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Global existence and asymptotic behavior for the full compressible Euler equations with damping in $\mathbb{R}^3$

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Abstract. We are concerned with the global existence and asymptotic behavior of classical solutions to the Cauchy problem for the full compressible Euler equations with damping in $\mathbb{R}^3$. We prove the global existence of the classical solutions by the delicate energy method under the condition that the initial data are close to the constant equilibrium state in H^3-framework. An energy estimate on $\|\nabla u\|_{L^1((0,t);\tilde{B}^{0,3/2}_{2,1}(\mathbb{R}^3))}$ enables us to close the energy estimates for the non-dissipative entropy. Moreover, the optimal time decay rate is also established.

1. Introduction. Consider the following full compressible Euler equations with damping in $\mathbb{R}^3$:

$$
\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P = -a\rho u, \\ (\rho\mathcal{E})_t + \operatorname{div}(\rho u\mathcal{E} + uP) = -a\rho u^2. \end{cases} \tag{1.1}
$$

Here ρ, $u = (u_1, u_2, u_3)^t$ and P represent the density, the velocity and the pressure respectively. The total energy $\mathcal{E}$ is $|u|^2/2 + \mathrm{e}$, where e is the internal energy. $a > 0$ models the damping constant and $1/a$ may be regarded as the relaxation time for some physical flows. In this paper, we will consider only polytropic fluids, so that the equations of state for the fluid are given by

$$
P = R\rho\theta, \qquad \mathrm{e} = c_\gamma\theta, \qquad P = A\exp\{S/c_\gamma\}\rho^\gamma, \tag{1.2}
$$

where $A, R > 0$ are the universal gas constants, $\gamma > 1$ is the adiabatic exponent, S is the specific entropy, and $c_\gamma = R/(\gamma - 1)$ is the specific heat at constant volume. We complete (1.1) with the Cauchy data

$$
(\rho, u, \theta)(0, x) = (\rho_0(x), u_0(x), \theta_0(x)), \qquad x \in \mathbb{R}^3. \tag{1.3}
$$

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The system (1.1) occurs in the mathematical modeling of compressible flow through a porous medium, which has spanned a large range of applications. The major feature of (1.1) is that shock waves develop in finite time for solutions with general initial data. Due to the physical importance and mathematical challenges, the study of system (1.1) has attracted many physicists and mathematicians.

In the case of isentropic flow where $S = \text{const}$, many results concerning the existence, uniqueness and large time behavior of (weak, strong or smooth) solutions in one dimension can be found in [2, 3, 10–12, 15, 17–20, 23, 25, 26, 28, 29, 42, 44] and the references cited therein. In higher dimensions, Wang and Yang [35] first proved the global existence and uniqueness of classical solutions and obtained pointwise estimates for them. Sideris, Thomases and Wang [32] studied the effect of damping on the large-time behavior of solutions to the Cauchy problem for the three-dimensional compressible Euler equations; they proved that damping prevents the development of singularities in small amplitude classical solutions, using an equivalent reformulation of the Cauchy problem to obtain effective energy estimates. Formation of singularities was also exhibited for large data. Fang and Xu [9] weakened the regularity assumptions on the initial data and obtained the existence and asymptotic behavior of C^1 solutions by the spectral localization method. Tan and Wu [34] proved the optimal time decay based on the Hodge decomposition technique, L^p-L^2 estimates for the linearized equations and an elaborate energy method. Later, Tan and Wang [33] studied the global existence and time-asymptotic behavior of small smooth solutions by a refined pure energy method. Jang and Masmoudi [21] studied well-posedness of compressible Euler equations in a physical vacuum. For the initial boundary value problem, refer for instance to [31, 39, 40] and the references therein.

For non-isentropic flow, the global existence of small smooth solutions to the Cauchy problem was proved in [16, 43] in one dimension, and the large time behavior of these solutions can be found in [13, 27]. In higher dimensions, Wu, Tan and Huang [38] proved the global existence of small smooth solutions under the additional assumption that the initial data are bounded in L^1. Wu [36] studied the relaxation limit of the relaxing Cauchy problem for non-isentropic compressible Euler equations with damping and proved that the velocity of the relaxing equations converges weakly to the velocity of the relaxed equations, while other variables of the relaxing equations converge strongly to the corresponding variables of the relaxed equations. For the initial boundary value problem, refer for instance to [14, 30, 37, 41] and the references therein.

The main motivation of this paper is to prove the global existence without boundedness of $\|(P-P_0, u_0)\|_{L^1}$ which is used in proving the global existence in [38], and show optimal time decay rates by an elaborate energy method.

To begin with, we note that all thermodynamical variables $\rho, \theta, \mathrm{e}, P$ as well as the entropy S can be represented by functions of any two of them. To overcome the difficulties arising from the non-isentropic case, we will rewrite system (1.1) as in [7, 38]. We take the two variables to be P and S; then the equation of state is replaced by

$$\rho = A^{-1/\gamma}P^{1/\gamma}\exp\left\{-\frac{(\gamma-1)S}{\gamma R}\right\}. \tag{1.4}$$

Under the aforementioned assumptions, we can rewrite (1.1) in terms of (P, u, S) as follows:

$$\begin{cases} \partial_t P + \gamma P \nabla\cdot u + u\cdot\nabla P = 0, \\ \rho\partial_t u + \rho(u\cdot\nabla)u + \nabla P = -au, \\ \partial_t S + (u\cdot\nabla)S = 0, \end{cases} \tag{1.5}$$

where $\rho = \rho(P, S)$ is given by (1.4). It should be mentioned that (1.5) is a hyperbolic system, while the dissipation property comes from the damping term. In this paper, we are concerned with the initial value problem for (1.5) with the following initial conditions:

$$(P, u, S)(0, x) = (P(x), u(x), S(x)) \to (\bar{P}, 0, \bar{S}) \quad \text{as } |x|\to\infty. \tag{1.6}$$

Notation. The norms in the Sobolev spaces $H^m(\mathbb{R}^3)$ and $W^{m,q}(\mathbb{R}^3)$ are denoted respectively by $\|\cdot\|_m$ and $\|\cdot\|_{m,q}$ for $m \ge 0$ and $q \ge 1$. In particular, for $m = 0$ we simply use $\|\cdot\|$ and $\|\cdot\|_{L^q}$. For conciseness, we do not specify in function space names whether they involve scalar-valued or vector-valued functions. Moreover, $\|(f, g)\|_X$ denotes $\|f\|_X + \|g\|_X$. We write $a \lesssim b$ to mean that $a \le Cb$ for a universal constant $C > 0$ that only depends on the parameters of the problem. We denote $\nabla = \partial_x = (\partial_1, \partial_2, \partial_3)$, where $\partial_i = \partial_{x_i}$, $\nabla_i = \partial_i$, and set $\partial_x^\ell f = \nabla^\ell f = \nabla(\nabla^{\ell-1}f)$. The integration domain $\mathbb{R}^3$ will always be omitted. Finally, $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2(\mathbb{R}^3)$.

We also recall the Littlewood–Paley decomposition. Choose a radial function $\varphi \in S(\mathbb{R}^3)$ supported in $\mathcal{C} = \{\xi \in \mathbb{R}^N : 3/4 \le |\xi| \le 8/3\}$ such that

$$\sum_{q\in\mathbb{Z}} \varphi(2^{-q}\xi) = 1 \quad \text{for all } \xi \ne 0.$$

For $q \in \mathbb{Z}$, define the dyadic blocks

$$\Delta_q f = \mathfrak{F}^{-1}(\varphi(2^{-q}\xi)\mathfrak{F}f).$$

We denote by $\mathcal{D}'(\mathbb{R}^N)$ the dual space of $\mathcal{D}(\mathbb{R}^N) = \{f \in S(\mathbb{R}^3) : D^\alpha\hat{f}(0) = 0, \forall\alpha \in \mathbb{N}^3\}$; it also can be identified with the quotient space $S'(\mathbb{R}^3)/\mathcal{P}$ with the polynomial space $\mathcal{P}$. The formal equality

$$f = \sum_{q\in\mathbb{Z}} \Delta_q f$$

holds true for $f \in \mathcal{D}'(\mathbb{R}^3)$ and is called the *homogeneous Littlewood–Paley decomposition.*

DEFINITION 1.1. Let $s \in \mathbb{R}$ and $1 \le p, r \le \infty$. The *homogeneous Besov space* $\dot{B}^s_{p,r}(\mathbb{R}^3)$ is defined by

$$\dot{B}^s_{p,r}(\mathbb{R}^3) = \{f \in D'(\mathbb{R}^3) : \|f\|_{\dot{B}^s_{p,r}} < \infty\},$$

where

$$\|f\|_{\dot{B}^s_{p,r}} \triangleq \big\|2^{qs}\|\Delta_q f\|_{L^p}\big\|_{l^r}.$$

For $s_1, s_2 \in \mathbb{R}$ and $1 \le p, r \le \infty$, we define the *hybrid Besov spaces* $\tilde{B}^{s_1,s_2}_{p,r}$ with norm $\tilde{B}^{s_1,s_2}_{p,r}$ given by

$$\|f\|_{\tilde{B}^{s_1,s_2}_{p,r}} \triangleq \Big(\sum_{q\le 0} \|2^{qrs_1}\Delta_q f\|^r_{L^p}\Big)^{1/r} + \Big(\sum_{q>0} \|2^{qrs_2}\Delta_q f\|^r_{L^p}\Big)^{1/r}.$$

Our main results are stated in the following theorem.

THEOREM 1.2. *Assume that* $\|(P_0 - \bar{P}, u_0, S_0 - \bar{S})\|_3$ *is sufficiently small. Then there exists a unique global solution* $(P(t,x), u(t,x), S(t,x))$ *to the Cauchy problem* (1.5)–(1.6) *such that for any* $t \in [0,\infty)$,

$$\|(P(\cdot,t) - \bar{P}, u(\cdot,t))\|_3^2 + \int_0^t (\|\nabla P(\cdot,\tau)\|_2^2 + \|u(\cdot,\tau)\|_3^2)\,d\tau \lesssim \|(P_0 - \bar{P}, u_0)\|_3^2, \tag{1.7}$$

$$\|S(\cdot,t) - \bar{S}\|_3 \lesssim \|S_0 - \bar{S}\|_3 \exp\{C\|(P_0 - \bar{P}, u_0)\|_3\}. \tag{1.8}$$

Moreover, if in addition there is some $p \in [1, 6/5)$ *such that*

$$\|P_0 - \bar{P}\|_{L^p} + \|u_0\|_{L^{3p/(3-p)}} < \infty,$$

then

$$\|(P(\cdot,t) - \bar{P})\|_{L^q} \lesssim (1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}, \qquad 2 \le q \le 6, \tag{1.9}$$

$$\|\nabla P(\cdot,t)\|_2 + \|u(\cdot,t)\|_3 \lesssim (1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}}, \tag{1.10}$$

$$\|\partial_t(P(\cdot,t), u(\cdot,t), S(\cdot,t))\| \lesssim (1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}}. \tag{1.11}$$

Finally, if $p = 1$, *denote*

$$(\mathcal{P}_0, m_0) = (P_0^{1/\gamma} - \bar{P}_0^{1/\gamma}, \rho_0 u_0)$$

and assume that the Fourier transform $(\hat{\mathcal{P}}_0, \hat{m}_0)$ *satisfies*

$$|\hat{\mathcal{P}}_0| \ge c_0|\xi|, \quad \hat{m}_0 = 0, \qquad \textit{for } 0 \le |\xi| \ll 1, \tag{1.12}$$

where c_0 is a positive constant. Then we also have the lower bounds

$$\|P(\cdot,t)-\bar{P}\| \ge c_1(1+t)^{-3/4}, \tag{1.13}$$

$$\|u(\cdot,t)\| \ge c_1(1+t)^{-5/4}, \tag{1.14}$$

where c_1 is a positive constant independent of time.

REMARK 1.3. Here we prove the global existence without the boundedness of $\|P_0-\bar{P}\|_{L^p}+\|u_0\|_{L^{3p/(3-p)}}$, unlike the previous work [38], and our results are also valid for the two-dimensional case. Compare this with the Navier–Stokes equations without heat conductivity [7], where the boundedness of $\|P_0-\bar{P}\|_{L^p}+\|u_0\|_{L^p}$ is used in proving the global existence. Moreover, $\|\nabla(P,u)(\cdot,t)\|_1$ for the linear solution to the system of Navier–Stokes equations without heat conductivity decays only as $(1+t)^{-1}$ in the Cauchy problem (see [7]), which is not integrable, in particular making the strategy of [7] difficult to apply; thus global existence and optimal convergence rates for the Cauchy problem for the system of Navier–Stokes equations without heat conductivity in the two-dimensional case are still an open problem.

The rest of this paper is devoted to proving Theorem 1.2. In Section 2, we first reformulate the system and give some careful a priori estimates for strong solutions. Then the global existence of strong solutions is established by the standard continuity argument. In Section 3, we derive decay-in-time estimates for the linearized system and use the energy method to derive a Lyapunov-type energy inequality for all the derivatives controlled by the first order derivatives; then we utilize the decay-in-time estimates for the linearized system to control the first order derivatives by the higher order derivatives. Hence, optimal decay rates of the global strong solutions follow from these two kinds of estimates. In Section 4, we establish the lower bound for the time decay rate of the global solution.

2. Global existence

2.1. Reformulation. In this subsection, we first reformulate system (1.5). Set

$$\kappa_1=\sqrt{\frac{c_\gamma}{(R+c_\gamma)\bar{\rho}\bar{P}}},\qquad \kappa_2=\sqrt{\frac{(R+c_\gamma)\bar{P}}{c_\gamma\bar{\rho}}},$$

where $\bar{\rho}=\rho(\bar{P},\bar{S})$. After the change of variables

$$(n,v,s)=\left(P-\bar{P},\frac{1}{\kappa_1}u,S-\bar{S}\right),$$

the initial value problem (1.5)–(1.6) is reformulated as

$$\begin{cases} \partial_t n + \kappa_2 \nabla \cdot v = F, \\ \partial_t v + \kappa_2 \nabla n + av = G, \\ \partial_t \mathcal{S} + \kappa_1 (v \cdot \nabla)\mathcal{S} = 0, \\ (n, v, \mathcal{S})|_{t=0} := (n_0, v_0, \mathcal{S}_0) \to (0,0,0) \quad \text{as } |x| \to \infty, \end{cases} \tag{2.1}$$

where

$$F(n, v, \mathcal{S}) = -\frac{(R + c_\gamma)\kappa_1}{c_\gamma} n \nabla \cdot v - \kappa_1 v \cdot \nabla n,$$
$$G(n, v, \mathcal{S}) = -\kappa_1 (v \cdot \nabla) v - \frac{1}{\kappa_1}\left(\frac{1}{\rho} - \frac{1}{\bar{\rho}}\right)\nabla n.$$

For later use, we list some elementary but useful inequalities of Sobolev type and properties of Besov spaces. First, we recall the following Sobolev inequalities.

Lemma 2.1. *Let $f \in H^2(\mathbb{R}^3)$. Then*

(i) $\|f\|_{L^\infty} \le C\|\nabla f\|^{1/2}\|\nabla^2 f\|^{1/2} \le C\|\nabla f\|_1$;
(ii) $\|f\|_{L^6} \le C\|\nabla f\|$;
(iii) $\|f\|_{L^q} \le C\|f\|_1,\ 2 \le q \le 6$.

Next, we state some product estimates in $\dot{B}^s_{2,1}$ (cf. [4–6]).

Lemma 2.2. *Let $s_1, s_2 \le 3/2$ with $s_1 + s_2 > 0$, and let $u \in \dot{B}^{s_1}_{2,1}$ and $v \in \dot{B}^{s_2}_{2,1}$. Then $uv \in \dot{B}^{s_1+s_2-3/2}_{2,1}$ and*

$$\|uv\|_{\dot{B}^{s_1+s_2-3/2}_{2,1}} \le C\|u\|_{\dot{B}^{s_1}_{2,1}}\|v\|_{\dot{B}^{s_2}_{2,1}}.$$

The next two lemmas give embedding estimates for hybrid Besov spaces (cf. [4–6]).

Lemma 2.3. *The following embeddings hold:*

(i) $\tilde{B}^{s,s}_{2,1} = \dot{B}^s_{2,1}$;
(ii) *if $s \le t$, then $\tilde{B}^{s,t}_{2,1} = \dot{B}^s_{2,1} \cap \dot{B}^t_{2,1}$; otherwise, $\tilde{B}^{s,t}_{2,1} = \dot{B}^s_{2,1} + \dot{B}^t_{2,1}$.*

Lemma 2.4. *There exists a positive constant C such that for all $s \in \mathbb{R}$,*

$$\frac{1}{C^{|s|+1}}\|u\|_{H^s} \le \|u\|_{\tilde{B}^{0,s}_{2,2}} \le C^{|s|+1}\|u\|_{H^s}.$$

The last lemma concerns estimates of solutions to the transport equation (cf. [4–6]).

Lemma 2.5. *Suppose f is a solution of*

$$\begin{cases} \partial_t f + \kappa u \cdot \nabla f = g, \\ f|_{t=0} = f_0. \end{cases}$$

Then there exists a constant $C > 0$ depending only on κ such that, for all $0 < t \le \infty$,

$$\|f\|_{L^\infty(0,t;\tilde{B}^{0,2}_{2,2})} \le \Big[\|f_0\|_{\tilde{B}^{0,2}_{2,2}} + \int_0^t \exp\Big(-C\int_0^\tau \|\nabla u(\tau')\|_{\tilde{B}^{0,3/2}_{2,1}} d\tau'\Big)\|g(\tau)\|_{\tilde{B}^{0,2}_{2,2}}\,d\tau\Big] \times \exp\Big(C\int_0^t \|\nabla u(\tau)\|_{\tilde{B}^{0,3/2}_{2,1}}\,d\tau\Big).$$

2.2. A priori estimate. By a classical argument, the global existence of solutions will be obtained by combining the local existence result with a priori estimates. Since the local existence and uniqueness in H^3 can be established by following the methods of Kato [22] or Majda [24] (we omit the details), the global solutions will follow by the standard continuity argument after we establish an a priori estimate. Therefore, we make an a priori assumption

$$\|(n,v,\mathcal{S})\|_3 \le \delta \ll 1. \tag{2.2}$$

Under the assumption (2.2), we give three energy estimates on $(n,v,\mathcal{S})$ which can be found in [38, Section 4 and Lemmas 2.2 and 2.5]; we omit their proofs for brevity.

Lemma 2.6. *Under the assumptions of Theorem* 1.2 *and* (2.2), *there exists a positive constant $D_1 > 0$ suitably large which is independent of δ such that*

$$\frac{d}{dt}\{D_1\|(n,v)(t)\|^2 + \langle\nabla n, v\rangle(t)\} + C\|(\nabla n, v)(t)\|^2 \lesssim \|\nabla v(t)\|^2, \tag{2.3}$$

$$\frac{d}{dt}\Big\{D_1 H_1^2(n(t),v(t)) + \sum_{1\le|\alpha|\le 2}\langle\partial_x^\alpha\nabla n, \partial_x^\alpha v\rangle(t)\Big\} + C\big(\|\nabla^2 n(t)\|_1^2 + \|\nabla v(t)\|_2^2\big) \lesssim \delta\|(\nabla n, v)(t)\|^2, \tag{2.4}$$

where $H_1(n,v)$ is equivalent to $\|\nabla(n,v)\|_2$, if δ is small enough. Moreover,

$$\|\mathcal{S}\|_{L^\infty(0,t;\tilde{B}^{0,2}_{2,2})} \le \|\mathcal{S}_0\|_{\tilde{B}^{0,2}_{2,2}} \exp\Big(C\int_0^t \|\nabla v(\tau)\|_{\tilde{B}^{0,3/2}_{2,1}} d\tau\Big), \tag{2.5}$$

$$\|\nabla\mathcal{S}\|_{L^\infty(0,t;\tilde{B}^{0,2}_{2,2})} \le \Big[\|\nabla\mathcal{S}_0\|_{\tilde{B}^{0,2}_{2,2}} + \int_0^t \exp\Big(-C\int_0^\tau \|\nabla v(\tau')\|_{\tilde{B}^{0,3/2}_{2,1}} d\tau'\Big) \times \|\nabla v(\tau)\|_{\tilde{B}^{0,3/2}_{2,2}} \cdot \|\nabla\mathcal{S}(\tau)\|_{\tilde{B}^{0,2}_{2,2}}\,d\tau\Big] \times \exp\Big(C\int_0^t \|\nabla v(\tau)\|_{\tilde{B}^{0,3/2}_{2,1}}\,d\tau\Big), \tag{2.6}$$

for any $t \ge 0$.

LEMMA 2.7. *Under the assumptions of Theorem* 1.2 *and* (2.2),

$$\|\nabla v(t)\|_{\tilde{B}^{0,3/2}_{2,1}} + a\int_0^t \|\nabla v(\tau)\|_{\tilde{B}^{0,3/2}_{2,1}}\,d\tau \lesssim \|(n_0,v_0)\|_3 + \int_0^t \|\nabla(n,v)(\tau)\|_2^2\,d\tau. \tag{2.7}$$

Proof. For $q \le 0$, applying $2^{s_1 q}\Delta_q$ to $(2.1)_1$ and $(2.1)_2$ and multiplying them by $\Delta_q n$ and $\Delta_q v$ respectively and then integrating over $\mathbb{R}^3$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}2^{s_1 q}\|(\Delta_q n,\Delta_q v)\|^2 + a2^{s_1 q}\|\Delta_q v\|^2\\
&\quad = \left\langle -\frac{(R+c_\gamma)\kappa_1}{c_\gamma}2^{s_1 q}\Delta_q(n\nabla\cdot v) - \kappa_1 2^{s_1 q}\Delta_q(v\cdot\nabla n), \Delta_q n\right\rangle\\
&\qquad + \left\langle -\kappa_1 2^{s_1 q}\Delta_q(v\cdot\nabla v) - \frac{1}{\kappa_1}2^{s_1 q}\Delta_q[(1/\rho - 1/\bar\rho)\nabla n], \Delta_q v\right\rangle.
\end{aligned}\tag{2.8}$$

Applying $2^{s_1 q}\Delta_q$ to $(2.1)_2$ and $2^{s_1 q}\Delta_q\nabla$ to $(2.1)_1$, multiplying them by $\Delta_q\nabla n$ and $\Delta_q v$ respectively, and then integrating over $\mathbb{R}^3$, we arrive at

$$\begin{aligned}
&\frac{d}{dt}2^{s_1 q}\langle\Delta_q v,\Delta_q\nabla n\rangle + \kappa_2 2^{s_1 q}\|\Delta_q\nabla n\|^2\\
&\quad = \langle -\kappa_2 2^{s_1 q}\Delta_q\nabla(\nabla\cdot v),\Delta_q v\rangle - \langle a2^{s_1 q}\Delta_q v,\Delta_q\nabla n\rangle\\
&\qquad + \left\langle -\frac{(R+c_\gamma)\kappa_1}{c_\gamma}2^{s_1 q}\Delta_q(n\nabla\cdot v) - \kappa_1 2^{s_1 q}\Delta_q(v\cdot\nabla n), \Delta_q v\right\rangle\\
&\qquad + \left\langle -\kappa_1 2^{s_1 q}\Delta_q(v\cdot\nabla v) - \frac{1}{\kappa_1}2^{s_1 q}\Delta_q[(1/\rho - 1/\bar\rho)\nabla n], \Delta_q\nabla n\right\rangle.
\end{aligned}\tag{2.9}$$

Multiplying (2.8) by D_2 suitably large and adding it to (2.9), we get

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}D_2 2^{s_1 q}\|(\Delta_q n,\Delta_q v)\|^2 + 2^{s_1 q}\langle\Delta_q v,\Delta_q\nabla n\rangle\right)\\
&\quad + aD_2 2^{s_1 q}\|\Delta_q v\|^2 + \kappa_2 2^{s_1 q}\|\Delta_q\nabla n\|^2\\
&= D_2\left\langle -\frac{(R+c_\gamma)\kappa_1}{c_\gamma}2^{s_1 q}\Delta_q(n\nabla\cdot v) - \kappa_1 2^{s_1 q}\Delta_q(v\cdot\nabla n), \Delta_q n\right\rangle\\
&\quad + D_2\left\langle -\kappa_1 2^{s_1 q}\Delta_q(v\cdot\nabla v) - \frac{1}{\kappa_1}2^{s_1 q}\Delta_q[(1/\rho - 1/\bar\rho)\nabla n], \Delta_q v\right\rangle\\
&\quad + \langle -\kappa_2 2^{s_1 q}\Delta_q\nabla(\nabla\cdot v),\Delta_q v\rangle - \langle a2^{s_1 q}\Delta_q v,\Delta_q\nabla n\rangle\\
&\quad \left\langle -\frac{(R+c_\gamma)\kappa_1}{c_\gamma}2^{s_1 q}\Delta_q(n\nabla\cdot v) - \kappa_1 2^{s_1 q}\Delta_q(v\cdot\nabla n), \Delta_q v\right\rangle\\
&\quad + \left\langle -\kappa_1 2^{s_1 q}\Delta_q(v\cdot\nabla v) - \frac{1}{\kappa_1}2^{s_1 q}\Delta_q[(1/\rho - 1/\bar\rho)\nabla n], \Delta_q\nabla n\right\rangle
\end{aligned}\tag{2.10}$$

$$\leq C2^{s_1 q}\|\big(\Delta_q(n\nabla\cdot v), \Delta_q(v\nabla\cdot n), \Delta_q(v\nabla\cdot v), \Delta_q[(1/\rho-1/\bar\rho)\nabla n]\big)\| \times \|(\Delta_q n, \Delta v)\| + 2\frac{a^2}{\kappa_2}2^{s_1 q}\|\Delta_q v\| + \frac{\kappa_2}{2}2^{s_1 q}\|\Delta_q \nabla n\|.$$

Now we define the temporal energy functional

$$\mathcal{H}_q^{s_1} = \frac{\frac{1}{2}D_2 2^{s_1 q}\|(\Delta_q n, \Delta_q v)\|^2 + 2^{s_1 q}\langle \Delta_q v, \Delta_q \nabla n\rangle}{\|(\Delta_q n, \Delta_q v)\|}.$$

Then $\mathcal{H}_q^{s_1}$ is equivalent to $2^{s_1 q}\|(\Delta_q n, \Delta_q v)\|$ since D_2 can be large enough. By (2.10), we obtain

$$\frac{d}{dt}\mathcal{H}_q^{s_1} + C2^q\mathcal{H}_q^{s_1} \lesssim 2^{s_1 q}\|\big(\Delta_q(n\nabla\cdot v), \Delta_q(v\nabla\cdot n), \Delta_q(v\nabla\cdot v), \Delta_q[(1/\rho-1/\bar\rho)\nabla n]\big)\|. \tag{2.11}$$

For $q \geq 0$, by the same argument as in (2.10), if we define the temporal energy functional

$$\mathcal{H}_q^{s_2} = \frac{\frac{1}{2}D_2 2^{s_2 q}\|(\Delta_q n, \Delta_q v)\|^2 + 2^{(s_2-1)q}\langle \Delta_q v, \Delta_q \nabla n\rangle}{\|(\Delta_q n, \Delta_q v)\|},$$

then $\mathcal{H}_q^{s_2}$ is equivalent to $2^{s_2 q}\|(\Delta_q n, \Delta_q v)\|$ and satisfies

$$\frac{d}{dt}\mathcal{H}_q^{s_2} + C\mathcal{H}_q^{s_2} \lesssim 2^{s_2 q}\|(\Delta_q(n\nabla\cdot v), \Delta_q(v\nabla\cdot n), \Delta_q(v\nabla\cdot v), \Delta_q[(1/\rho-1/\bar\rho)\nabla n])\|. \tag{2.12}$$

Taking $s_1 = 1$ in (2.11) and $s_2 = 5/2$ in (2.12), we conclude after summation over q in $\mathbb{Z}$ that

$$\begin{aligned}
&\|(n,v)\|_{\tilde B_{2,1}^{1,5/2}} + \int_0^t \|(n,v)\|_{\tilde B_{2,1}^{2,5/2}}\,d\tau \\
&\lesssim \|(n_0,v_0)\|_{\tilde B_{2,1}^{1,5/2}} + \int_0^t \|\big(n\nabla\cdot v, v\nabla\cdot n, v\nabla\cdot v, (1/\rho-1/\bar\rho)\nabla n\big)\|_{\tilde B_{2,1}^{1,5/2}}\,d\tau \\
&\lesssim \|(n_0,v_0)\|_{\tilde B_{2,1}^{1,3/2}} + \int_0^t \big(\|(n,v)\|_{\tilde B_{2,1}^{3/2,5/2}}\|(n,v)\|_{\tilde B_{2,1}^{2,5/2}} \\
&\qquad + \|(1/\rho-1/\bar\rho)\|_{\dot B_{2,1}^{3/2,5/2}}\|n\|_{\tilde B_{2,1}^{2,5/2}}\big)\,d\tau \\
&\lesssim \|(n_0,v_0)\|_3 + \int_0^t \big(\|\nabla(n,v)\|_2^2 + \|\nabla(1/\rho-1/\bar\rho)\|_2\|n\|_{\tilde B_{2,1}^{2,5/2}}\big)\,d\tau \\
&\lesssim \|(n_0,v_0)\|_3 + \int_0^t \|\nabla(n,v)\|_2^2\,d\tau + \delta\int_0^t \|n\|_{\tilde B_{2,1}^{2,5/2}}\,d\tau.
\end{aligned} \tag{2.13}$$

Since δ is sufficiently small, the inequality (2.13) gives

$$\|(n,v)\|_{\tilde{B}^{1,5/2}_{2,1}} + \int_0^t \|(n,v)\|_{\tilde{B}^{2,5/2}_{2,1}}\,d\tau \lesssim \|(n_0,v_0)\|_3 + \int_0^t \|\nabla(n,v)\|_2^2\,d\tau. \tag{2.14}$$

Using $(2.1)_2$, (2.14) and the same argument as in (2.10), we have

$$\begin{aligned}\|\nabla v\|_{\tilde{B}^{0,3/2}_{2,1}} &+ a\int_0^t \|\nabla v\|_{\tilde{B}^{0,3/2}_{2,1}}\,d\tau \\ &\lesssim \|\nabla v_0\|_{\tilde{B}^{0,3/2}_{2,1}} + \int_0^t \|\nabla(\nabla n, v\cdot\nabla v, (1/\rho - 1/\bar{\rho})\nabla n)\|_{\tilde{B}^{0,3/2}_{2,1}}\,d\tau \\ &\lesssim \|\nabla v_0\|_{\tilde{B}^{0,3/2}_{2,1}} + \int_0^t (\|\nabla(n,v)\|_2^2 + \|n\|_{\tilde{B}^{2,5/2}_{2,1}})\,d\tau \\ &\lesssim \|(n_0,v_0)\|_3 + \int_0^t \|\nabla(n,v)\|_2^2\,d\tau,\end{aligned}$$

which is (2.7). ∎

Proof of Theorem 1.2. Since $\delta > 0$ is sufficiently small, from (2.3) and (2.4) we have a function $H_2(n,v)$ which is equivalent to $\|(n,v)\|_3$ and satisfies

$$\frac{d}{dt}H_2^2(n(t),v(t)) + C\big(\|\nabla n(t)\|_2^2 + \|v(t)\|_3^2\big) \le 0. \tag{2.15}$$

Integrating over $[0,t]$, we obtain

$$H_2^2(n(t),v(t)) + C\int_0^t \big(\|\nabla n(\tau)\|_2^2 + \|v(\tau)\|_3^2\big)\,d\tau \lesssim H_2^2(n(0),v(0)), \tag{2.16}$$

which gives (1.7). By (2.5), (2.15), Lemma 2.4, Lemma 2.7 and (2.16), we arrive at

$$\begin{aligned}\|\mathcal{S}(t)\|_3 &\lesssim \|\mathcal{S}(0)\|_3 \exp\Big\{C\int_0^t \|\nabla v\|_{\tilde{B}^{0,3/2}_{2,1}}\,d\tau\Big\} \\ &\lesssim \|\mathcal{S}(0)\|_3 \exp\{C\|(n(0),v(0))\|_3\},\end{aligned} \tag{2.17}$$

which gives (1.8). Thus we have proved the global existence result of Theorem 1.2.

3. Convergence rate of the solution. We now prove the decay part of Theorem 1.2. In Section 3.1, we establish decay-in-time estimates for the linearized system and an elementary inequality. In Section 3.2, we show the optimal decay rates by delicate energy estimates.

3.1. Linear estimates. We consider the following linear system:

$$U(t) = K(t)U_0, \qquad t \geq 0,$$

where

$$U = [n, v]^t, \qquad U_0 = [n_0, v_0]^t$$

and $K(t) = e^{tB}$ with B being the matrix-valued differential operator given by

$$B := \begin{pmatrix} 0 & -\kappa_2 \operatorname{div} \\ -\kappa_2 \nabla & -a \end{pmatrix}.$$

Then the solution semigroup $K(t)$ has the following decay in time, which has been proved in [1, 34].

LEMMA 3.1. *Let $1 \leq p \leq 2$ and $k \geq 0$ be an integer. Suppose $U(t) = (n(t), v(t)) = (K_1(t)U_0, K_2(t)U_0)$ is a solution of the linear system $U_t = BU$. Then for any $t \geq 0$,*

$$\|\nabla^k K_1(t)U_0\| \lesssim (1+t)^{-\sigma(p,2;k)}(\|n_0\|_{L^p} + \|v_0\|_{L^{3p/(3-p)}} + \|(n_0, v_0)\|_k), \tag{3.1}$$

$$\|\nabla^k K_2(t)U_0\| \lesssim (1+t)^{-\sigma(p,2;k+1)}(\|n_0\|_{L^p} + \|v_0\|_{L^{3p/(3-p)}} + \|(n_0, v_0)\|_k), \tag{3.2}$$

where the decay rate is measured by

$$\sigma(p, 2; k) = \frac{3}{2}\left(\frac{1}{p} - \frac{1}{2}\right) + \frac{k}{2}.$$

Finally, if $|\hat{n}_0| \geq c_0$ and $\hat{v}_0 = 0$ for $0 \leq |\xi| \ll 1$, then there exists a positive constant c_2 such that

$$\|n(t)\| \geq c_2(1+t)^{-3/4}, \tag{3.3}$$

$$\|v(t)\| \geq c_2(1+t)^{-5/4}. \tag{3.4}$$

Finally, the following elementary inequality will also be used [8]:

LEMMA 3.2. *If $r_1 > 1$ and $r_2 \in [0, r_1]$, then there exists a constant $C(r_1, r_2)$ such that*

$$\int_0^\tau (1+t-\tau)^{-r_1}(1+\tau)^{-r_2}\, d\tau \leq C(r_1, r_2)(1+t)^{-r_2}.$$

3.2. Convergence rates. We use Lemma 3.1 to show a priori decay-in-time estimates on $(\nabla n, v)$. Hence the decay-in-time estimate on n can be obtained. First we have the following lemma.

Lemma 3.3. *Let (n,v,s) be the solution of (2.1). Then*

$$\|\nabla n(t)\| \lesssim (1+t)^{-\sigma(p,2;1)}(\|n_0\|_{L^p\cap H^1}+\|v_0\|_{L^{3p/(3-p)}\cap H^1}) + \delta\int_0^t (1+t-\tau)^{-\sigma(p,2;1)}(\|\nabla n(\tau)\|_2+\|v(\tau)\|_3)\,d\tau \tag{3.5}$$

for any $t \ge 0$.

Proof. From Duhamel's principle and Lemma 3.1, we have

$$\|\nabla n(t)\| \lesssim (1+t)^{-\sigma(p,2;1)}(\|n_0\|_{L^p\cap H^1}+\|v_0\|_{L^{3p/(3-p)}\cap H^1}) + \int_0^t (1+t-\tau)^{-\sigma(p,2;1)}\|(F,G)(\tau)\|_{L^1\cap H^1}\,d\tau. \tag{3.6}$$

The nonlinear terms F and G can be estimated as follows:

$$\begin{aligned}\|(F,G)\|_{L^1} &\lesssim \|(n,v,\mathcal{S})\|_1(\|\nabla n\|_1+\|v\|_2)\\ &\lesssim \delta(\|\nabla n\|_1+\|v\|_2),\end{aligned} \tag{3.7}$$

$$\begin{aligned}\|(F,G)\|_1 &\lesssim \|(n,v,\mathcal{S})\|_{W^{1,\infty}}(\|\nabla n\|_2+\|v\|_3)\\ &\lesssim \delta(\|\nabla n\|_2+\|v\|_3).\end{aligned} \tag{3.8}$$

Plugging the above two inequalities into (3.6), we arrive at (3.5). ■

Now, we use Lemma 3.1 to derive decay-in-time estimates on $(\nabla n, v)$. Multiplying $(2.1)_2$ by v, integrating over $\mathbb{R}^3$ and using the Cauchy–Schwarz inequality, we obtain

$$\frac{d}{dt}\|v(t)\|^2 + C\|v(t)\|^2 \lesssim \|\nabla n(t)\|^2 + \delta\|\nabla v(t)\|^2. \tag{3.9}$$

Now we define the temporal energy functional

$$M(t) = \|u(t)\|^2 + D_1H_1(n(t),v(t)) + \sum_{1\le|\alpha|\le 2}\langle\partial_x^\alpha\nabla n, \partial_x^\alpha v\rangle(t)$$

for any $t\ge 0$; notice that $M(t)$ is equivalent to $\|\nabla n(t)\|_2^2+\|v(t)\|_3^2$ since D_1 can be large enough. Adding (2.4) to (3.9), we obtain

$$\frac{d}{dt}M(t) + C(\|\nabla^2 n(t)\|_1^2 + \|v(t)\|_3^2) \lesssim \|\nabla n(t)\|^2.$$

Adding $\|\nabla n\|^2$ to both sides gives

$$\frac{d}{dt}M(t) + M(t) \lesssim \|\nabla n(t)\|^2. \tag{3.10}$$

Set

$$N(t) = \sup_{0\le\tau\le t}(1+\tau)^{2\sigma(p,2;1)}M(\tau), \tag{3.11}$$

and notice that

$$\|\nabla n(\tau)\|_2+\|v(\tau)\|_3 \lesssim \sqrt{M(\tau)} \lesssim (1+\tau)^{-\sigma(p,2;1)}\sqrt{N(t)},\ 0 \le \tau \le t. \tag{3.12}$$

Then it follows from Lemma 3.3 that

$$\begin{aligned}\|\nabla n(t)\| &\lesssim (1+t)^{-\sigma(p,2;1)}(\|n_0\|_{L^p}+\|v_0\|_{L^{3p/(3-p)}}) \\ &\quad + \delta\int_0^t (1+t-\tau)^{-\sigma(p,2;1)}(1+\tau)^{-\sigma(p,2;1)}\,d\tau\,\sqrt{N(t)} \\ &\lesssim (1+t)^{-\sigma(p,2;1)}(\|n_0\|_{L^p}+\|v_0\|_{L^{3p/(3-p)}}+\delta\sqrt{N(t)}).\end{aligned} \tag{3.13}$$

Hence, by Gronwall's inequality, (3.10) and (3.13) lead to

$$\begin{aligned}M(t) &\lesssim e^{-t}M(0)+\int_0^t e^{-(t-\tau)}\|\nabla n(\tau)\|^2\,d\tau \\ &\lesssim e^{-t}M(0)+\int_0^t e^{-(t-\tau)}(1+\tau)^{-2\sigma(p,2;1)} \\ &\qquad \times(\|n_0\|_{L^p}+\|v_0\|_{L^{3p/(3-p)}}+\delta\sqrt{N(\tau)})\,d\tau \\ &\lesssim (1+t)^{-2\sigma(p,2;1)}(\|n_0\|^2_{L^p}+\|v_0\|^2_{L^{3p/(3-p)}}+\delta^2 N(t)).\end{aligned} \tag{3.14}$$

Since $N(t)$ is non-decreasing, from (3.12) and (3.14) we have

$$N(t) \lesssim (\|n_0\|^2_{L^p}+\|v_0\|^2_{L^{3p/(3-p)}}+\delta^2 N(t)) \tag{3.15}$$

for any $t \ge 0$, which implies that

$$N(t) \lesssim \|n_0\|^2_{L^p}+\|v_0\|^2_{L^{3p/(3-p)}}, \tag{3.16}$$

since $\delta > 0$ is small enough. Thus we obtain (1.10) from (3.12) and (3.16).

Next, by Duhamel's principle, we arrive at

$$\begin{aligned}\|n(t)\| &\lesssim (1+t)^{-\sigma(p,2;0)}(\|n_0\|_{L^p\cap L^2}+\|v_0\|_{L^{3p/(3-p)}\cap L^2}) \\ &\quad + \int_0^t (1+t-\tau)^{-\sigma(p,2;0)}\|(F,G)(\tau)\|_{L^1\cap L^2}\,d\tau \\ &\lesssim (1+t)^{-\sigma(p,2;0)}(\|n_0\|_{L^p\cap L^2}+\|v_0\|_{L^{3p/(3-p)}\cap L^2}) \\ &\quad + \int_0^t (1+t-\tau)^{-\sigma(p,2;0)}(\|\nabla n(\tau)\|_2+\|v\|_3)\,d\tau \\ &\lesssim \Big[(1+t)^{-\sigma(p,2;0)}+\int_0^t (1+t-\tau)^{-\sigma(p,2;0)}(1+\tau)^{-\sigma(p,2;1)}\,d\tau\Big] \\ &\quad \times(\|n_0\|_{L^p\cap H^3}+\|v_0\|_{L^{3p/(3-p)}\cap H^3}) \\ &\lesssim (1+t)^{-\sigma(p,2;0)}(\|n_0\|_{L^p\cap H^3}+\|v_0\|_{L^{3p/(3-p)}\cap H^3})\end{aligned}$$

for any $t \ge 0$. Thus (1.9) is proved. For (1.11), using the above decay estimates on (n, v), we have

$$\|\partial_t(n, v, \mathcal{S})(t)\| \lesssim (\|n\|_1 + \|\nabla n\|) \lesssim (1+t)^{-\sigma(p,2;1)}.$$

Thus the proof of the convergence rate in Theorem 1.2 is complete. ∎

4. Lower bound on the time decay rate. In this section, we consider the lower bound on the time decay for global solutions. Define

$$\mathcal{P}(t,x) = P^{1/\gamma}(t,x) - \bar{P}^{1/\gamma}, \qquad m(t,x) = \rho(t,x)u(t,x).$$

The Cauchy problem for $(\mathcal{P}, m)$ is

$$\begin{cases} \partial_t \mathcal{P} + \dfrac{\bar{P}^{1/\gamma}}{\bar{\rho}} \operatorname{div} m = -\operatorname{div}\left(\mathcal{P}u + \dfrac{\bar{P}^{1/\gamma}(\bar{\rho}-\rho)m}{\rho\bar{\rho}}\right), \\ \partial_t m + \eta\nabla\mathcal{P} + am = -\nabla(P(\mathcal{P},\bar{P}) - \bar{P} - \eta\mathcal{P}) - \operatorname{div}(\rho u \otimes u), \\ (\mathcal{P}, m)|_{t=0} := (\mathcal{P}_0, m_0) \to (0,0) \quad \text{as } |x| \to \infty, \end{cases} \tag{4.1}$$

where $\eta = P_{\mathcal{P}}(0, \bar{P})$.

Denote $V = (\mathcal{P}, m)$ and

$$\mathcal{F} = \left[-\operatorname{div}\left(\mathcal{P}u + \frac{\bar{P}^{1/\gamma}(\bar{\rho}-\rho)m}{\rho\bar{\rho}}\right), -\nabla(P(\mathcal{P},\bar{P}) - \bar{P} - \eta\mathcal{P}) - \operatorname{div}(\rho u \otimes u)\right].$$

Then by Duhamel's principle, Lemma 3.1 and (1.12), we have

$$\begin{aligned} \|\mathcal{P}\| &\ge c_2\|K_1(t)V_0\| - C\int_0^t \|K_1(t-\tau)\mathcal{F}(\tau)\|\,d\tau \\ &\ge c_3(1+t)^{-3/4} - C\int_0^t (1+t-\tau)^{-5/4} \\ &\qquad \times \left\|\left(\mathcal{P}u + \frac{\bar{P}^{1/\gamma}(\bar{\rho}-\rho)m}{\rho\bar{\rho}}, P(\mathcal{P},\bar{P}) - \bar{P} - \eta\mathcal{P}, \rho u\otimes u\right)(\tau)\right\|_{L^1\cap H^1} d\tau \\ &\ge c_3(1+t)^{-3/4} - C\int_0^t (1+t-\tau)^{-5/4}(1+\tau)^{-5/4}\,d\tau \ge c_4(1+t)^{-3/4}, \end{aligned}$$

which gives (1.13). For m, we have

$$\begin{aligned} \|m\| &\ge c_2\|K_2(t)V_0\| - C\int_0^t \|K_2(t-\tau)\mathcal{F}(\tau)\|\,d\tau \\ &\ge c_3(1+t)^{-5/4} - C\int_0^t (1+t-\tau)^{-7/4} \\ &\qquad \times \left\|\left(\mathcal{P}u + \frac{\bar{P}^{1/\gamma}(\bar{\rho}-\rho)m}{\rho\bar{\rho}}, P(\mathcal{P},\bar{P}) - \bar{P} - \eta\mathcal{P}, \rho u\otimes u\right)(\tau)\right\|_{L^1\cap H^1} d\tau \end{aligned}$$

$$\geq c_3(1+t)^{-5/4} - C\int_0^t (1+t-\tau)^{-7/4}[(1+\tau)^{-3/2} + (1+\tau)^{-5/4}\delta]\,d\tau$$
$$\geq c_4(1+t)^{-5/4},$$

since $\delta > 0$ is sufficiently small. Thus we get (1.14), and the proof of Theorem 1.2 is complete. ∎

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