q)} \frac{|u(x)|m(x)}{\sqrt{Q(u)}} = G(x, x)m(x).$$

Since $D(Q)$ is dense in $\ell^2(V, m)$, for every $x \in V$ there exists $v_x \in D(Q)$ such that $|v_x(x)| > 0$. Therefore the above equation shows $G(x, x) > 0$ for every $x \in V$. We need to find a reference function g as in Definition 2.4. Define

$$g(x) = \frac{a_x}{G(x, x)m(x)},$$

where (a_x) is a sequence of strictly positive numbers such that $g \in \ell^1(V,m) \cap \ell^\infty(V)$ and

$$\sum_{x \in V} a_x = 1.$$

For $u \in D(Q)$ we obtain

$$\sum_{x \in V} |u(x)| g(x) m(x) = \sum_{x \in V} \frac{|u(x)| a_x}{G(x,x)} \leq \sum_{x \in V} a_x \sqrt{Q(u)} = \sqrt{Q(u)},$$

as was to be shown. ■

COROLLARY 2.9. *Let (b,c) be connected. Then Q is recurrent if and only if e^{-tL} is recurrent.*

Proof. This follows from the previous theorem and the fact that recurrence/transience of Q and of e^{-tL} are mutually exclusive when the underlying graph is connected (see Proposition 2.2 and Lemma 2.5). ■

REMARK 2.10. The previous corollary is also true without (b,c) being connected. In the following chapters we only deal with connected graphs. Therefore, we refrain from giving details.

Next we introduce the extended Dirichlet space $D(Q)_e$ of the form Q. It will turn out that properties of $D(Q)_e$ are related to recurrence and transience of Q.

DEFINITION 2.11 (Extended Dirichlet space). We call the set

$$D(Q)_e := \{u : V \to \mathbb{R} \mid \text{there exists a } Q\text{-Cauchy sequence } (u_n) \text{ in } D(Q) \text{ such that } u_n \to u \text{ pointwise}\}$$

the *extended Dirichlet space* of Q. A sequence (u_n) as in the definition of $D(Q)_e$ is an *approximating sequence* for u.

The Dirichlet form Q can be extended to the extended Dirichlet space. Indeed, for $u \in D(Q)_e$ and an approximating sequence (u_n) of u the sequence $(Q(u_n))$ has a limit which could serve as a definition for the form on the extended space. The following lemma shows that this is well-defined and compatible with the form $\widetilde{Q}$.

LEMMA 2.12. *Let $u \in D(Q)_e$ and let (u_n) be an approximating sequence for u. Then $u \in \widetilde{D}$ and*

$$\lim_{n \to \infty} \widetilde{Q}(u - u_n) = 0.$$

In particular,

$$\widetilde{Q}(u) = \lim_{n \to \infty} Q(u_n).$$

Proof. By Fatou's lemma we obtain

$$\widetilde{Q}(u - u_n)^{1/2} \leq \liminf_{l \to \infty} \widetilde{Q}(u_l - u_n)^{1/2} = \liminf_{l \to \infty} Q(u_l - u_n)^{1/2}.$$

Since (u_n) is a Q-Cauchy sequence, this yields $u \in \widetilde{D}$ and $\lim \widetilde{Q}(u - u_n) = 0$. ■

For $u, v \in D(Q)_e$ we set

$$Q(u,v) := \widetilde{Q}(u,v)$$

to extend Q to $D(Q)_e$. The previous lemma shows that $D(Q)$ is dense in $D(Q)_e$ with respect to the pseudometric induced by Q on $D(Q)_e$. Whenever the underlying graph (b,c) is connected, $D(Q)_e$ can be computed as follows.

PROPOSITION 2.13. *Let (b,c) be connected. The extended Dirichlet space of a Dirichlet form Q associated to (b,c) is given by the closure of $D(Q)$ in $\mathbf{D}$, i.e.,*

$$D(Q)_e = \overline{D(Q)}^{\|\cdot\|_o}.$$

Proof. Let $u \in D(Q)_e$ and $(u_n) \subseteq D(Q)$ be an approximating sequence for u. Then (u_n) is a Cauchy sequence with respect to $\|\cdot\|_o$. Since $\mathbf{D}$ is complete, (u_n) converges in $\mathbf{D}$ to some function v. By the pointwise convergence of the u_n to u we infer that $u = v$. This shows $u \in \overline{D(Q)}^{\|\cdot\|_o}$. The other inclusion follows from Proposition 1.6. ∎

COROLLARY 2.14. *Let (b,c) be connected and $Q^{(D)}$ be the regular Dirichlet form associated with (b,c) on $\ell^2(X,m)$. Its extended Dirichlet space is given by*

$$D(Q^{(D)})_e = \mathbf{D}_0.$$

Proof. This is a consequence of

$$D(Q)_e = \overline{D(Q^{(D)})}^{\|\cdot\|_o} = \overline{\overline{C_c(V)}^{\|\cdot\|_Q}}^{\|\cdot\|_o} = \overline{C_c(V)}^{\|\cdot\|_o},$$

where the first equality follows from the previous proposition, and the last from the fact that the norm $\|\cdot\|_o$ is continuous with respect to $\|\cdot\|_Q = (\widetilde{Q}(\cdot) + \|\cdot\|^2)^{1/2}$. ∎

REMARK 2.15. That the extended Dirichlet space of Q is the closure of $D(Q)$ in the Yamasaki space $\mathbf{D}$ seems to be a new observation. The equality $D(Q^{(D)})_e = \mathbf{D}_0$ is a link between [4] and [23] which we could not find in the literature.

We now turn to criteria for recurrence and transience. First let us record a necessary condition (see [4, Lemma 1.5.5]).

LEMMA 2.16. *Suppose Q is transient. Then $(D(Q)_e, Q)$ is a Hilbert space.*

Proof. By transience there exists a strictly positive $g \in \ell^1(V,m) \cap \ell^\infty(V)$ such that

$$\langle |u|, g\rangle \leq \sqrt{Q(u)} \quad \text{for all } u \in D(Q). \tag{2.3}$$

By the definition of $D(Q)_e$ and Lemma 2.12 this inequality extends to all $u \in D(Q)_e$. Thus, Q is non-degenerate on $D(Q)_e$. To prove completeness let (u_n) be a Cauchy sequence in $(D(Q)_e, Q)$. Inequality (2.3) implies the pointwise convergence of (u_n) to a function u. This yields $u \in D(Q)_e$ with approximating sequence (u_n). We infer that $Q(u_n - u) \to 0$ as $n \to \infty$ from Lemma 2.12. ∎

Below we prove characterizations for recurrence and transience. In the recurrent case an important step will be to construct a sequence of functions in $D(Q)$ converging to 1 with respect to Q. To obtain such a sequence we study a transient Dirichlet form Q^g which is defined by perturbing Q in the following way. For a strictly positive $g \in \ell^1(V,m) \cap \ell^\infty(V)$ we set

$$Q^g : D(Q) \times D(Q) \to \mathbb{R}, \qquad Q^g(u,v) := Q(u,v) + \langle gu, v\rangle.$$

The following proposition shows that Q^g is indeed a transient Dirichlet form.

PROPOSITION 2.17. *Q^g is a transient Dirichlet form.*

Proof. Obviously, Q^g has the Markov property. As the sum of a closed and a continuous form, it is also closed. To show its transience we compute

$$\sum_{x\in V} |u(x)|g(x)m(x) \le \sqrt{\|g\|_1}\sqrt{\langle gu, u\rangle} \le \sqrt{\|g\|_1}\sqrt{Q^g(u)}.$$

Thus, we can choose $\tilde{g} = g/\sqrt{\|g\|_1}$ as a strictly positive reference function. ∎

Let L^g denote the self-adjoint operator associated with Q^g. The following lemma is the key ingredient for characterizing recurrence. It is a variant of [4, Lemma 1.6.6].

LEMMA 2.18. *Let Q be recurrent and let g be as above. For any $x \in V$,*

$$\lim_{\alpha\to 0+} (L^g + \alpha)^{-1} g(x) = 1.$$

Proof. We write

$$Q_\alpha(u, v) := Q(u, v) + \alpha\langle u, v\rangle.$$

Recall that

$$Q_\alpha(w, u) = \langle f, u\rangle \quad \text{for all } u \in D(Q)$$

if and only if $w = (L + \alpha)^{-1} f$ (see Appendix A.1). The proof will be in three steps.

STEP 1. We compute the resolvent of Q^g in terms of the resolvent of Q. Using the characterization of the resolvent of Q_α, observe that for any $f \in \ell^2(V, m)$ and $u \in D(Q)$,

$$\begin{aligned} Q_\alpha((L^g + \alpha)^{-1} f, u) &= Q^g_\alpha((L^g + \alpha)^{-1} f, u) - \langle g(L^g + \alpha)^{-1} f, u\rangle \\ &= \langle f - g(L^g + \alpha)^{-1} f, u\rangle. \end{aligned}$$

This implies

$$(L^g + \alpha)^{-1} f = (L + \alpha)^{-1}[f - g(L^g + \alpha)^{-1} f]. \tag{2.4}$$

STEP 2. We show $(L^g + \alpha)^{-1} g \le 1$ by considering Q as a Dirichlet form on the perturbed space $\ell^2(V, (g + \alpha)m)$. Since we have assumed that g is positive and bounded, the measures $(g + \alpha)m$ and m are equivalent. Thus, Q is closed on $\ell^2(V, (g + \alpha)m)$. For $f \in \ell^2(V, (g + \alpha)m)$ and $v \in D(Q)$,

$$\begin{aligned} &Q((L^g + \alpha)^{-1}[\alpha f + fg], v) + \langle (L^g + \alpha)^{-1}[\alpha f + fg], v\rangle_{(g+\alpha)m} \\ &\qquad = Q^g((L^g + \alpha)^{-1}[\alpha f + fg], v) + \alpha\langle (L^g + \alpha)^{-1}[\alpha f + fg], v\rangle \\ &\qquad = Q^g_\alpha((L^g + \alpha)^{-1}[\alpha f + fg], v) = \langle \alpha f + gf, v\rangle = \langle f, v\rangle_{(\alpha+g)m}. \end{aligned}$$

From this calculation we infer that $(L^g + \alpha)^{-1}[\alpha f + fg]$ is the first-order resolvent of f associated to Q as a Dirichlet form on $\ell^2(V, (g + \alpha)m)$. The Markov property of this resolvent (see Appendix A.1) implies that for any $f \in \ell^2(V, (g + \alpha)m)$ with $0 \le f \le 1$,

$$(L^g + \alpha)^{-1}[\alpha f + fg] \le 1. \tag{2.5}$$

Pick a sequence (f_n) in $\ell^2(V, (g + \alpha)m)$ such that $0 \le f_n \le f_{n+1} \le 1$ and $\lim f_n(x) = 1$ for all $x \in V$. Then $f_n g \to g$ in $\ell^2(V, m)$. Inequality (2.5) together with the non-negativity of $(L^g + \alpha)^{-1}(\alpha f_n)$ yields

$$(L^g + \alpha)^{-1} g(x) = \lim_{n\to\infty} (L^g + \alpha)^{-1}(f_n g)(x) \le \limsup_{n\to\infty} [1 - (L^g + \alpha)^{-1}(\alpha f_n)(x)] \le 1.$$

STEP 3. Equation (2.4) applied to g and Step 2 imply

$$\begin{aligned}1 &\geq \limsup_{\alpha\to 0+}(L^g+\alpha)^{-1}g(x) = \limsup_{\alpha\to 0+}(L+\alpha)^{-1}[g-g(L^g+\alpha)^{-1}g](x)\\ &= \limsup_{\alpha\to 0+}\frac{1}{m(x)}\langle g(1-(L^g+\alpha)^{-1}g),(L+\alpha)^{-1}\delta_x\rangle\\ &\geq \limsup_{\alpha\to 0+}\frac{g(x)(1-(L^g+\alpha)^{-1}g(x))}{m(x)}\langle \delta_x,(L+\alpha)^{-1}\delta_x\rangle \geq 0.\end{aligned}$$

Since $(L+\alpha)^{-1}\delta_x(x)$ converges to infinity by our assumptions (Q was supposed to be recurrent), $(L^g+\alpha)^{-1}g(x)$ must tend to 1. ∎

We can now prove the main theorems of this chapter. For general Dirichlet forms they can be found in [4, Theorems 1.6.2 and 1.6.3].

THEOREM 2.19 (Abstract characterization of transience). *Let (b,c) be connected and Q a Dirichlet form associated with (b,c). The following conditions are equivalent:*

(i) e^{-tL} *is transient.*
(ii) $u\in D(Q)_e$ *and* $Q(u)=0$ *implies* $u=0$.
(iii) $(D(Q)_e,Q)$ *is a Hilbert space.*

Proof. (i)⇒(iii) has already been shown in Lemma 2.16.

(iii)⇒(ii) is trivial.

(ii)⇒(i). Assume e^{-tL} is recurrent. Then, for a strictly positive $g\in\ell^1(V,m)\cap\ell^\infty(V)$, Lemma 2.18 implies $u_n := (L^g+1/n)^{-1}g\to 1$ pointwise. Furthermore, by the correspondence $(L^g+\alpha)^{-1}\leftrightarrow Q^g$ we obtain

$$Q(u_n,u_n)=\langle g,u_n\rangle-\frac{1}{n}\langle u_n,u_n\rangle-\langle gu_n,u_n\rangle.$$

Because $g\in\ell^1(V,m)$ and u_n is uniformly bounded by 1 (see Step 2 of the previous proof), we infer from Lebesgue's theorem that

$$\lim_{n\to\infty}\langle g,u_n\rangle=\lim_{n\to\infty}\langle gu_n,u_n\rangle=\sum_{x\in V}g(x)m(x).$$

This shows

$$Q(u_n,u_n)\leq\langle g,u_n\rangle-\langle gu_n,u_n\rangle\to 0 \quad \text{as } n\to\infty.$$

The above computations imply $1\in D(Q)_e$ and Lemma 2.12 yields $Q(1)=0$. This contradiction to (ii) shows that e^{-tL} is not recurrent. Since (b,c) is connected, this implies transience. ∎

THEOREM 2.20 (Abstract characterization of recurrence). *Let (b,c) be connected and Q a Dirichlet form associated with (b,c). The following assertions are equivalent:*

(i) e^{-tL} *is recurrent.*
(ii) *There exists a sequence (u_n) in $D(Q)$ with* $\lim u_n=1$ *pointwise and* $\lim Q(u_n)=0$.
(iii) $1\in D(Q)_e$ *and* $Q(1)=0$.

Proof. (i)⇒(ii) has already been shown in the proof of Theorem 2.19.

(ii)⇒(iii) follows from the definition of $D(Q)_e$ and Lemma 2.12.

(iii)⇒(i). By the second statement of Theorem 2.19, (iii) implies non-transience of e^{-tL}. Connectedness of (b, c) implies recurrence. ■

As a consequence of these theorems, let us note that a non-vanishing potential c implies transience.

Corollary 2.21. *Let (b, c) be a connected graph such that $c \not\equiv 0$. Assume Q is a Dirichlet form associated with (b, c). Then Q is transient.*

Proof. If 1 belonged to $D(Q)_e$, we would obtain

$$Q(1) = \widetilde{Q}(1) = \sum_{x \in V} c(x) \neq 0.$$

Then Theorem 2.20 implies that Q is not recurrent since condition (iii) would fail. Because (b, c) is connected, we infer the transience of Q. ■

Remark 2.22. The previous result is certainly well-known, as it immediately follows from [4, Theorem 1.6.5]. Because of it we will assume $c \equiv 0$ in what follows whenever we deal with recurrence/transience.

When dealing with the regular Dirichlet form associated to a graph, the previous theorems show that recurrence does not depend on the underlying measure m. Namely, the following holds.

Corollary 2.23. *Let (b, c) be connected. Let m_1 and m_2 be two measures of full support on V. Let $Q^{(D)}_{m_i}$ be the regular Dirichlet form associated with (b, c) on $\ell^2(V, m_i)$, $i = 1, 2$. Then $Q^{(D)}_{m_1}$ is recurrent if and only if $Q^{(D)}_{m_2}$ is recurrent.*

Proof. We have seen in Corollary 2.14 that the extended Dirichlet space of $Q^{(D)}_{m_1}$ and of $Q^{(D)}_{m_2}$ is given by $\mathbf{D}_0$. As $\mathbf{D}_0$ does not depend on the underlying measure and the previous theorems show that recurrence can be characterized in terms of the extended Dirichlet space, the claim follows. ■

Remark 2.24. The previous corollary is certainly well-known to experts. Nevertheless, it is quite remarkable, as the domain of $Q^{(D)}$ heavily depends on m. In fact, the other global properties which are studied in Chapter 5 also depend on the underlying measure.

3. Discrete time vs. continuous time

In this chapter we compare the notion of recurrence of a discrete-time Markov chain with the one developed in the previous chapter. This notion is usually used in textbooks for defining a recurrent graph (see e.g. [23, 25]).

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. A sequence of random variables $(X_n)_{n \geq 0}$ on Ω with values in V is called a *random walk associated with* $(b, 0)$ if

$$\mathbb{P}(X_{n+1} = y \mid X_n = x) = \frac{b(x, y)}{\deg(x)} \quad \text{for all } x, y \in V,\ n \geq 0$$

and

$$\mathbb{P}(X_{n+1} = x_{n+1} \mid X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) = \mathbb{P}(X_{n+1} = x_{n+1} \mid X_n = x_n)$$

for all $n \geq 0$ and all $x_0, \ldots, x_{n+1} \in V$. We skip the proof of the existence of such a Markov chain since this is standard.

Given a random walk associated with $(b, 0)$ one might ask about the long-time behavior of $(X_n)_{n\geq 0}$. In particular the question whether $(X_n)_{n\geq 0}$ returns to a particular point infinitely often is of interest.

Definition 3.1 (Recurrent/transient random walk). A random walk $(X_n)_{n\geq 0}$ associated with $(b, 0)$ is called *recurrent* if

$$\mathbb{P}(X_n = y \text{ infinitely often} \mid X_0 = x) = 1$$

for all $x, y \in V$. It is called *transient* if for all $x, y \in V$ the above probability is strictly less than 1.

Remark 3.2. Usually one calls a graph $(b, 0)$ recurrent whenever the random walk associated with it is recurrent (see e.g. [23, 25]). The known criteria for recurrence and transience are usually proven in this context.

Let P be the transition matrix of the random walk associated with $(b, 0)$, i.e., the infinite matrix with entries

$$P(x, y) = b(x, y)/\deg(x).$$

The following theorem relates recurrence of the random walk of a graph to the matrix P. It is a standard exercise in Markov chain theory (see e.g. [3]).

Theorem 3.3. *Let $(b, 0)$ be connected and let $(X_n)_{n\geq 0}$ be the random walk associated with it. Then $(X_n)_{n\geq 0}$ is recurrent if and only if*

$$\sum_{n=0}^{\infty} P^{(n)}(x, y) = \infty$$

for all $x, y \in V$. Here $P^{(n)}$ denotes powers of the matrix P.

We now discuss the relation of recurrence of Dirichlet forms and recurrence of the random walk associated to a weighted graph. Assume $m \equiv 1$. We write $\widetilde{L} = D - A$, where D and A are the two infinite matrices with entries given by

$$D(x, y) = \begin{cases} \deg(x) & \text{if } x = y, \\ 0 & \text{if } x \neq y, \end{cases} \qquad A(x, y) = \begin{cases} 0 & \text{if } x = y, \\ b(x, y) & \text{if } x \neq y. \end{cases}$$

Then $P = D^{-1}A$ is the transition matrix of the random walk associated to $(b, 0)$ (here D^{-1} is the *formal inverse* of D, i.e., the infinite matrix having $\deg^{-1}$ on its diagonal). The following theorem relates Green's function of $Q^{(D)}$ to the transition matrix P.

Theorem 3.4. *Let $(b, 0)$ be connected and let $e^{-tL^{(D)}}$ be the semigroup associated with $Q^{(D)}$ on $\ell^2(V, 1)$. For any $x, y \in V$,*

$$G(x, y) = \int_0^{\infty} e^{-tL^{(D)}} \delta_x(y)\, dt = \frac{1}{\deg(x)} \sum_{n=0}^{\infty} P^{(n)}(x, y).$$

Proof. Let $x, y \in V$. By Proposition 2.3 we obtain

$$\int_0^\infty e^{-tL^{(D)}} \delta_x(y)\, dt = \lim_{\alpha \to 0+} (L^{(D)} + \alpha)^{-1} \delta_x(y).$$

Pick an exhaustion (K_i) of V (i.e. $K_i \subseteq K_{i+1}$ and $\bigcup K_i = V$) with finite K_i and $x \in K_1$. Theorem A.5 shows that the resolvent of $Q^{(D)}$ can be approximated by

$$(L^{(D)} + \alpha)^{-1} \delta_x = \lim_{i \to \infty} (L_{K_i} + \alpha)^{-1} \delta_x,$$

where L_{K_i} is the restriction of $\widetilde{L}$ to $C(K_i)$ (in the sense described in Theorem A.5). We now compute the right-hand side of the previous equation.

We denote by A_{K_i}, D_{K_i} the restrictions of A, D to $C(K_i)$ in the sense of Theorem A.5, i.e., $A_{K_i} := p_{K_i} A i_{K_i}$ and $D_{K_i} := p_{K_i} A i_{K_i}$, where $i_{K_i} : C(K_i) \to C(V)$ is the canonical inclusion and $p_{K_i} : C(V) \to C(K_i)$ is the restriction of a function to K_i. With this notation we obtain the matrix identity

$$L_{K_i} + \alpha = D_{K_i} - A_{K_i} + \alpha = (D_{K_i} + \alpha)(I - (D_{K_i} + \alpha)^{-1} A_{K_i}) \tag{3.1}$$

on the finite-dimensional space $C(K_i)$. The matrix $I - (D_{K_i} + \alpha)^{-1} A_{K_i}$ is invertible since the operator norm of $(D_{K_i} + \alpha)^{-1} A_{K_i}$ considered as an operator on $\ell^\infty(K_i)$ is strictly less than 1. To see this we let $f \in \ell^\infty(K_i)$ with $\|f\|_\infty \leq 1$ and observe

$$\begin{aligned}\|(D_{K_i} + \alpha)^{-1} A_{K_i} f\|_\infty &\leq \max_{x \in K_i} \frac{1}{\deg(x) + \alpha} \sum_{y \in K_i} b(x, y)|f(y)| \\ &\leq \max_{x \in K_i} \frac{\deg(x)}{\deg(x) + \alpha} < 1.\end{aligned}$$

Inverting both sides of (3.1) and using the von Neumann series expansion for the inverse of $(I - (D_{K_i} + \alpha)^{-1} A_{K_i})$, yields for, $y \in K_i$

$$\begin{aligned}(L_{K_i} + \alpha)^{-1} \delta_x(y) &= (I - (D_{K_i} + \alpha)^{-1} A_{K_i})^{-1} (D_{K_i} + \alpha)^{-1} \delta_x(y) \\ &= \frac{1}{\deg(x) + \alpha} (I - (D_{K_i} + \alpha)^{-1} A_{K_i})^{-1} \delta_x(y) \\ &= \frac{1}{\deg(x) + \alpha} \sum_{n=0}^\infty [(D_{K_i} + \alpha)^{-1} A_{K_i}]^n \delta_x(y). \end{aligned} \tag{3.2}$$

Furthermore, a direct calculation implies

$$\lim_{\alpha \to 0+} \lim_{i \to \infty} [(D_{K_i} + \alpha)^{-1} A_{K_i}]^n \delta_x(y) = P^{(n)}(x, y).$$

Hence, in order to obtain the desired formula we need to pass to the limits under the sum in (3.2). For this it suffices to show that convergence in i and afterwards convergence in α are monotone. We show by induction over n that for $y \in K_i$,

$$[(D_{K_i} + \alpha)^{-1} A_{K_i}]^n \delta_x(y) \leq [(D_{K_{i+1}} + \alpha)^{-1} A_{K_{i+1}}]^n \delta_x(y).$$

The case $n = 0$ is clear. Now assume we have shown the statement for all indices up to $n-1$. Using the non-negativity of $[(D_{K_{i+1}} + \alpha)^{-1} A_{K_{i+1}}]^{n-1} \delta_x(z)$ for $z \in K_{i+1}$, we obtain

$$\begin{aligned}[(D_{K_i}+\alpha)^{-1}A_{K_i}]^n\delta_x(y) &= \frac{1}{\deg(y)+\alpha}\sum_{z\in K_i} b(y,z)[(D_{K_i}+\alpha)^{-1}A_{K_i}]^{n-1}\delta_x(z)\\ &\le \frac{1}{\deg(y)+\alpha}\sum_{z\in K_i} b(y,z)[(D_{K_{i+1}}+\alpha)^{-1}A_{K_{i+1}}]^{n-1}\delta_x(z)\\ &\le \frac{1}{\deg(y)+\alpha}\sum_{z\in K_{i+1}} b(y,z)[(D_{K_{i+1}}+\alpha)^{-1}A_{K_{i+1}}]^{n-1}\delta_x(z)\\ &= [(D_{K_{i+1}}+\alpha)^{-1}A_{K_{i+1}}]^n\delta_x(y).\end{aligned}$$

The above implies monotone convergence of the summands in i. A similar computation shows monotone convergence in α. ∎

Remark 3.5. • The previous theorem is a version of [1, Theorem 4.34]. However, our proof uses a different approach.

• The operator $L^{(D)}$ is a restriction of $\widetilde{L} = D - A$. In this sense $L^{(D)}$ equals the difference of certain ℓ^2-restrictions of D and A. As $L^{(D)}$, D and A can be unbounded on $\ell^2(V,1)$ (and will be if deg is an unbounded function—see [16, Theorem 11]), the relation of their domains is not so clear. This is the reason why we had to use finite-dimensional approximations in the previous proof.

• The proof of Theorem 3.4 provides a rigorous version of the computation suggested in [11, Remark 4.14].

Corollary 3.6. *Let $(b,0)$ be connected and let m be an arbitrary measure of full support. Let $Q^{(D)}$ be the regular Dirichlet form associated with $(b,0)$ on $\ell^2(V,m)$. The random walk associated with $(b,0)$ is recurrent if and only if $Q^{(D)}$ is recurrent.*

Proof. This is an immediate consequence of the previous two theorems and the fact that recurrence of $Q^{(D)}$ does not depend on the underlying measure (see Corollary 2.23). ∎

Remark 3.7. The previous corollary seems to be widely believed in. However, we could not find a reference which gives the result in its full generality. A form of it which is valid under some additional conditions can be found in [12].

Remark 3.8. Let us briefly discuss the stochastic interpretation of the formula in Theorem 3.4 by computing the quantities involved from a probabilistic point of view. We will omit technical details (such as construction of the related processes and measurability issues) and do computations at a formal level.

First suppose we are given a Markov process $(X_t)_{t\ge 0}$ in continuous time with values in $V\cup\{\infty\}$ satisfying

$$\mathbb{P}_x(X_t = y) = e^{-tL^{(D)}}\delta_y(x).$$

Here $\mathbb{P}_x$ denotes the probability under the condition that $X_0 = x$ (for a detailed discussion of the relation of Markov processes and regular Dirichlet forms see [4]). Let λ be the

Lebesgue measure on $\mathbb{R}$. We then obtain

$$\begin{aligned}\mathbb{E}_x[\lambda\{t>0 \mid X_t=y\}] &= \int_\Omega \int_0^\infty 1_{\{t>0|X_t(\omega)=y\}}(s)\,ds\,d\mathbb{P}_x(\omega)\\ &= \int_0^\infty \int_\Omega 1_{\{t>0|X_t(\omega)=y\}}(s)\,d\mathbb{P}_x(\omega)\,ds\\ &= \int_0^\infty \mathbb{P}_x(X_s=y)\,ds = \int_0^\infty e^{-sL^{(D)}}\delta_y(x)\,ds.\end{aligned}$$

Therefore, the integral $\int_0^\infty e^{-tL^{(D)}}\delta_y(x)\,dt$ is equal to the expected time that $(X_t)_{t\geq 0}$ spends in y provided that it started at x.

Now assume $(X_n)_{n\geq 0}$ is a random walk associated with $(b,0)$. Let $\sharp$ denote the counting measure on $\mathbb{N}$. Then

$$\mathbb{E}_x[\sharp\{n\geq 0 \mid X_n=y\}] = \int_\Omega \sum_{n=0}^\infty 1_{\{\omega\in\Omega|X_n(\omega)=y\}}d\mathbb{P}_x(\omega) = \sum_{n=0}^\infty \mathbb{P}_x(X_n=y) = \sum_{n=0}^\infty P^{(n)}(x,y).$$

This shows that $\sum_{n=0}^\infty P^{(n)}(x,y)$ coincides with the expected number of visits of $(X_n)_{n\geq 0}$ to y whenever it started at x.

4. Characterizations of recurrence and transience

4.1. Classical characterizations of recurrence. In this section we prove continuous-time analogues to characterizations of recurrence and transience which are known in the discrete-time setting. In contrast to the discrete-time theory we use the Dirichlet form methods developed above to deduce them and point out where the 'classical' results can be found.

First we show that $Q^{(D)}$ is transient whenever the graph $(b,0)$ supports a monopole of finite energy (Theorem 4.2). Afterwards, we characterize recurrence in terms of properties of the Yamasaki space $\mathbf{D}$ and the capacity of points (Theorem 4.3). As a last classical characterization of recurrence, we show that it is equivalent to each superharmonic function of finite energy being constant (Theorem 4.5). The proofs of Theorems 4.2 and 4.5 seem to be new while that of Theorem 4.3 was suggested by Daniel Lenz [20].

As seen in Corollary 2.23, recurrence of $Q^{(D)}$ is independent of the underlying measure. Therefore, we will not indicate the ℓ^2-space on which $Q^{(D)}$ is considered in the statements of the theorems in this section.

Definition 4.1 (Monopole). A function $u\in\widetilde{D}$ is called a *monopole of finite energy* if there exists some $x\in V$ such that

$$\widetilde{L}u=\delta_x.$$

The first theorem we prove deals with the existence of such monopoles. Its discrete time analogue is due to Lyons [21] (see [23, Theorem 3.33] for related material).

Theorem 4.2. *Let $(b,0)$ be connected. The Dirichlet form $Q^{(D)}$ is transient if and only if there exists a monopole of finite energy.*

Proof. Assume $u \in \widetilde{D}$ satisfies $\widetilde{L}u = \delta_w$. By Lemma 2.5 it suffices to show that $Q^{(D)}$ has a non-trivial reference function. To this end let $v \in C_c(V)$ be given. By Lemma 1.4 and the Cauchy–Schwarz inequality we obtain

$$\langle v, \delta_w \rangle = \langle v, \widetilde{L}u \rangle = \widetilde{Q}(v,u) = Q^{(D)}(u,v) \leq Q^{(D)}(v)^{1/2} \widetilde{Q}(u)^{1/2}.$$

Since $C_c(V)$ is dense in $D(Q^{(D)})$ with respect to the form norm $\|\cdot\|_Q$, this inequality extends to all $v \in D(Q^{(D)})$. Hence, $Q^{(D)}(v)^{-1/2}\delta_x$ is a reference function for Q and transience is shown.

Conversely, assume that $Q^{(D)}$ is transient. By Definition 2.4 there exists a strictly positive $g \in \ell^\infty(V) \cap \ell^1(V,m)$ with

$$\langle |u|, g \rangle \leq Q^{(D)}(u)^{1/2} \quad \text{for every } u \in D(Q^{(D)}).$$

By the definition of $D(Q^{(D)})_e$ and Lemma 2.12 this inequality extends to all $u \in D(Q^{(D)})_e$. For a fixed $w \in V$ we obtain the continuity of the linear functional

$$F_w : D(Q^{(D)})_e \to \mathbb{R}, \quad u \mapsto u(w),$$

with respect to the inner product $Q^{(D)}$. Theorem 2.19 shows that $(D(Q^{(D)})_e, Q^{(D)})$ is a Hilbert space. Thus, by Riesz' representation theorem there exists a function $v \in D(Q^{(D)})_e$ such that

$$u(w) = F_w(u) = Q^{(D)}(u,v)$$

for all $u \in D(Q^{(D)})_e$. By Lemma 2.12 the inclusion $D(Q^{(D)})_e \subseteq \widetilde{D}$ holds. With the help of Lemma 1.4 we compute

$$(\widetilde{L}v)(x) = \frac{1}{m(x)} \langle \widetilde{L}v, \delta_x \rangle = \frac{1}{m(x)} \widetilde{Q}(v, \delta_x) = \frac{1}{m(x)} Q^{(D)}(v, \delta_x) = \frac{\delta_x(w)}{m(x)}.$$

This shows $\widetilde{L}(m(w)v) = \delta_w$ and finishes the proof. ∎

For locally finite graphs the next classical characterization of recurrence is due to Yamasaki [27]. It deals with the structure of the space $\mathbf{D}$ and shows that recurrence is equivalent to points having capacity zero, where the *capacity* of $x \in V$ is defined by

$$\operatorname{cap}(x) = \inf\{Q^{(D)}(v) \mid v \in C_c(V),\, v(x) = 1\}.$$

Theorem 4.3. *Let $(b,0)$ be connected. The following assertions are equivalent:*

(i) *$Q^{(D)}$ is recurrent.*
(ii) *$C_c(V)$ is dense in $\mathbf{D}$, i.e., $\mathbf{D} = \mathbf{D}_0$.*
(iii) *The constant function 1 can be approximated in $\mathbf{D}$ by functions in $C_c(V)$. In this case, the approximating functions e_n can be chosen to satisfy $0 \leq e_n \leq 1$.*
(iv) $\operatorname{cap}(o) = \inf\{Q^{(D)}(v) \mid v \in C_c(V),\, v(o) = 1\} = 0$.

Proof. (i)⇒(iii). Theorem 2.20 yields the existence of a sequence $(f_n) \subseteq D(Q^{(D)})$ such that $f_n \to 1$ with respect to $\|\cdot\|_o$. Since $C_c(V)$ is dense in $D(Q^{(D)})$ with respect to $\|\cdot\|_Q$ there exist $\tilde{e}_n \in C_c(V)$ such that

$$\|\tilde{e}_n - f_n\|_Q \to 0 \quad \text{as } n \to \infty.$$

Let $e_n = (0 \vee \tilde{e}_n) \wedge 1$. By the Markov property of $Q^{(D)}$ we obtain

$$Q^{(D)}(e_n)^{1/2} \leq Q^{(D)}(\tilde{e}_n)^{1/2} \leq Q^{(D)}(\tilde{e}_n - f_n)^{1/2} + Q^{(D)}(f_n)^{1/2}.$$

Because $c \equiv 0$, the right side of the above inequality needs to converge to zero as $n \to \infty$. It is straightforward that $e_n \to 1$ pointwise. This implies

$$\|1 - e_n\|_o \to 0 \quad \text{as } n \to \infty,$$

and shows (iii).

(iii)⇒(ii). This proof will be done in two steps.

STEP 1. Let $u \in \mathbf{D}$ satisfy $0 \le u \le 1$ and let $(e_n) \subseteq C_c(V)$ be a sequence approximating 1 in $\mathbf{D}$ such that $0 \le e_n \le 1$. Then by Corollary 1.11 the sequence $u \wedge e_n$ converges to u with respect to $\|\cdot\|_o$. Furthermore, $u \wedge e_n \in C_c(V)$, showing that

$$u \in \overline{C_c(V)}^{\|\cdot\|_o}.$$

STEP 2. Let $u \in \mathbf{D}$ be such that $u \ge 0$. Then Corollary 1.10 yields the convergence of $u \wedge N$ to u with respect to $\|\cdot\|_o$ as $N \to \infty$. Step 1 allows us to approximate $u \wedge N$ by functions in $C_c(V)$. For general $u \in \mathbf{D}$ we can split u into its positive and negative parts which both belong to $\mathbf{D}$. This shows (ii).

(ii)⇒(i). Proposition 2.13 shows that $D(Q^{(D)})_e$ is the closure of $D(Q^{(D)})$ in $\mathbf{D}$. Since $C_c(V) \subseteq D(Q^{(D)})$, condition (ii) implies $D(Q^{(D)})_e = \mathbf{D}$. From this and $c \equiv 0$ we infer $1 \in D(Q^{(D)})_e$ and $Q^{(D)}(1) = 0$. Now Theorem 2.20 yields (i).

(iii)⇒(iv). This is obvious upon noting that the sequence in (iii) can be chosen to satisfy $e_n(o) = 1$.

(iv)⇒(i). Assume $Q^{(D)}$ is transient. By Definition 2.4 there exists a constant $C > 0$ such that $Q^{(D)}(v)^{1/2} \ge C|v(o)|$ for any $v \in D(Q^{(D)})$. In particular,

$$\inf\{Q^{(D)}(v) \mid v \in C_c(V),\, v(o) = 1\} \ge C^2 > 0. \ \blacksquare$$

REMARK 4.4. For further references on the history of the previous theorem and related results see [23, Section 3.7].

The last classical characterization which we prove deals with *superharmonic functions of finite energy*, i.e., functions $u \in \widetilde{D}$ satisfying

$$\widetilde{L}u \ge 0.$$

For its discrete-time version see [23, Theorem 3.34].

THEOREM 4.5. *Let $(b, 0)$ be connected. The Dirichlet form $Q^{(D)}$ is recurrent if and only if any superharmonic function of finite energy is constant.*

Proof. Assume $Q^{(D)}$ is recurrent and let $u \in \widetilde{D}$ with $\widetilde{L}u \ge 0$. As a first step we show that u is harmonic, i.e., $\widetilde{L}u = 0$. Assume the contrary, i.e., there exists a $w \in V$ such that $\widetilde{L}u(w) > 0$. By Lemma 1.4 we obtain, for all $v \in C_c(V)$,

$$|v(w)|\widetilde{L}u(w)m(w) \le \langle |v|, \widetilde{L}u\rangle = \widetilde{Q}(|v|, u) \le Q^{(D)}(v)^{1/2}\widetilde{Q}(u)^{1/2}.$$

Therefore, the function $C\delta_w$ with

$$C = \frac{1}{\widetilde{L}u(w)Q^{(D)}(v)^{1/2}}$$

is a reference function for $Q^{(D)}$ and Lemma 2.5 implies its transience. As this contradicts our assumption, we conclude $\widetilde{L}u = 0$.

Next, we show $|u(x) - u(y)| = 0$ for all $x, y \in V$. Since $Q^{(D)}$ is recurrent, part (ii) of Theorem 4.3 implies the existence of a sequence $(u_n) \subseteq C_c(V)$ with $\|u - u_n\|_o \to 0$. Furthermore, Lemma 1.5 shows that for each $x, y \in V$ there exists a constant $K_{x,y} > 0$ with

$$|u(x) - u(y)| \leq K_{x,y} \widetilde{Q}(u)^{1/2}.$$

Combining these observations and Lemma 1.4 we obtain

$$|u(x) - u(y)| \leq K_{x,y} \widetilde{Q}(u)^{1/2} = K_{x,y} \lim_{n\to\infty} \widetilde{Q}(u, u_n)^{1/2} = K_{x,y} \lim_{n\to\infty} \langle \widetilde{L}u, u_n \rangle^{1/2} = 0.$$

This proves one implication.

Conversely, assume $Q^{(D)}$ is transient. By Theorem 4.2 there exists a monopole of finite energy. This monopole is clearly superharmonic and non-constant. ■

Remark 4.6. In the literature the normalized Laplacian, i.e., the operator $\widetilde{L}_{b,0,\deg}$, is used to state analogous results to Theorems 4.2 and 4.5 for the discrete-time case.

4.2. New criteria for recurrence. In this section we provide two more criteria for recurrence, which seem to be new. The first one asks whether certain integrals vanish (Theorem 4.7), while the second deals with the validity of Green's formula for a different situation than in Lemma 1.4 (Theorem 4.8). Both criteria were motivated by recent works. The first one is an analogue to a result of [6], while the second is a version of [11, Theorem 4.6] for not necessarily locally finite graphs.

Theorem 4.7. *Let $(b, 0)$ be connected. The form $Q^{(D)}$ is recurrent if and only if*

$$\sum_{x\in V} \widetilde{L}u(x)m(x) = 0$$

for all $u \in \widetilde{D}$ with $\widetilde{L}u \in \ell^1(V, m)$.

Proof. Let $Q^{(D)}$ be recurrent and $u \in \widetilde{D}$ be such that $\widetilde{L}u \in \ell^1(V, m)$. By Theorem 4.3 there exists a sequence e_n in $C_c(V)$ satisfying $\|e_n - 1\|_o \to 0$ and $0 \leq e_n \leq 1$. We infer by Lebesgue's theorem and Lemma 1.4 that

$$\sum_{x\in V} \widetilde{L}u(x)m(x) = \lim_{n\to\infty} \sum_{x\in V} e_n(x)\widetilde{L}u(x)m(x) = \lim_{n\to\infty} \widetilde{Q}(e_n, u) = 0.$$

Conversely, assume $Q^{(D)}$ is transient. Then Theorem 4.2 yields the existence of a function $v \in \widetilde{D}$ satisfying $\widetilde{L}v = \delta_w \in \ell^1(V, m)$ for some $w \in V$. Obviously

$$\sum_{x\in V} \widetilde{L}v(x)m(x) = m(w) \neq 0. \ ■$$

To prove Green's formula in Lemma 1.4 we needed that one of the functions had compact support. As a last characterization for recurrence we show that recurrence is equivalent to the validity of Green's formula for a different class of functions. We introduce the *boundary term*

$$R : D_\infty \times D^1 \to \mathbb{R}, \qquad R(u, v) = \widetilde{Q}(u, v) - \langle u, \widetilde{L}v \rangle.$$

Here we use the notation $D_\infty = \widetilde{D} \cap \ell^\infty(V)$ and $D^1 = \{v \in \widetilde{D} \mid \widetilde{L}v \in \ell^1(V, m)\}$.

Theorem 4.8. *Let $(b, 0)$ be connected. The Dirichlet form $Q^{(D)}$ is recurrent if and only if $R \equiv 0$.*

Proof. Assume $Q^{(D)}$ is recurrent. Let $u \in D_\infty$ and $v \in D^1$. Theorem 4.3 yields a sequence $(u_n) \subseteq C_c(V)$ converging to u with respect to $\|\cdot\|_o$. Without loss of generality this sequence can be chosen to be uniformly bounded by $\|u\|_\infty$. Lebesgue's theorem and Lemma 1.4 yield

$$\widetilde{Q}(u,v) = \lim_{n\to\infty} \widetilde{Q}(u_n, v) = \lim_{n\to\infty} \langle u_n, \widetilde{L}v\rangle = \langle u, \widetilde{L}v\rangle.$$

This implies $R \equiv 0$.

Conversely, assume $Q^{(D)}$ is transient. By Theorem 4.2 there exists $v \in D^1$ such that

$$\sum_{x\in V} \widetilde{L}v(x)m(x) \neq 0.$$

Since $1 \in D_\infty$ and $\widetilde{Q}(1,v) = 0$, this implies $R(1,v) \neq 0$. ∎

REMARK 4.9. As mentioned earlier, the previous theorem was motivated by [11, Theorem 4.6] which considers a boundary term pairing certain monopoles and dipoles with functions of finite energy. However, the boundary representation that is used by the authors of [11] seems to hold true only for locally finite graphs. We will explain some details of their computation below.

Let us now consider the case when $(b, 0)$ is locally finite. We can then compute the boundary term R by a limiting procedure. First we fix some notation. For a subgraph $W \subseteq V$ let

$$\operatorname{bd} W = \{x \in W \mid \text{there exists } y \in V \setminus W \text{ such that } x \sim y\}$$

be the set of all vertices in W which are connected to the complement of W. Furthermore, let

$$\operatorname{int} W = W \setminus \operatorname{bd} W.$$

Note that $x \in \operatorname{int} W$ and $y \sim x$ imply $y \in W$. For $u \in \widetilde{D}$ and $x \in \operatorname{bd} W$ we let the *outward normal derivative* with respect to W be defined by

$$(\partial_W u)(x) = \sum_{y\in W} b(x,y)(u(x) - u(y)).$$

Using it we can compute the boundary term R as in [11].

PROPOSITION 4.10. *Let $(b, 0)$ be locally finite. For $u \in D_\infty$ and $v \in D^1$ the boundary term R is given by*

$$R(u,v) = \lim_{n\to\infty} \sum_{x\in \operatorname{bd} V_n} u(x)(\partial_{V_n} v)(x),$$

where (V_n) is an increasing sequence of finite subsets of V with $\bigcup_n V_n = V$.

Proof. Let V_n be as above. Then a simple calculation shows

$$\frac{1}{2}\sum_{x,y\in V_n} b(x,y)(u(x) - u(y))(v(x) - v(y))$$
$$= \sum_{x\in \operatorname{int} V_n} u(x)(\widetilde{L}v)(x)m(x) + \sum_{x\in \operatorname{bd} V_n} u(x)(\partial_{V_n} v)(x).$$

Because $(b, 0)$ is locally finite, we obtain $\bigcup_n \operatorname{int} V_n = V$. Furthermore, our assumptions imply that the sum on the left and

$$\sum_{x \in V} u(x)(\widetilde{L}v)(x)m(x)$$

are absolutely convergent. Letting $n \to \infty$ gives the desired statement, on noting that absolute convergence yields independence from the choice of the V_n. ∎

REMARK 4.11. The local finiteness is crucial for the above computations, which are taken from [11, proof of Theorem 4.6]. Otherwise one cannot control $\operatorname{int} W$ for finite sets W. In the non-local finite case it might even happen that $\operatorname{int} W = \emptyset$ for all finite $W \subseteq V$.

5. Further global properties

In this chapter we discuss two other issues: stochastic completeness and the validity of $Q^{(D)} = Q^{(N)}$. It turns out that characterizations of these two properties are similar to the ones obtained for recurrence and transience. We first introduce the notion of stochastic completeness (Definition 5.1) and then prove a characterization analogous to Theorem 4.7 (see Theorem 5.3), which is motivated by results of [6]. Afterwards, we present a criterion for stochastic completeness in terms of the unique solvability of the equation $(\widetilde{L}+\alpha)u = 0$ on ℓ^∞ (Theorem 5.5). This criterion is taken from [15]. We then characterize when the Neumann form $Q^{(N)}$ and the regular Dirichlet form $Q^{(D)}$ coincide. We show that this is related to unique solvability of $(\widetilde{L} + \alpha)u = 0$ on $\widetilde{D} \cap \ell^2(V, m)$ and the validity of Green's formula for ℓ^2-functions (Theorem 5.6). The connection between $Q^{(D)} = Q^{(N)}$ and the validity of $\widetilde{Q}(u, v) = \langle \widetilde{L}u, v\rangle$ for certain ℓ^2-functions seems to be new.

5.1. Stochastic completeness. Using the extension of a Markovian resolvent to $\ell^\infty(V)$ (see Appendix A.1) we introduce the concept of stochastic completeness.

DEFINITION 5.1. Let Q be a Dirichlet form associated with (b, c). Then Q is called *stochastically complete* if

$$(L + 1)^{-1}1 = 1.$$

Otherwise Q is called *stochastically incomplete.*

REMARK 5.2. By general principles (the correspondence of $(L + \alpha)^{-1}$ and e^{-tL}) the definition of stochastic completeness is equivalent to the validity of

$$e^{-tL}1 = 1$$

for all $t > 0$. This equation is important whenever one investigates a Markov process $(X_t)_{t\geq 0}$ on $V \cup \{\infty\}$ which satisfies

$$\mathbb{P}(X_t = y \mid X_0 = x) = e^{-tL}\delta_y(x).$$

In view of this equation, stochastic completeness is equivalent to

$$\mathbb{P}(X_t \in V \mid X_0 = x) = 1 \quad \text{for all } t > 0.$$

In other words, stochastic completeness describes the property that X_t does not leave V in finite time.

The next result is similar to Theorem 4.7. It seems to be new in this context.

THEOREM 5.3. *Let (b,c) be connected and let m be a measure of full support. The associated regular Dirichlet form $Q^{(D)}$ on $\ell^2(V,m)$ is stochastically complete if and only if*

$$\sum_{x\in V}(\widetilde{L}u)(x)m(x)=0$$

for all $u\in D(Q^{(D)})\cap\ell^1(V,m)$ with $\widetilde{L}u\in\ell^1(V,m)\cap\ell^2(V,m)$.

Proof. Let (e_n) be a sequence in $C_c(V)$ which satisfies $0\le e_n\le e_{n+1}\le 1$ and $e_n\to 1$ pointwise. Let $u_n=(L^{(D)}+1)^{-1}e_n\in D(L^{(D)})$. The way we extended the resolvent to ℓ^∞ yields pointwise convergence of (u_n) towards $(L^{(D)}+1)^{-1}1$. Furthermore, since $(L^{(D)}+1)^{-1}$ is Markovian, we infer $0\le u_n\le 1$.

Assume $Q^{(D)}$ is stochastically complete and let $u\in D(Q^{(D)})\cap\ell^1(V,m)$ with $\widetilde{L}u\in\ell^1(V,m)\cap\ell^2(V,m)$. Proposition 1.13 shows that $u\in D(L^{(D)})$. Furthermore, stochastic completeness yields pointwise convergence of (u_n) towards 1. Using the self-adjointness of $L^{(D)}$ and Lebesgue's theorem we may compute

$$\begin{aligned}\sum_{x\in V}(\widetilde{L}u)(x)m(x)&=\sum_{x\in V}(L^{(D)}u)(x)m(x)=\lim_{n\to\infty}\sum_{x\in V}u_n(x)(L^{(D)}u)(x)m(x)\\&=\lim_{n\to\infty}\sum_{x\in V}(L^{(D)}u_n)(x)u(x)m(x)=\lim_{n\to\infty}\sum_{x\in V}(e_n(x)-u_n(x))u(x)m(x)\\&=\sum_{x\in V}\big(1-(L^{(D)}+1)^{-1}1(x)\big)u(x)m(x)=0.\end{aligned}$$

This shows one implication.

Conversely, if the sum is always vanishing, set $u=(L^{(D)}+1)^{-1}v$, where $v\in\ell^1(V,m)\cap\ell^2(V,m)$ is chosen strictly positive. This implies that $u\in D(Q^{(D)})\cap\ell^1(V,m)$ (see Appendix, extension of the resolvent to ℓ^1) and $\widetilde{L}u\in\ell^1(V,m)\cap\ell^2(V,m)$. Then our assumptions, Lebesgue's theorem and the self-adjointness of the resolvent yield

$$\begin{aligned}0=\sum_{x\in V}(\widetilde{L}u)(x)m(x)&=\lim_{n\to\infty}\sum_{x\in V}e_n(x)(L^{(D)}u)(x)m(x)\\&=\lim_{n\to\infty}\sum_{x\in V}e_n(x)\big(v(x)-(L^{(D)}+1)^{-1}v(x)\big)m(x)\\&=\lim_{n\to\infty}\sum_{x\in V}\big(e_n(x)-(L^{(D)}+1)^{-1}e_n(x)\big)v(x)m(x)\\&=\sum_{x\in V}\big(1-(L^{(D)}+1)^{-1}1(x)\big)v(x)m(x).\end{aligned}$$

Since v was chosen strictly positive and $(L^{(D)}+1)^{-1}1(x)\le 1$ (see the discussion in Appendix A.1), this shows $(L^{(D)}+1)^{-1}1=1$. ■

REMARK 5.4. Just as for recurrence, one can show that on a connected graph a non-vanishing potential c implies stochastic incompleteness (see e.g. [15]). Therefore, we will assume $c\equiv 0$ in the following sections.

The next theorem is a characterization of stochastic completeness in terms of the unique solvability of $(\widetilde{L}+\alpha)u = 0$ on $\ell^\infty(V)$. We need it in the next chapter.

THEOREM 5.5. *Let $Q^{(D)}$ be the regular Dirichlet form associated with $(b,0)$ on $\ell^2(X,m)$. The following assertions are equivalent:*

(i) *$Q^{(D)}$ is stochastically complete.*
(ii) *For any $\alpha > 0$ the equation $(\widetilde{L}+\alpha)u = 0$ is uniquely solvable on $\ell^\infty(V)$.*

Proof. This is an immediate consequence of [15, Theorem 1]. ∎

5.2. Regularity of the Neumann form. From the definition of $Q^{(D)}$ and $Q^{(N)}$ it is not clear whether these two forms coincide or not. The theorem below provides a characterization of this in terms of unique solvability of $(\widetilde{L}+\alpha)u = 0$ on $\widetilde{D}\cap\ell^2(V,m)$ and the validity of Green's formula for ℓ^2-functions.

THEOREM 5.6. *Let $Q^{(D)}$ be the regular Dirichlet form associated with (b,c) on $\ell^2(X,m)$ and let $Q^{(N)}$ be the Neumann form associated with (b,c) on $\ell^2(X,m)$. The following assertions are equivalent:*

(i) $Q^{(D)} = Q^{(N)}$.
(ii) *For any $\alpha > 0$ the equation $(\widetilde{L}+\alpha)u = 0$ is uniquely solvable in $\widetilde{D}\cap\ell^2(V,m)$.*
(iii) *For all $u \in \widetilde{D}\cap\ell^2(V,m)$ and $v \in \widetilde{D}\cap\ell^2(V,m)$ with $\widetilde{L}v \in \ell^2(V,m)$,*

$$\widetilde{Q}(u,v) = \langle u, \widetilde{L}v\rangle.$$

Proof. (ii)⇒(i). By the general theory (see Appendix A.1) it suffices to show that the resolvents $(L^{(D)}+\alpha)^{-1}$ and $(L^{(N)}+\alpha)^{-1}$ coincide. Let $u \in \ell^2(V,m)$. Set

$$v = (L^{(N)}+\alpha)^{-1}u - (L^{(D)}+\alpha)^{-1}u.$$

Since both $L^{(D)}$ and $L^{(N)}$ are restrictions of $\widetilde{L}$ to their corresponding domains, we infer that

$$(\widetilde{L}+\alpha)v = 0.$$

Hence, (ii) implies $v = 0$ and we conclude $(L^{(D)}+\alpha)^{-1} = (L^{(N)}+\alpha)^{-1}$.

(i)⇒(ii). By Proposition 1.13 the domain of $L^{(D)}$ is given by

$$D(L^{(D)}) = \{u \in D(Q^{(D)}) \mid \widetilde{L}u \in \ell^2(V,m)\}.$$

Since we have assumed $Q^{(D)} = Q^{(N)}$, this implies

$$D(L^{(D)}) = \{u \in \widetilde{D}\cap\ell^2(V,m) \mid \widetilde{L}u \in \ell^2(V,m)\}.$$

Let $\alpha > 0$ and $u \in \widetilde{D}\cap\ell^2(V,m)$ with $\widetilde{L}u = -\alpha u$. By the above characterization of $D(L^{(D)})$ we obtain $u \in D(L^{(D)})$. Since the spectrum of $L^{(D)}$ is contained in $[0,\infty)$, we infer $u = 0$.

(i)⇒(iii). Assume $Q^{(D)} = Q^{(N)}$. Proposition 1.13 implies

$$\begin{aligned} D(L^{(N)}) = D(L^{(D)}) &= \{v \in D(Q^{(D)}) \mid \widetilde{L}v \in \ell^2(V,m)\} \\ &= \{v \in \widetilde{D}\cap\ell^2(V,m) \mid \widetilde{L}v \in \ell^2(V,m)\}. \end{aligned}$$

This shows (iii).

(iii)⇒(i). Assume $\widetilde{Q}(u,v) = \langle u, \widetilde{L}v\rangle$ for all $u \in \widetilde{D} \cap \ell^2(V,m)$ and $v \in \widetilde{D} \cap \ell^2(V,m)$ with $\widetilde{L}v \in \ell^2(V,m)$. By the correspondence of $L^{(N)}$ and $Q^{(N)}$ the domain of $L^{(N)}$ satisfies

$$D(L^{(N)}) \supseteq \{v \in \widetilde{D} \cap \ell^2(V,m) \mid \widetilde{L}v \in \ell^2(V,m)\}.$$

Hence, Proposition 1.13 shows $L^{(D)} \subseteq L^{(N)}$. Taking adjoints yields the statement. ∎

Remark 5.7. The equivalence (i) and (ii) was already shown in [9, Corollary 3.3], with a different proof. The equivalence of (i) and (iii) seems to be new.

Part (iii) of the theorem above can be considered as a boundary term characterization of $Q^{(D)} = Q^{(N)}$ which is analogous to Theorem 4.8. To see this we introduce the boundary term

$$\hat{R} : \widetilde{D} \cap \ell^2(V,m) \times \{u \in \widetilde{D} \cap \ell^2(V,m) \mid \widetilde{L}u \in \ell^2(V,m)\} \to \mathbb{R}$$

acting by

$$\hat{R}(u,v) = \widetilde{Q}(u,v) - \langle u, \widetilde{L}u\rangle.$$

Corollary 5.8. *$Q^{(D)} = Q^{(N)}$ if and only if $\hat{R} \equiv 0$.*

6. Consequences of recurrence

In this chapter we discuss the relationships between all the global properties which were introduced above. We prove that recurrence of $Q^{(D)}$ always implies stochastic completeness and $Q^{(D)} = Q^{(N)}$ (see Theorem 6.3), and that all properties coincide in the case when m is finite (Theorem 6.5). Using these results we show that recurrence of $Q^{(D)}$ is related to the unique solvability of the eigenvalue equation $(\widetilde{L} + \alpha)u = 0$ on the space $\widetilde{D}$ (Theorem 6.6).

Lemma 6.1. *Let $(b, 0)$ be connected and assume that $Q^{(D)}$ is recurrent. Let $\alpha > 0$ and let $u \in \widetilde{D}$ satisfy $u \leq 0$ and $(\widetilde{L} + \alpha)u \geq 0$. Then $u \equiv 0$.*

Proof. Let u be as above. From $\widetilde{L}u \geq -\alpha u$ we infer that u is superharmonic. Theorem 4.5 implies that u is constant. We obtain

$$0 \leq (\widetilde{L} + \alpha)u = \alpha u \leq 0.$$

This shows $u \equiv 0$. ∎

From this lemma we can deduce the following uniqueness statement for solutions to the equation $(\widetilde{L} + \alpha)u = 0$ in $\widetilde{D}$.

Lemma 6.2. *Let $(b, 0)$ be connected and assume that $Q^{(D)}$ is recurrent. Let $\alpha > 0$ and let $u \in \widetilde{D}$ with $(\widetilde{L} + \alpha)u = 0$. Then $u \equiv 0$.*

Proof. Let $u_+ = u \wedge 0$ and $u_- = (-u) \wedge 0$ denote the positive/negative part of u. Since $u_+, u_- \in \widetilde{D}$, it suffices to show $(\widetilde{L}+\alpha)u_+ \leq 0$ and $(\widetilde{L}+\alpha)u_- \leq 0$ to obtain the statement by the previous lemma. The assumption $(\widetilde{L} + \alpha)u = 0$ implies

$$(\widetilde{L} + \alpha)u_+ = (\widetilde{L} + \alpha)u_-,$$

which is equivalent to

$$(\widetilde{L}+\alpha)u_+(x) = \frac{\deg(x)}{m(x)}u_-(x) - \frac{1}{m(x)}\sum_{y\in V} b(x,y)u_-(y) + \alpha u_-(x).$$

For $x \in V$ with $u(x) \geq 0$ we obtain

$$(\widetilde{L}+\alpha)u_+(x) = -\frac{1}{m(x)}\sum_{y\in V} b(x,y)u_-(y) \leq 0.$$

For $x \in V$ with $u(x) < 0$ the definition of $\widetilde{L}$ yields

$$(\widetilde{L}+\alpha)u_+(x) = -\frac{1}{m(x)}\sum_{y\in V} b(x,y)u_+(y) \leq 0. \blacksquare$$

We are now able to prove that recurrence implies stochastic completeness and $Q^{(D)} = Q^{(N)}$.

THEOREM 6.3. *Let $(b,0)$ be connected and let $Q^{(D)}$ be recurrent. Then $Q^{(D)}$ is stochastically complete and $Q^{(D)} = Q^{(N)}$.*

Proof. Stochastic completeness is an immediate consequence of Theorems 4.7 and 5.3. The equality $Q^{(D)} = Q^{(N)}$ follows readily from Lemma 6.2 and Theorem 5.6. ■

REMARK 6.4. Theorem 6.3 is well-known and valid in much more general situations. That recurrence implies stochastic completeness immediately follows from [4, Theorems 1.6.5 and 1.6.6]. That it also implies $Q^{(D)} = Q^{(N)}$ was shown in [19, Theorem 6.3].

THEOREM 6.5. *Let $(b,0)$ be connected and $m(V) < \infty$. The following assertions are equivalent:*

(i) *$Q^{(D)}$ is recurrent.*
(ii) *$Q^{(D)}$ is stochastically complete.*
(iii) *$Q^{(D)} = Q^{(N)}$.*

Proof. (i)⇒(ii). This is part of Theorem 6.3.

(iii)⇒(i). The finiteness of m implies $1 \in D(Q^{(N)}) = D(Q^{(D)})$. Since $c \equiv 0$, we conclude $Q^{(D)}(1) = 0$, and recurrence of $Q^{(D)}$ follows from Theorem 2.20.

(ii)⇒(iii). Assume $Q^{(D)} \neq Q^{(N)}$. Then the resolvents $(L^{(N)}+1)^{-1}$ and $(L^{(D)}+1)^{-1}$ must be different. Because $\ell^\infty(V) \cap \ell^2(V,m)$ is dense in $\ell^2(V,m)$, there exists a bounded function u with

$$(L^{(D)}+1)^{-1}u \neq (L^{(N)}+1)^{-1}u.$$

Both $(L^{(D)}+1)^{-1}u$ and $(L^{(N)}+1)^{-1}u$ are bounded solutions to the equation

$$(\widetilde{L}+1)v = u.$$

Therefore, $\widetilde{L}+1$ is not injective on $\ell^\infty(V)$. This implies stochastic incompleteness by Theorem 5.5. ■

We have already seen that stochastic completeness and the equality $Q^{(D)} = Q^{(N)}$ are related to the uniqueness of solutions of the equation $(\widetilde{L}+\alpha)u = 0$ on certain function spaces. With the help of the last theorem we prove a similar statement for recurrence. However, as we pointed out before, recurrence does not depend on the choice of m. Thus, the uniqueness statement which is equivalent to recurrence must be stronger than the ones for the other concepts. Let us write $Q^{(D)}_m$ whenever we refer to $Q^{(D)}$ on $\ell^2(V,m)$, and let $\widetilde{L}_m$ be the corresponding formal operator. Furthermore, we denote by $\widetilde{\Delta}$ the operator on $\widetilde{D}$ defined by

$$(\widetilde{\Delta}u)(x) := (\widetilde{L}_{b,0,1}u)(x) = \sum_{y\in V} b(x,y)(u(x)-u(y)).$$

THEOREM 6.6. *Let $(b,0)$ be connected. The following assertions are equivalent:*

(i) *For some measure m of full support on V the form $Q^{(D)}_m$ is recurrent.*
(ii) *For all measures m of full support on V the form $Q^{(D)}_m$ is recurrent.*
(iii) *For all measures m of full support and for any $\alpha > 0$ the equation $(\widetilde{L}_m + \alpha)u = 0$ has a unique solution in $\widetilde{D}$.*
(iv) *For some finite measure m of full support and for any $\alpha > 0$ the equation $(\widetilde{L}_m+\alpha)u = 0$ has a unique solution in $\widetilde{D}$.*
(v) *For all $v : V \to (0,\infty)$ the equation $(\widetilde{\Delta} + v)u = 0$ has a unique solution in $\widetilde{D}$.*
(vi) *For some $v : V \to (0,\infty)$ which belongs to $\ell^1(V,1)$ the equation $(\widetilde{\Delta} + v)u = 0$ has a unique solution in $\widetilde{D}$.*

Proof. (i)⇒(ii). It suffices to show the statement for transience. Let m be a measure such that $Q^{(D)}_m$ is transient and let m' be another measure of full support. Theorem 4.2 shows that there exist $v \in \widetilde{D}$ and $w \in V$ such that

$$\widetilde{L}_m v = \delta_w.$$

Then

$$\widetilde{L}_{m'} v = \frac{m(w)}{m'(w)}\delta_w.$$

This implies the existence of a monopole with respect to $\widetilde{L}_{m'}$ and Theorem 4.2 shows transience of $Q^{(D)}_{m'}$.

(ii)⇒(iii). This follows from Lemma 7.2.

(iii)⇒(iv). This is obvious.

(iv)⇒(i). By the uniqueness of the solution, we infer $Q^{(D)}_m = Q^{(N)}_m$ from Theorem 5.6. Since m is finite, Theorem 6.5 implies recurrence of $Q^{(D)}_m$.

(v)⇔(iii). This is clear since any $v : V \to (0,\infty)$ can be written in the form $v = \alpha m$ and vice versa. Then $(\widetilde{\Delta} + v)u = 0$ if and only if $(\widetilde{L}_m + \alpha)u = 0$.

(v)⇒(vi). This is clear.

(vi)⇒(i). Let $v : V \to (0,\infty)$ belong to $\ell^1(V,1)$ and suppose

$$(\widetilde{\Delta} + v)u = 0$$

has a unique solution in $\widetilde{D}$. Then the above is obviously equivalent to

$$(\widetilde{L}_v + 1)u = 0$$

being uniquely solvable in $\widetilde{D}$. We infer that $(L^{(D)} + 1)^{-1}$ and $(L^{(N)} + 1)^{-1}$ must agree (as resolvents associated with $Q^{(D)}, Q^{(N)}$ on $\ell^2(V,v)$). This shows $L^{(D)} = L^{(N)}$, which implies $Q^{(D)} = Q^{(N)}$. Since v is a finite measure on V, we conclude by Theorem 6.5 that $Q^{(D)}_v$ is recurrent, arriving at (i). ∎

Remark 6.7. That all the discussed global properties coincide in the finite-measure case (Theorem 6.5) seems to be new. Also the relation of recurrence to the spectral theory of $\widetilde{L}$ on $\widetilde{D}$ is new.

Appendix: General results

In this appendix we provide some known results. The first part is devoted to general theory of Dirichlet forms, while the second one deals with some results about Dirichlet forms on graphs. The last part provides two theorems about vector-valued integrals.

A.1. Dirichlet forms and associated objects

Definition A.1. Let $D(Q)$ be a dense subspace of $\ell^2(V,m)$. A map

$$Q : D(Q) \times D(Q) \to \mathbb{R}$$

is called a *Dirichlet form* if the following conditions are satisfied:

(Q1) $Q(f,f) \geq 0$, $Q(f,g) = Q(g,f)$ and $Q(\alpha f + g, h) = \alpha Q(f,h) + Q(g,h)$ for all $f,g,h \in D(Q)$ and $\alpha \in \mathbb{R}$. (Linearity)

(Q2) $D(Q)$ equ ipped with with the inner product

$$\langle f,g\rangle_Q = Q(f,g) + \langle f,g\rangle$$

is a Hilbert space. (Closedness)

(Q3) For any *normal contraction* F (i.e. $F : \mathbb{R} \to \mathbb{R}$ with $F(0) = 0$ and $|F(x) - F(y)| \leq |x-y|$ for all $x,y \in \mathbb{R}$) and any $u \in D(Q)$ the function $F \circ u$ belongs to $D(Q)$ and

$$Q(F \circ u) \leq Q(u). \qquad \text{(Markov property)}$$

We call a Dirichlet form *regular* if $C_c(V) \subseteq D(Q)$ and

$$\overline{C_c(V)}^{\|\cdot\|_Q} = D(Q),$$

where $\|\cdot\|_Q$ is the norm given by

$$\|\cdot\|_Q = \sqrt{\langle\cdot,\cdot\rangle_Q}.$$

Remark A.2. This definition of regularity differs from the one in [4] which requires $D(Q) \cap C_c(V)$ to be dense in $D(Q)$ with respect to $\|\cdot\|_Q$ and in $C_c(V)$ with respect to $\|\cdot\|_\infty$. However, it was shown in [15, Lemma 2.1] that a Dirichlet form on a countable discrete space is regular in the sense of [4] if and only if the above is satisfied.

Definition A.3. A family $(T_t)_{t>0}$ of bounded linear operators on the Hilbert space $\ell^2(V,m)$ is called a *strongly continuous Markovian semigroup* if the following conditions are satisfied:

(S1) For any $t > 0$ the operator T_t is self-adjoint. (Symmetry)

(S2) $T_{t+s} = T_t T_s$ for every $t,s > 0$. (Semigroup property)

(S3) $\|T_t f\|_2 \leq \|f\|_2$ for every $t > 0$, $f \in \ell^2(V,m)$. (Contractivity)

(S4) $\|T_t f - f\|_2 \to 0$ as $t \to 0$ for every $f \in \ell^2(V,m)$. (Strong continuity)

(S5) $0 \leq T_t f \leq 1$ for $f \in \ell^2(V,m)$ with $0 \leq f \leq 1$. (Markov property)

A family $(G_\alpha)_{\alpha>0}$ of bounded linear operators on $\ell^2(V,m)$ is called a *strongly continuous Markovian resolvent* if the following conditions are satisfied:

(R1) For any $\alpha > 0$ the operator G_α is self-adjoint. (Symmetry)
(R2) $G_\alpha - G_\beta + (\alpha-\beta)G_\alpha G_\beta = 0$ for every $\alpha, \beta > 0$. (Resolvent equation)
(R3) $\|\alpha G_\alpha f\|_2 \le \|f\|_2$ for every $\alpha > 0$, $f \in \ell^2(V,m)$. (Contractivity)
(R4) $\|\alpha G_\alpha f - f\|_2 \to 0$ as $\alpha \to \infty$ for every $f \in \ell^2(V,m)$. (Strong continuity)
(R5) $0 \le \alpha G_\alpha f \le 1$ for $f \in \ell^2(V,m)$ with $0 \le f \le 1$. (Markov property)

Every Dirichlet form Q is in-one-to one correspondence with a non-negative self-adjoint operator L, a strongly continuous Markovian resolvent G_α and a strongly continuous Markovian semigroup T_t. Given any one of those four objects, one can reconstruct the others. We want to give a short discussion about the connection between those objects. For detailed proofs of the statements below see [4, Sections 1.3–1.4].

Given a Dirichlet form Q, the domain of its associated operator L is given by

$$D(L) = \{u \in D(Q) \mid \exists w \in \ell^2(V,m)\ \forall v \in D(Q) : Q(u,v) = \langle w, v\rangle\},$$

on which it acts by

$$Lu = w.$$

This operator is self-adjoint and non-negative. Furthermore, its square root satisfies $D(L^{1/2}) = D(Q)$ and

$$Q(u,v) = \langle L^{1/2}u, L^{1/2}v\rangle$$

for all $u, v \in D(Q)$. Because L is non-negative, its spectrum is contained in $[0,\infty)$. Thus, for positive α the operators $(L+\alpha)^{-1}$ exist and are bounded. They satisfy (R1)–(R5). The spectral calculus of L allows us to define e^{-tL}, which is a semigroup satisfying (S1)–(S5).

Let us stress some more relations of the above objects.

(i) Let $u \in \ell^2(V,m)$. Then

$$Q(w,v) + \alpha\langle w, v\rangle = \langle u, v\rangle$$

holds for all $v \in D(Q)$ if and only if $w = (L+\alpha)^{-1}u$.

(ii) The domain of L is given by

$$D(L) = \left\{ u \in \ell^2(V,m) \;\middle|\; \lim_{t\to 0} \frac{u - e^{-tL}u}{t} \text{ exists in } \ell^2(V,m) \right\}.$$

For $u \in D(L)$,

$$Lu = \lim_{t\to 0} \frac{u - e^{-tL}u}{t},$$

where the limit is taken in $\ell^2(V,m)$.

Resolvents and semigroups associated with Dirichlet forms can be uniquely extended to bounded operators on $\ell^1(V,m)$ and $\ell^\infty(V)$. We discuss this extension for the resolvents, the semigroups can be treated similarly. Let $u \in \ell^1(V,m) \cap \ell^2(V,m)$ and let $K \subseteq V$ be finite. Using the self-adjointness and the Markov property of $(L+\alpha)^{-1}$ we obtain

$$\begin{aligned}\sum_{x\in K} |(L+\alpha)^{-1}u(x)|m(x) &\le \sum_{x\in V}(L+\alpha)^{-1}|u|(x)1_K(x)m(x)\\ &= \sum_{x\in V}|u|(x)(L+\alpha)^{-1}1_K(x)m(x) \le \sum_{x\in V}\alpha^{-1}|u|(x)m(x).\end{aligned}$$

Here 1_K denotes the indicator function of the set K. Since K was arbitrary we infer

$$\|(L+\alpha)^{-1}u\|_1 \le \alpha^{-1}\|u\|_1.$$

Thus, $(L+\alpha)^{-1}$ extends uniquely to $\ell^1(V,m)$. Now let $u \in \ell^\infty(V)$ be positive. We choose a sequence of non-negative functions $(u_n) \subseteq \ell^2(V,m)$ converging monotonically towards u. Because $(L+\alpha)^{-1}$ maps non-negative functions to non-negative functions we infer

$$0 \le (L+\alpha)^{-1}u_n \le (L+\alpha)^{-1}u_{n+1}.$$

By (R5) we obtain

$$(L+\alpha)^{-1}u_n \le \alpha^{-1}\|u\|_\infty.$$

Hence the limit as $n\to\infty$ exists and is bounded by $\alpha^{-1}\|u\|_\infty$. It is easy to verify that this limit is independent of the choice of the sequence u_n. Now set

$$(L+\alpha)^{-1}u(x) := \lim_{n\to\infty}(L+\alpha)^{-1}u_n(x).$$

For an arbitrary $u \in \ell^\infty(V)$ split u into its positive and negative parts and repeat the above procedure. We then obtain a linear operator $(L+\alpha)^{-1} : \ell^\infty(V) \to \ell^\infty(V)$ which satisfies

$$\|\alpha(L+\alpha)^{-1}u\|_\infty \le \|u\|_\infty.$$

A.2. Dirichlet forms associated with graphs

Theorem A.4 ([15, Theorem 7]). *Let Q be a regular Dirichlet form on $\ell^2(V,m)$. Then there exists a graph (b,c) over V such that $Q = Q^{(D)}_{b,c}$.*

The next theorem is an approximation result for the resolvent $(L^{(D)}+\alpha)^{-1}$. We fix some notation. For finite $W \subseteq V$ let $L_W := p_W\widetilde{L}i_W : C(W) \to C(W)$. Here i_W is the canonical embedding of $C(W)$ into $C(V)$ and p_W is the projection of $C(V)$ onto $C(W)$. In some sense L_W is the restriction of $\widetilde{L}$ to $C(W)$, i.e., for $u \in C(W)$ and $x \in W$ it satisfies

$$L_W u(x) = \frac{u(x)}{m(x)}\sum_{y\in V} b(x,y) - \frac{1}{m(x)}\sum_{y\in W} b(x,y)u(y) + \frac{c(x)}{m(x)}u(x).$$

Theorem A.5 ([15, Proposition 2.7]). *Let $Q^{(D)}$ be the regular Dirichlet form associated with (b,c) and let $L^{(D)}$ be the associated operator. Let (K_n) be a sequence of finite subsets of V such that $K_n \subseteq K_{n+1}$ and $\bigcup K_n = V$. For any $u \in C(K_1)$,*

$$\lim_{n\to\infty}\|(L^{(D)}+\alpha)^{-1}u - (L_{K_n}+\alpha)^{-1}u\|_2 = 0.$$

(Here u and $(L_{K_n}+\alpha)^{-1}u$ are extended by 0 outside of their domains.)

Definition A.6. A bounded operator T on $\ell^2(V,m)$ is called *positivity improving* if $u \ge 0$ and $u \not\equiv 0$ implies $Tu(x) > 0$ for all $x \in V$.

THEOREM A.7 ([9, Theorem 6.3]). *Let (b,c) be connected and Q be a Dirichlet form associated with (b,c). Its resolvent $(L+\alpha)^{-1}$ and its semigroup e^{-tL} are positivity improving.*

A.3. Vector-valued integration. The following theorems are special cases of statements for Bochner integrals. To avoid Hilbert-space-valued integration we include elementary proofs for them.

Let $f : V \times [a,b] \to \mathbb{R}$ be such that for each $x \in V$ the function $f(x,\cdot)$ is integrable. We define

$$\int_a^b f(\cdot,t)\,dt : V \to \mathbb{R}$$

pointwise via

$$\int_a^b f(\cdot,t)\,dt(x) := \int_a^b f(x,t)\,dt.$$

THEOREM A.8. *Assume $a,b \in \mathbb{R}$. Let $f : V \times [a,b] \to \mathbb{R}$ be such that for each $t \in [a,b]$ the function $f(\cdot,t)$ belongs to $\ell^2(V,m)$ and $t \mapsto f(\cdot,t)$ is continuous as a mapping from $[a,b]$ to $\ell^2(V,m)$. Then*

$$\left\| \int_a^b f(\cdot,t)\,dt \right\|_2 \le \int_a^b \|f(\cdot,t)\|_2\,dt.$$

Proof. The continuity assumption ensures that all occurring integrals exist. Without loss of generality we can assume $m \equiv 1$. By monotone convergence it suffices to show the statement for finite sets V. This case can be reduced to $|V| = 2$ by induction. Assume we have shown the result for all sets of cardinality less than or equal to n, where $n \ge 2$, and let $|V| = n+1$. Fix $o \in V$. We obtain

$$\begin{aligned}
\Big(\sum_{x\in V}\Big|\int_a^b f(x,t)\,dt\Big|^2\Big)^{1/2} &= \Big(\Big|\int_a^b f(o,t)\,dt\Big|^2 + \sum_{x\in V\setminus\{o\}}\Big|\int_a^b f(x,t)\,dt\Big|^2\Big)^{1/2} \\
&\le \Big(\Big|\int_a^b f(o,t)\,dt\Big|^2 + \Big\{\int_a^b \Big[\sum_{x\in V\setminus\{o\}} |f(x,t)|^2\Big]^{1/2} dt\Big\}^2\Big)^{1/2} \\
&\le \int_a^b \Big(\sum_{x\in V} |f(x,t)|^2\Big)^{1/2} dt = \int_a^b \|f(\cdot,t)\|_2\,dt.
\end{aligned}$$

It remains to treat the case $|V| = 2$ to finish the proof. Our continuity assumptions ensure that all of the above integrals can be computed via Riemann sums. Therefore, it suffices to show the statement for simple functions f,g of the form

$$f = \sum_i \alpha_i 1_{A_i} \quad \text{and} \quad g = \sum_i \beta_i 1_{A_i}$$

with pairwise disjoint sets A_i. We need to show

$$\Big(\Big|\int_a^b f(t)\,dt\Big|^2 + \Big|\int_a^b g(t)\,dt\Big|^2\Big)^{1/2} \le \int_a^b (f(t)^2 + g(t)^2)^{1/2}\,dt.$$

Plugging in f and g and taking the square on both sides, we conclude that this is equivalent to

$$\sum_{i,j}(\alpha_i\alpha_j+\beta_i\beta_j)\lambda(A_i)\lambda(A_j)\le\sum_{i,j}(\alpha_i^2+\beta_i^2)^{1/2}(\alpha_j^2+\beta_j^2)^{1/2}\lambda(A_i)\lambda(A_j),$$

where λ denotes the Lebesgue measure. But this inequality holds since

$$\alpha_i\alpha_j+\beta_i\beta_j\le(\alpha_i^2+\beta_i^2)^{1/2}(\alpha_j^2+\beta_j^2)^{1/2}$$

is always true (use $(c-d)^2\ge 0$ for arbitrary $c,d\in\mathbb{R}$). ∎

Theorem A.9. *Assume $a,b\in\mathbb{R}$. Let $f:V\times[a,b]\to\mathbb{R}$ be such that for each $t\in[a,b]$ the function $f(\cdot,t)$ belongs to $\ell^2(V,m)$ and $t\mapsto f(\cdot,t)$ is continuous as a mapping from $[a,b]$ to $\ell^2(V,m)$. Furthermore, let T be a bounded linear operator on $\ell^2(V,m)$. Then $\int_a^b f(\cdot,t)\,dt\in\ell^2(V,m)$ and*

$$T\int_a^b f(\cdot,t)\,dt=\int_a^b Tf(\cdot,t)\,dt.$$

Proof. $\int_a^b f(\cdot,t)\,dt\in\ell^2(V,m)$ follows from Theorem A.8 because $t\mapsto\|f(\cdot,t)\|_2$ is continuous and

$$\Big\|\int_a^b f(\cdot,t)\,dt\Big\|_2\le\int_a^b\|f(\cdot,t)\|_2\,dt.$$

Let $g\in\ell^2(V,m)$. Then Lebesgue's theorem yields

$$\int_a^b\langle f(\cdot,t),g\rangle\,dt=\Big\langle\int_a^b f(\cdot,t)\,dt,g\Big\rangle.$$

Now the statement follows from

$$\begin{aligned}\Big\langle T\int_a^b f(\cdot,t)\,dt,g\Big\rangle&=\Big\langle\int_a^b f(\cdot,t)\,dt,T^*g\Big\rangle=\int_a^b\langle f(\cdot,t),T^*g\rangle\,dt=\int_a^b\langle Tf(\cdot,t),g\rangle\,dt\\&=\Big\langle\int_a^b Tf(\cdot,t)\,dt,g\Big\rangle,\end{aligned}$$

where T^* denotes the adjoint of T. ∎

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