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PROJECTIVE TENSOR PRODUCT OF PROTO-QUANTUM SPACES

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Abstract. A proto-quantum space is a (general) matricially normed space in the sense of Effros and Ruan presented in a 'matrix-free' language. We show that these spaces have a special (projective) tensor product possessing the universal property with respect to completely bounded bilinear operators. We study some general properties of this tensor product (among them a kind of adjoint associativity), and compute it for some tensor factors, notably for L_1-spaces. In particular, we obtain what could be called the proto-quantum version of the Grothendieck theorem about classical projective tensor products by L_1-spaces. Finally, we compare the new tensor product with the known projective tensor product of operator spaces, and show that the standard construction of the latter is not fit for general proto-quantum spaces.

1. Introduction. In their paper [7], Effros and Ruan introduced and investigated the important notion of a matricially normed space. Very soon, after the discovery of Ruan's Representation Theorem [21], the great majority of papers and monographs were dedicated only to an outstanding special class of these structures, the $\mathcal{L}^\infty$-matricially normed spaces. Now the latter are called abstract operator spaces (or just operator spaces), and sometimes quantum spaces. The theory of operator spaces is very rich and well developed. It is presented in widely known textbooks [11, 19, 20, 2].

On the other hand, already in [7, 21] it was demonstrated that matricially normed spaces are of considerable interest even outside the class of operator spaces, that is, without assuming the second axiom of Ruan. In this paper we return to general matricially normed spaces, but presented in the equivalent 'non-coordinate' or 'matrix-free' language. (The latter seems to be more convenient for us in this circle of questions.) We hope that our observations, together with some results in the cited papers, also show that general matricially normed spaces deserve independent interest. Moreover, in their study we sometimes come to things that look very different from what we know about operator spaces.

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Our main point is that the general matricially normed spaces, which we call proto-quantum spaces, have a tensor product, possessing the universal property relative to the class of completely bounded bilinear operators. In the context of operator spaces, such a tensor product was discussed in [11, II.7.1], and in the non-coordinate language in [14, Ch. 7.2]. Therefore, following the terminology of those textbooks, we call this tensor product projective. (Note that another kind of tensor product, the Haagerup tensor product, was already discussed in [7, 21].)

The contents of the paper are as follows. The second and third sections contain initial definitions, notably of a proto-quantum space, of a completely bounded operator and of a completely bounded bilinear operator. Also the simplest examples are presented, in particular the maximal and minimal proto-quantization of a given normed space. We show that the maximal proto-quantization always gives rise to an $\mathcal{L}^1$-space, and that every bounded functional with domain an $\mathcal{L}^p$-space and with range $\mathbb{C}$ made into an $\mathcal{L}^q$-space is 'automatically' completely bounded provided $p \leq q$. (The notation $\mathcal{L}^p$, here and hereafter, refers to the non-coordinate counterpart of the matricially normed $\mathcal{L}^p$-space, initially introduced in [7].)

In Section 4 we consider several further examples of proto-quantum non-quantum spaces. In particular, we introduce what we call the standard proto-quantization of the space $L_p(X, E)$, where E is a proto-quantum space. (Among the examples one can find the $\mathcal{L}^p$-space $L_p(X, E)$, where E is an $\mathcal{L}^p$-space.) Also we show that some related bilinear operators are completely contractive; this will be used in subsequent sections.

In Section 5 we define the non-completed projective tensor product of proto-quantum spaces, denoted by $\otimes_{\text{pop}}$, and its 'completed' version, denoted by $\widehat{\otimes}_{\text{pop}}$. We prove the corresponding existence theorems by displaying explicit constructions.

In Section 6 we present some examples of the computation of the tensor product introduced. It turns out that, just as in the case of the classical projective tensor product of normed spaces, L_1-spaces are especially nice tensor factors. As the base of most applications, we show that for all proto-quantum spaces E and F we have

$$L_1(X, E) \,\widehat{\otimes}_{\text{pop}}\, L_1(Y, F) \simeq L_1(X \times Y, E \,\widehat{\otimes}_{\text{pop}}\, F).$$

(Here and hereafter, $\simeq$ means a completely isometric isomorphism.) Another frequently used fact is that, under some assumptions on E and F,

$$E \,\widehat{\otimes}_{\text{pop}}\, F \simeq E \,\widehat{\otimes}_{\text{pr}}\, F,$$

where the right side is the classical projective tensor product, made a proto-quantum space according to a recipe in Section 4. Combining these two theorems, we find that for a p-convex proto-quantum space E (in particular, an $\mathcal{L}^p$-space) and the complex plane, considered as an $\mathcal{L}^p$-space, we have

$$L_1(X, \mathbb{C}) \widehat{\otimes}_{\text{pop}} E \simeq L_1(X, E).$$

This result can be considered as a version, for proto-quantum spaces, of the Grothendieck theorem on tensoring by L_1-spaces (cf., e.g., [12, §2, Chap. I, n°2]).

At the beginning of the next section we extend to general proto-quantum spaces the method of quantization of a given space of completely bounded operators, first suggested in [8, p. 140], [3], [9] (see also the textbooks [11, I.3.2] or [14, Ch. 8.7]). Then we establish the suitable form of the so-called law of adjoint associativity, connecting spaces of operators with tensor products. (The form of that law in the classical functional analysis is presented, e.g., in [14, Ch. 6.1].) Namely, for proto-quantum spaces E, F, G, the space $\mathcal{CB}(E \otimes_{\text{pop}} F, G)$ is, in a natural way, (completely) isometrically isomorphic to $\mathcal{CB}(F, \mathcal{CB}(E, G))$ and to $\mathcal{CB}(E, \mathcal{CB}(F, G))$.

(We recall that the above mentioned method essentially differs from the initial approach to what we call a dual matricially normed space. This approach was considered in [7], and shown to have some advantages. However, its essential drawback is that it does not lead to the adjoint associativity.)

In the concluding Section 8 we compare the tensor product $\otimes_{\text{pop}}$ with what could be called its prototype. By this we mean the well known projective tensor product of operator spaces, denoted here by $\otimes_{\text{op}}$, discovered independently by Blecher/Paulsen [3] and Effros/Ruan [9]. For operator spaces (i.e. when the second axiom of Ruan is fulfilled) both tensor products coincide. However, for general proto-quantum spaces the standard formulae for the norms give different numbers: in the case of $\otimes_{\text{op}}$ they are essentially greater than $\otimes_{\text{pop}}$. As an example, we consider the projective tensor square of a certain proto-quantum space, and for every n we display an element of its amplification for which the first number is n^2, whereas the second is n.

2. Proto-quantum spaces and their first examples. As already mentioned, we use the so-called non-coordinate approach to the structures in question, and not the more widespread 'matrix' approach, as in [11, 19, 20, 2]. Some of our terms and notation are contained in [14], where practically only (abstract) operator spaces, called quantum spaces there, were considered. For the convenience of the reader, we briefly repeat some of the needed definitions.

To begin with, we fix an arbitrary separable infinite-dimensional Hilbert space L. We write $\mathcal{B}$ instead of $\mathcal{B}(L)$, the Banach algebra of all bounded operators on L with the operator norm, usually denoted just by $\|\cdot\|$. The symbol $\otimes$ is used for the (algebraic) tensor product of linear spaces and for elementary tensors. The symbols $\otimes_{\text{pr}}$ and $\otimes_{\text{in}}$ denote the non-completed projective and injective tensor products of normed spaces, respectively. The complex-conjugate space of a linear space E is denoted by E^{cc}. The identity operator on a linear space E is denoted by $\mathbf{1}_E$, and we write $\mathbf{1}$ instead of $\mathbf{1}_L$.

For $\xi, \eta \in L$ we denote by $\xi \circ \eta$ the rank 1 operator on L taking ζ to $\langle \zeta, \eta \rangle \xi$. Recall that $\|x \circ y\| = \|x\| \, \|y\|$.

Denote by $\mathcal{F}$ the (non-closed) two-sided ideal of $\mathcal{B}$ consisting of all finite rank bounded operators. Recall that there is a linear isomorphism $L \otimes L^{\mathrm{cc}} \to \mathcal{F}$, well defined by taking $\xi \otimes \eta$ to $\xi \circ \eta$. For $p \in [1, \infty]$ we denote by $\| \cdot \|_p$ the norm of the pth Schatten class on $\mathcal{F}$, and write $\mathcal{F}_p := (\mathcal{F}, \| \cdot \|_p)$; in particular, $\mathcal{F}_\infty$ is $\mathcal{F}$ with the operator norm.

In what follows we need a triple notion of amplification: first, we amplify linear spaces, then linear operators and finally bilinear operators.

The *amplification* of a given linear space E is the tensor product $\mathcal{F} \otimes E$. Usually we briefly denote it by $\mathcal{F}E$, and an elementary tensor $a \otimes x$ with $a \in \mathcal{F}$ and $x \in E$ by ax. Note that $\mathcal{F}E$ is a bimodule over the algebra $\mathcal{B}$ with the outer multiplications denoted by '$\cdot$' and well defined by $a \cdot (bx) := (ab)x$ and $(ax) \cdot b := (ab)x$.

Definition 2.1. A seminorm on $\mathcal{F}E$ is called a *proto-quantum seminorm*, or briefly a *PQ-seminorm*, on E if the $\mathcal{B}$-bimodule $\mathcal{F}E$ is contractive, that is, $\|a \cdot u \cdot b\| \leq \|a\| \, \|u\| \, \|b\|$. A PQ-seminorm on E is called a *quantum seminorm*, or briefly a *Q-seminorm*, on E if for all $u, v \in \mathcal{F}E$ and (ortho)projections $P, Q \in \mathcal{B}$ with $PQ = 0$ we have

$$\|P \cdot u \cdot P + Q \cdot v \cdot Q\| = \max\{\|P \cdot u \cdot P\|, \|Q \cdot v \cdot Q\|\}.$$

The space E endowed with a PQ-seminorm is called a *seminormed proto-quantum space*, or briefly a *seminormed PQ-space*. In the case of a normed PQ-space we usually omit the word 'normed'.

In a similar way we use the terms *seminormed Q-space* and *Q-space*.

Remark 2.2. By their definition, Q-spaces can be treated as a special case of Ruan bimodules, considered in [15] and [23].

Remark 2.3. Let us recall, for the convenience of the reader, the way of translating 'matrix language' to 'non-coordinate language'. Let E be a matricially normed space in the sense of [7], and let $u \in \mathcal{F}E$. Clearly, there exists a finite rank projection P such that u has the form $\sum_{k=1}^n a_k x_k$ with $a_k = P \cdot a_k \cdot P$ and $x_k \in E$. We choose an arbitrary orthonormal basis in $P(L)$ and denote by (a_{ij}^k) the matrix, in this basis, of the restriction of a_k to $P(L)$. Then we take the matrix $(u_{ij} := \sum_k a_{ij}^k x_k)$ with entries in E and set $\|u\| := \|(u_{ij})\|$. It is easy to show that $\|u\|$ does not depend on the choice of P and of the basis in $P(L)$, and that the function $u \mapsto \|u\|$ is a PQ-norm on E.

A seminormed PQ-space E becomes a seminormed space (in the usual sense) if for $x \in E$ we set $\|x\| := \|Qx\|$, where Q is an arbitrary rank 1 operator of norm 1. Obviously, this seminorm does not depend on the choice of Q. The resulting seminormed space is called the *underlying space* of a given PQ-space, and the latter is called a *proto-quantization* (briefly a P-quantization) or, if we deal with a Q-space, a *quantization* of the former. (The term 'quantization' comes from Effros' seminal lecture [6]. Indeed, in the space $E = \mathbb{C} \otimes E$ the commutative scalars from $\mathbb{C}$ are replaced by the 'non-commutative scalars' from $\mathcal{F}$, a typical device of 'quantum mathematics'.)

Proposition 2.4. *Let E be a seminormed PQ-space with a normed underlying space. Then the seminorm on $\mathcal{F}E$ is a norm.*

The proof, given in [14, Prop. 1.2.2] for Q-spaces, is valid without any modification.

Example 2.5. Every non-zero normed space, say E, has a lot of P-quantizations. We distinguish the *maximal* and *minimal* ones, denoted by $E_{\max}$ and $E_{\min}$, respectively. The first is obtained by endowing $\mathcal{F}E$ with the norm of $L \otimes_{\mathrm{pr}} L^{\mathrm{cc}} \otimes_{\mathrm{pr}} E = \mathcal{F}_1 \otimes_{\mathrm{pr}} E$, and the second with the norm of $L \otimes_{\mathrm{in}} L^{\mathrm{cc}} \otimes_{\mathrm{in}} E = \mathcal{F}_\infty \otimes_{\mathrm{in}} E$. (Evidently, the first of these PQ-norms is never a Q-norm, whereas it is not difficult to show that the second one is always a Q-norm.)

As a matter of fact, the first norm is maximal in the sense that it is the greatest of all norms of P-quantizations of E. Indeed, we easily see that the norm on $L \otimes L^{\mathrm{op}} \otimes E$, corresponding to any given PQ-norm on E, is a cross-norm. But among all cross-norms there is a greatest one, and it is exactly the norm on $L \otimes_{\mathrm{pr}} L^{\mathrm{op}} \otimes_{\mathrm{pr}} E$. In a similar sense the second norm is minimal, but this statement will be justified a little later.

As to $E_{\max}$, it is a member of the whole family of P-quantizations of E, denoted by ${}^{(p)}E$, $1 \leq p \leq \infty$; they are obtained by endowing $\mathcal{F}E$ with the norm of $\mathcal{F}_p \otimes_{\mathrm{pr}} E$. Clearly, ${}^{(1)}E = E_{\max}$. In particular, among various P-quantizations of $\mathbb{C}$ we distinguish PG-spaces ${}^{(p)}\mathbb{C}$; we see that the amplification of such a space is identified with $\mathcal{F}_p$. Moreover, there is only one P-quantization of $\mathbb{C}$ which is a quantization, and this is ${}^{(\infty)}\mathbb{C}$.

In what follows, if numbers $\lambda_k \geq 0$, $k = 1, \dots, n$, are given, we shall understand $(\sum_{k=1}^n \lambda_k^p)^{1/p}$ as $\max\{\lambda_1, \dots, \lambda_n\}$ in the case $p = \infty$.

We shall say that a projection $P \in \mathcal{B}$ is a *support* of an element $u \in \mathcal{F}E$ if $P \cdot u \cdot P = u$.

For $p \in [1, \infty]$ we shall say that a PQ-space E is an *$\mathcal{L}^p$-space*, respectively, a *p-convex space* or a *p-concave space*, if for any $u_1, \dots, u_n \in \mathcal{F}E$ with pairwise orthogonal supports we have $\|\sum_{k=1}^n u_k\| = (\sum_{k=1}^n \|u_k\|^p)^{1/p}$, respectively $\|\sum_{k=1}^n u_k\| \leq (\sum_{k=1}^n \|u_k\|^p)^{1/p}$, $\|\sum_{k=1}^n u_k\| \geq (\sum_{k=1}^n \|u_k\|^p)^{1/p}$. Obviously, it is sufficient to have similar relations for $n = 2$. We see that '$\mathcal{L}^\infty$-space' is just another name for 'Q-space'. Clearly, every PQ-space is 1-convex and ∞-concave. Moreover, ${}^{(p)}\mathbb{C}$ is evidently an $\mathcal{L}^p$-space.

Throughout the paper, for $a, b \in \mathcal{F}$ we shall write $a \approx b$ provided $SaT = b$ for some unitary $S, T \in \mathcal{B}$. Similarly, for $u, v \in \mathcal{F}E$, where E is a PQ-space, we shall write $u \approx v$ provided $S \cdot u \cdot T = v$ for S, T as above. Clearly, $a \approx b$ implies $\|a\|_p = \|b\|_p$ for all $p \in [1, \infty]$, and $u \approx v$ implies $\|u\| = \|v\|$. It is well known (and easy to show) that for every $a \in \mathcal{F}$ we have

$$a \approx h, \quad \text{where} \quad h = \sum_{k=1}^n s_k P_k \tag{2.1}$$

for some pairwise orthogonal rank 1 projections $P_k \in \mathcal{F}$, and some $s_k \geq 0$.

Proposition 2.6. *If E is a p-convex PQ-space, a p-concave PG-space or an $\mathcal{L}^p$-space, then for all $a \in \mathcal{F}$ and $x \in E$ we have $\|ax\| \leq \|a\|_p\|x\|$, $\|ax\| \geq \|a\|_p\|x\|$ or $\|ax\| = \|a\|_p\|x\|$, respectively. In particular, $\|ax\| \leq \|a\|_1\|x\|$ for every E.*

Proof. Let h be as in (2.1). Then $ax \approx hx$. But $hx = \sum_{k=1}^n s_k P_k x$, where the summands have pairwise orthogonal supports, namely P_k. Therefore in the 'convex' case we have $\|ax\| = \|hx\| \leq (\sum_{k=1}^n \|s_k P_k x\|^p)^{1/p} = (\sum_{k=1}^n s_k^p)^{1/p}\|x\|$, where (recall) $(\sum_{k=1}^n s_k^p)^{1/p}$ is just $\|h\|_p$, that is, $\|a\|_p$. The remaining cases are treated in a similar way. ∎

Example 2.7. We shall show that the maximal P-quantization of a given normed space E is an $\mathcal{L}^1$-space. (And thus every normed space can be made an $\mathcal{L}^1$-space.)

Indeed, consider orthogonal projections $P, Q \in \mathcal{B}$ and the subspaces $\mathcal{F}_1^P := \{PaP : a \in \mathcal{F}\}$, $\mathcal{F}_1^Q := \{QaQ : a \in \mathcal{F}\}$ and $\mathcal{F}_1^{P,Q} = \{PaP + QaQ : a \in \mathcal{F}\}$ in $\mathcal{F}_1$. Clearly, $\mathcal{F}_1^{P,Q} = \mathcal{F}_1^P \oplus_1 \mathcal{F}_1^Q \subseteq \mathcal{F}_1$, where $\oplus_1$ means the ℓ_1-sum of normed spaces.

It is well known (and easy to check) that the operator $j : \mathcal{F}_1 \to \mathcal{F}_1^{P,Q} : a \mapsto Pap + QaQ$ is contractive (in fact, it is a norm 1 projection). Consider the operators $j \otimes_{\text{pr}} \mathbf{1}_E : \mathcal{F}_1 \otimes_{\text{pr}} E \to \mathcal{F}_1^{P,Q} \otimes_{\text{pr}} E$ and $i \otimes_{\text{pr}} \mathbf{1}_E : \mathcal{F}_1^{P,Q} \otimes_{\text{pr}} E \to \mathcal{F}_1 \otimes_{\text{pr}} E$, where i is the natural embedding. Both of them, being projective tensor products of contractive operators, are contractive themselves. But their composition is evidently the identity operator on $\mathcal{F}_1^{P,Q} \otimes_{\text{pr}} E$. It follows that $i \otimes_{\text{pr}} \mathbf{1}_E$ is an isometry (whereas $j \otimes_{\text{pr}} \mathbf{1}_E$ is a strict coisometry).

Now suppose that $u, v \in \mathcal{F}_1 \otimes_{\text{pr}} E$ have P and Q as their respective supports. Observe that for every $w \in \mathcal{F}_1 \otimes_{\text{pr}} E$ the equality $w = P \cdot w \cdot P + Q \cdot w \cdot Q$ means exactly that $w \in \mathcal{F}_1^{P,Q}$. Consequently, the elements u, v and $u + v$ have the same norms in $\mathcal{F}_1^{P,Q} \otimes_{\text{pr}} E$ and in $\mathcal{F}_1 \otimes_{\text{pr}} E$.

Finally, recall the known connection between $\oplus_1$ and $\otimes_{\text{pr}}$. In our situation we have an isometric isomorphism $I : \mathcal{F}_1^{P,Q} \otimes_{\text{pr}} E \to (\mathcal{F}_1^P \otimes_{\text{pr}} E) \oplus_1 (\mathcal{F}_1^Q \otimes_{\text{pr}} E)$, well defined by taking $(a+b) \otimes x$ to $a \otimes x + b \otimes x$. But for $u, v \in \mathcal{F}_1^{P,Q} \otimes_{\text{pr}} E$ we see that $I(u) \in \mathcal{F}_1^P \otimes_{\text{pr}} E$ and $I(v) \in \mathcal{F}_1^Q \otimes_{\text{pr}} E$. Therefore

$$\|u+v\| = \|I(u+v)\| = \|I(u)+I(v)\| = \|I(u)\| + \|I(v)\| = \|u\| + \|v\|,$$

and we are done.

Note, however, that the PG-space ${}^{(p)}E$ for $E \neq \mathbb{C}$ and $p > 1$ is not, generally speaking, an $\mathcal{L}^p$-space.

Remark 2.8. Recall that numerous examples, all of them concerning Q-spaces, are presented in the cited textbooks. They include an example, which is the most important in the whole theory of Q-spaces, being, in a sense, universal [21, 10]. This is the so-called concrete quantization of a space, consisting of operators. But we do not need this in the present paper.

3. Completely bounded linear and bilinear operators. Suppose that we are given an operator $\varphi : E \to F$ between linear spaces. The *amplification* of φ is the operator $\varphi_\infty : \mathcal{F}E \to \mathcal{F}F$ well defined on elementary tensors by $ax \mapsto a\varphi(x)$. Clearly, φ_∞ is a morphism of $\mathcal{B}$-bimodules.

Definition 3.1. An operator φ between seminormed PQ-spaces is called *completely bounded*, respectively *completely contractive*, if its amplification is bounded, respectively contractive, in the usual sense. We set $\|\varphi\|_{\text{cb}} := \|\varphi_\infty\|$.

In a similar way we define the notions of a *completely isometric operator* and of a *completely isometric isomorphism.*

If φ is bounded as a map of the underlying seminormed spaces, we say that it is (just) *bounded* and denote its operator seminorm as usual by $\|\varphi\|$. Every completely bounded linear operator is obviously bounded, and we have $\|\varphi\| \leq \|\varphi\|_{\text{cb}}$.

Denote by $\mathcal{CB}(E,F)$ the subspace of $\mathcal{B}(E,F)$ consisting of completely bounded linear operators. It is a normed space with respect to $\|\cdot\|_{\text{cb}}$.

Some linear operators between PQ-spaces that are bounded are 'automatically' completely bounded. Here is an observation of that kind.

Proposition 3.2. *Let E be an $\mathcal{L}^p$-space or, more generally, a p-concave PQ-space for some $p \in [1,\infty]$ Then every bounded functional $f : E \to {}^{(q)}\mathbb{C}$, where $q \geq p$, is completely bounded, and $\|f\|_{\text{cb}} = \|f\|$.*

Proof. Let $u \in \mathcal{F}E$. If in (2.1) we set $a := f_\infty(u)$ and $v := S \cdot u \cdot T$, where S, T are unitary operators, we see that for some pairwise orthogonal rank 1 projections P_k and $s_k \geq 0$ we have

(3.1)
$$f_\infty(v) = \sum_k s_k P_k, \quad \|u\| = \|v\| \quad \text{and} \quad \|f_\infty(u)\|_q = \|f_\infty(v)\|_q = \Big(\sum_{k=1}^n s_k^q\Big)^{1/q}.$$

Therefore it suffices to prove that $(\sum_{k=1}^n s_k^q)^{1/q} \leq \|f\|\,\|v\|$.

Denote by ζ a primitive nth root of 1 and set, for $m = 1, \dots, n$, $W_m := \sum_{k=1}^n \zeta^{mk} P_k$ and $W'_m := \sum_{k=1}^n \zeta^{-mk} P_k$. Then a routine calculation shows that $\sum_{m=1}^n W'_m a W_m = n \sum_{k=1}^n P_k a P_k$ for all $a \in \mathcal{B}$. Hence, representing v as a sum of elementary tensors, we see that $\sum_{m=1}^n W'_m \cdot v \cdot W_m = n \sum_{k=1}^n P_k \cdot v \cdot P_k$. From this we have

(3.2)
$$\Big\|\sum_{k=1}^n P_k \cdot v \cdot P_k\Big\| \leq \frac{1}{n} \sum_{m=1}^n \|W'_m \cdot v \cdot W_m\| \leq \frac{1}{n} \sum_{m=1}^n \|W'_m\|\,\|v\|\,\|W_m\| \leq \|v\|.$$

Since $P_k b P_k$ is proportional to P_k for all $b \in \mathcal{B}$ and $k = 1, \dots, n$, we easily see that $P_k \cdot v \cdot P_k = P_k x_k$ for some $x_k \in E$.

Therefore, by (3.1), for all k,

$$s_k P_k = P_k f_\infty(v) P_k = f_\infty(P_k \cdot v \cdot P_k) = f_\infty(P_k x_k) = f(x_k) P_k,$$

hence $f(x_k) = s_k$, and consequently

$$s_k \leq \|f\|\,\|x_k\| = \|f\|\,\|P_k x_k\| = \|f\|\,\|P_k \cdot v \cdot P_k\|.$$

Therefore, taking into account that E is p-concave, we have

$$\Big(\sum_{k=1}^n s_k^p\Big)^{1/p} \leq \|f\| \Big(\sum_{k=1}^n \|P_k \cdot v \cdot P_k\|^p\Big)^{1/p} \leq \|f\| \Big(\sum_{k=1}^n \|P_k \cdot v \cdot P_k\|^q\Big)^{1/q} \leq \|f\| \Big\|\sum_{k=1}^n P_k \cdot v \cdot P_k\Big\|.$$

It remains to apply (3.2). ∎

In particular (cf. [7]), *for every PQ-space E every bounded functional $f : E \to {}^{(\infty)}\mathbb{C}$ is completely bounded, and $\|f\|_{\mathrm{cb}} = \|f\|$.*

Note that the latter assertion immediately implies that for a normed space E the norm on $\mathcal{F}E$ given by $\|u\| := \sup\{\|f_\infty(u)\|_\infty : f \in E^*,\ \|f\| \leq 1\}$ is the smallest among all norms of P-quantizations of E. But this is exactly the norm on $L \otimes_{\mathrm{in}} L^{\mathrm{cc}} \otimes_{\mathrm{in}} E$. This justifies the word 'minimal' for the latter norm (see above). Also we see that we have got a Q-norm.

On the other hand, in contrast to the space of all bounded operators, the space $\mathcal{CB}(E, F)$ can be very scanty. The following observation is taken from [21].

PROPOSITION 3.3. *Let E be a p-convex PQ-space, F a q-concave PQ-space, and $p > q$. Then there is no non-zero completely bounded operator from E into F.*

Proof. Let $\varphi : E \to F$ be a non-zero operator; our task is to show that it is not completely bounded. Take $x \in E$ with $\varphi(x) \neq 0$. Since $p > q$, for every $n \in \mathbb{N}$ there exist $m \in \mathbb{N}$ such that $m^{1/q} > (n\|x\|/\|\varphi(x)\|)m^{1/p}$. Take pairwise orthogonal rank 1 projections P_k, $k = 1, \ldots, m$, and set $u := \sum_k P_k x \in \mathcal{F}E$; then $\varphi_\infty(u) = \sum_k P_k\varphi(x) \in \mathcal{F}F$. We see that the elements $P_k x \in \mathcal{F}E$ as well as $P_k\varphi(x) \in \mathcal{F}F$ have pairwise orthogonal supports. Therefore

$$\begin{aligned}\|\varphi_\infty(u)\| &\geq \Big(\sum_k \|P_k\varphi(x)\|^q\Big)^{1/q} = (m\|\varphi(x)\|^q)^{1/q} = \|\varphi(x)\|m^{1/q} \\ &> \|\varphi(x)\|\frac{n\|x\|}{\|\varphi(x)\|}m^{1/p} = n\|x\|m^{1/p} \geq n\Big\|\sum_k P_k x\Big\| = n\|u\|.\end{aligned}$$

Since n is arbitrary, this means that the operator φ_∞ is not bounded. ∎

However, most of the various known counter-examples (one of the earliest is due to Tomiyama [22]) concern Q-spaces (see the textbooks cited above).

To amplify bilinear operators (in what follows, we shall say, for brevity, 'bioperators'), we shall use a certain operation that imitates the tensor product of operators on our Hilbert space L but does not lead out of L. (Within the 'matrix' approach we would have to use the Kronecker product of matrices.)

In what follows, the symbol $\dot{\otimes}$ is used for the Hilbert tensor product of Hilbert spaces, as well as of bounded operators acting on these spaces. By the Riesz–Fisher theorem, we can fix an arbitrary unitary isomorphism $\iota : L \dot{\otimes} L \to L$. Following [13], for $\xi, \eta \in L$ we denote the vector $\iota(\xi \otimes \eta) \in L$ by $\xi \diamondsuit \eta$, and for $a, b \in \mathcal{B}$ we denote the operator $\iota(a \dot{\otimes} b)\iota^{-1}$ on L by $a \diamondsuit b$; obviously, the latter is well defined by $(a \diamondsuit b)(\xi \diamondsuit \eta) = a(\xi) \diamondsuit b(\eta)$. Evidently,

$$(3.3)\quad (a \diamondsuit b)(c \diamondsuit d) = ac \diamondsuit bd, \quad \|\xi \diamondsuit \eta\| = \|\xi\|\,\|\eta\|, \quad \|a \diamondsuit b\| = \|a\|\,\|b\|.$$

Now suppose that we are given a bioperator $\mathcal{R} : E \times F \to G$ between linear spaces. Its *amplification* is the bioperator $\mathcal{R}_\infty : \mathcal{F}E \times \mathcal{F}F \to \mathcal{F}G$ well defined on elementary tensors by $\mathcal{R}_\infty(ax, by) = (a \diamondsuit b)\mathcal{R}(x, y)$.

REMARK 3.4. We do not consider here a different version of the amplification of a bioperator, which would lead to the important notion of the Haagerup tensor product of PQ-spaces (cf. [7] and, in the context of Q-spaces, [1, 3] and also the textbooks [11, 14]).

DEFINITION 3.5. A bioperator $\mathcal{R}$ between seminormed PQ-spaces is called *completely bounded*, respectively *completely contractive*, if its amplification is bounded, respectively contractive, in the usual sense. We set $\|\mathcal{R}\|_{\mathrm{cb}} := \|\mathcal{R}_\infty\|$.

It is easy to see that after restricting ourselves to Q-spaces and translating back to the 'matrix language', we shall obtain the standard definition of completely bounded (and completely contractive) bioperator between operator spaces (see [11, p. 126]).

Here is another example of 'automatic complete boundedness'. *If E, F are PQ-spaces, and $f : E \to \mathbb{C}$ and $g : F \to \mathbb{C}$ are bounded functionals, then the bilinear functional $f \times g : E \times F \to {}^{(\infty)}\mathbb{C} : (x, y) \mapsto f(x)g(y)$ is completely bounded, and $\|f \times g\|_{\mathrm{cb}} = \|f \times g\| = \|f\|\,\|g\|$.* This can be easily deduced from Proposition 3.2 with the help of the formula $(f \times g)_\infty(u, v) = f_\infty(u) \diamondsuit g_\infty(v)$ for $u \in \mathcal{F}E$ and $v \in \mathcal{F}F$.

As a good exercise, we mention the inner product bilinear functional $H \times H^{\mathrm{cc}} \to {}^{(\infty)}\mathbb{C} : (x, y) \mapsto \langle x, y\rangle$, where H is a Hilbert space. It is completely contractive if we endow both H and H^{cc} with the maximal PQ-norm (cf. Example 2.5), and it is not completely bounded if we endow them with the minimal Q-norm.

4. Further examples of proto-quantum spaces and related bilinear operators. We introduce several examples of PQ-spaces. Later some of them will show especially good behavior as tensor factors.

Example 4.1. Let (X, μ) be a measure space and F be an arbitrary PQ-space. We want to endow the normed space $L_p(X, F)$, $1 \le p \le \infty$, of p-integrable F-valued measurable functions on X with a PQ-norm.

As a preliminary step, consider the (non-completed) normed space $L_p(X, \mathcal{F}F)$ and note that it is a $\mathcal{B}$-bimodule with the outer multiplications defined by

$$[a\cdot\bar{x}](t) := a\cdot[\bar{x}(t)], \quad [\bar{x}\cdot b](t) := [\bar{x}(t)]\cdot b, \quad a, b \in \mathcal{B},\ \bar{x} \in L_p(X, \mathcal{F}F),\ t \in X.$$

A routine calculation shows that this bimodule is contractive.

Now consider the operator $\alpha : \mathcal{F}(L_p(X, F)) \to L_p(X, \mathcal{F}F)$ well defined on elementary tensors by taking ax to the $\mathcal{F}F$-valued function $\bar{x}(t) := a(x(t))$. Introduce the seminorm on $\mathcal{F}(L_p(X, F))$ by setting $\|u\| := \|\alpha(u)\|$. Observe that α is a $\mathcal{B}$-bimodule morphism: to show this, it is sufficient to consider appropriate elementary tensors.

Thus, there is an isometric morphism of the seminormed bimodule $\mathcal{F}(L_p(X, F))$ into a contractive $\mathcal{B}$-bimodule. It follows immediately that the former bimodule is itself contractive, hence the above seminorm on $\mathcal{F}(L_p(X, F))$ is a PQ-seminorm on $L_p(X, F)$. Further, for an arbitrary rank 1 operator $Q \in \mathcal{F}$ with $\|Q\| = 1$, and $x \in L_p(X, F)$, we have $\|Q[x(t)]\| = \|x(t)\|$ for all $t \in X$. Therefore for $Qx \in \mathcal{F}(L_p(X, F))$ we easily get

$\|Qx\| = \|x\|$. This means that the underlying seminormed space of the PQ-space constructed is the 'classical' $L_p(X, F)$. Consequently, Proposition 2.4 guarantees that the PQ-seminorm on $L_p(X, F)$ is actually a norm.

It is easy to verify that the PQ-space $L_p(X, F)$ is p-convex or p-concave provided F has the same property. In particular, if F is an $\mathcal{L}_p$-space, then so is $L_p(X, F)$. Note also that the PQ-space $L_p(X, F)$ is not a Q-space whenever $p < \infty$ and X is not a single atom.

EXAMPLE 4.2. Now we want to introduce a P-quantization of the 'classical' tensor product $E \otimes_{\mathrm{pr}} F$ of normed spaces when one of the tensor factors, say F, is a PQ-space.

Consider the linear isomorphism $\beta : \mathcal{F}(E \otimes F) \to E \otimes_{\mathrm{pr}} \mathcal{F}F$ well defined by taking $a(x \otimes y)$ to $x \otimes ay$, and introduce a norm on $\mathcal{F}(E \otimes F)$ by setting $\|U\| := \|\beta(U)\|$. The space $E \otimes_{\mathrm{pr}} \mathcal{F}F$, as a projective tensor product of a normed space and a contractive $\mathcal{B}$-bimodule, has itself a standard structure of a contractive $\mathcal{B}$-bimodule. The same is true of $\mathcal{F}(E \otimes F)$, because β is obviously a $\mathcal{B}$-bimodule morphism. Thus $E \otimes F$ becomes a PQ-space, and we must show that its underlying normed space is exactly $E \otimes_{\mathrm{pr}} F$.

Denote the norm on $E \otimes_{\mathrm{pr}} F$ and on $E \otimes_{\mathrm{pr}} \mathcal{F}F$ by $\| \cdot \|_{\mathrm{pr}}$, and the above PQ-norm, as well as the norm of the underlying space, just by $\| \cdot \|$.

Take $u \in E \otimes F$. It is easy to check that the norm on the underlying space is a cross-norm, hence $\|u\| \leq \|u\|_{\mathrm{pr}}$. Therefore our task is to show that $\|Pu\| \geq \|u\|_{\mathrm{pr}}$ for every rank 1 projection $P \in \mathcal{F}$.

Take an arbitrary representation of $\beta(Pu)$ as $\sum_{k=1}^n x_k \otimes w_k$ with $x_k \in E$ and $w_k \in \mathcal{F}F$. Obviously, $P \cdot w_k \cdot P = Py_k$ for some $y_k \in F$, $k = 1, \dots, n$. Therefore $\sum_{k=1}^n \|x_k\| \, \|w_k\| \geq \sum_{k=1}^n \|x_k\| \, \|P \cdot w_k \cdot P\| = \sum_{k=1}^n \|x_k\| \, \|y_k\|$.

But $\beta(Pu) = P \cdot \beta(Pu) \cdot P = \sum_{k=1}^n x_k \otimes P \cdot w_k \cdot P = \beta(P[\sum_{k=1}^n x_k \otimes y_k])$. It follows that $u = \sum_{k=1}^n x_k \otimes y_k$, and consequently $\sum_{k=1}^n \|x_k\| \, \|w_k\| \geq \|u\|_{\mathrm{pr}}$. From this, by the definition of the norm on $E \otimes_{\mathrm{pr}} \mathcal{F}F$, we get $\|Pu\| = \|\beta(Pu)\| \geq \|u\|_{\mathrm{pr}}$.

The above PQ-spaces feature in some bioperators that we shall essentially use. Their study needs a certain extended version of the operation $\diamondsuit$. Namely, if E is a linear space, $a \in \mathcal{F}$ and $u \in \mathcal{F}E$, then we introduce in $\mathcal{F}E$ the elements $a \diamondsuit u$ and $u \diamondsuit a$ defined by setting $a \diamondsuit (\sum_k b_k x_k) := \sum_k (a \diamondsuit b_k) x$ and $(\sum_k b_k x_k) \diamondsuit a := \sum_k (b_k \diamondsuit a) x_k$. We shall use the following properties of $\diamondsuit$ that may be of independent interest.

As a preparatory step, for a given $e \in L$ with $\|e\| = 1$ we introduce the operator S on L acting as $\zeta \mapsto e \diamondsuit \zeta$; it is of course an isometry. It is easy to verify that for all $b \in \mathcal{F}$ and $P := e \circ e$ we have

$$b = S^*(P \diamondsuit b)S \quad \text{and} \quad P \diamondsuit b = SbS^*. \tag{4.1}$$

PROPOSITION 4.3. *Let E be a PQ-space and $u \in \mathcal{F}E$. Then:*

(i) *For every $a \in \mathcal{F}$ we have $\|a \diamondsuit u\| = \|u \diamondsuit a\|$.*
(ii) *For every $Q \in \mathcal{F}$ of rank 1 we have $\|Q \diamondsuit u\| = \|Q\| \, \|u\|$.*
(iii) *For every $a \in \mathcal{F}$ we have $\|a \diamondsuit u\| \leq \|a\|_p \|u\|$ provided E is p-convex, $\|a \diamondsuit u\| \geq \|a\|_p \|u\|$ provided E is p-concave, and as a corollary, $\|a \diamondsuit u\| = \|a\|_p \|u\|$ provided $E \in \mathcal{L}_p$. In particular, $\|a \diamondsuit u\| \leq \|a\|_1 \|u\|$ for all E.*

Proof. (i) Consider the unitary operator $\triangle$ on L, well defined by taking $\xi \diamondsuit \eta$, $\xi, \eta \in L$, to $\eta \diamondsuit \xi$. Obviously, $b \diamondsuit a = \triangle(a \diamondsuit b)\triangle$ for all $a, b \in \mathcal{F}$. From this we easily deduce that $\triangle \cdot (a \diamondsuit u) \cdot \triangle = u \diamondsuit a$ for all $a \in \mathcal{F}$. It remains to recall that the $\mathcal{B}$-bimodule $\mathcal{F}E$ is contractive.

(ii) We can assume that $\|Q\| = 1$. Then $Q = \xi \circ \eta$ for some $\xi, \eta \in L$ with $\|\xi\| = \|\eta\| = 1$. Hence, for e and P as above, the formulae (4.1), combined with $Q = R_1 P R_2$ and $P = R_1^* Q R_2^*$, where $R_1 := \xi \circ e$ and $R_2 := e \circ \eta$, imply that

$$Q \diamondsuit b = (R_1 \diamondsuit \mathbf{1}) S b S^* (R_2 \diamondsuit \mathbf{1}), \qquad b = S^* (R_1^* \diamondsuit \mathbf{1})(q \diamondsuit b)(R_2^* \diamondsuit \mathbf{1}) S.$$

Therefore, representing u as a sum of elementary tensors, we obtain

$$Q \diamondsuit u = [(R_1 \diamondsuit \mathbf{1}) S] \cdot u \cdot [S^* (R_2 \diamondsuit \mathbf{1})], \qquad u = [S^* (R_1^* \diamondsuit \mathbf{1})] \cdot (q \diamondsuit u) \cdot [(R_2^* \diamondsuit \mathbf{1}) S].$$

But all operators featuring in these equalities have norm 1, and the bimodule $\mathcal{F}E$ is contractive. Consequently, we have the estimate $\|Q \diamondsuit u\| \leq \|u\|$ and its inverse.

(iii) By (2.1), for our a there exist h, P_k and s_k with the above mentioned properties. If S, T are appropriate operators, then $a \diamondsuit u = (S \diamondsuit \mathbf{1}) \cdot (h \diamondsuit u) \cdot (T \diamondsuit \mathbf{1})$, hence $\|a \diamondsuit u\| = \|h \diamondsuit u\| = \|\sum_k s_k P_k \diamondsuit u\|$. Further, the elements $P_k \diamondsuit u$ have pairwise orthogonal supports, namely $P_k \diamondsuit \mathbf{1}$. Combining this with (ii) and remembering what is $\|h\|_p$, we have, in the 'convex' case, $\|a \diamondsuit u\| \leq (\sum_{k=1}^n (s_k \|u\|)^p)^{1/p} = \|a\|_p \|u\|$. Similarly, in the 'concave' case we obtain the inverse estimate. ∎

Here are several applications. In the following proposition $p \in [1, \infty]$, and we consider $L_p(X, F)$, where F is a given PQ-space, and also $L_p(X, {}^{(p)}\mathbb{C})$, as PQ-spaces according to Example 4.1.

PROPOSITION 4.4. *Let F be p-convex. Then the bioperator*

$$\mathcal{R} : L_p(X, {}^{(p)}\mathbb{C}) \times F \to L_p(X, F)$$

taking a pair (z, y) to the F-valued function $t \mapsto z(t)y$, $t \in X$, is completely contractive.

Proof. Recall the isometric operator $\alpha : \mathcal{F}(L_p(X, F)) \to L_p(X, \mathcal{F}F)$ and its particular case $\alpha_0 : \mathcal{F}(L_p(X), {}^{(p)}\mathbb{C}) \to L_p(X, \mathcal{F}_p)$. Also introduce the bioperator $\mathcal{S} : L_p(X, \mathcal{F}_p) \times \mathcal{F}F \to L_p(X, \mathcal{F}F)$ taking a pair (ω, v) to the $\mathcal{F}F$-valued function $t \mapsto \omega(t) \diamondsuit v$ for $t \in X$. Consider the diagram

$$\begin{array}{ccc} \mathcal{F}(L_p(X, {}^{(p)}\mathbb{C})) \times \mathcal{F}F & \xrightarrow{\mathcal{R}_\infty} & \mathcal{F}(L_p(X, F)) \\ {\scriptstyle \alpha_0 \times \mathbf{1}_{\mathcal{F}F}} \downarrow & & \downarrow {\scriptstyle \alpha} \\ L_p(X, \mathcal{F}_p) \times \mathcal{F}F & \xrightarrow{\mathcal{S}} & L_p(X, \mathcal{F}F) \end{array}$$

It is commutative: this is easy to check on elementary tensors in the respective amplifications. Therefore, for $w \in \mathcal{F}(L_p(X, {}^{(p)}\mathbb{C}))$ and $v \in \mathcal{F}E$,

$$\|\mathcal{R}_\infty(w, v)\| = \|\alpha(\mathcal{R}_\infty(w, v))\| = \|\mathcal{S}(\alpha_0(w), v)\|. \tag{4.2}$$

But it follows from Proposition 4.3(iii) that for all $\omega \in L_p(X, \mathcal{F}_p)$ and $v \in \mathcal{F}F$ we have

$$\|\mathcal{S}(\omega, v)\| = \Big(\int_X \|\omega(t) \diamondsuit v\|^p \, dt\Big)^{1/p} \le \Big(\int_X (\|\omega(t)\|_p \|v\|)^p \, dt\Big)^{1/p} = \|\omega\| \, \|v\|.$$

Setting $\omega := \alpha_0(w)$ in (4.2) and remembering that α_0 is an isometry, we obtain $\|\mathcal{R}_\infty(w, v)\| \le \|w\| \, \|v\|$. ∎

In the following proposition, E is a normed space, ${}^{(p)}E$ is its P-quantization from Example 2.5, and F and $E \otimes_{\mathrm{pr}} F$ are PQ-spaces from Example 4.2.

Proposition 4.5. *Let F be p-convex. Then the canonical bioperator $\vartheta : {}^{(p)}E \times F \to E \otimes_{\mathrm{pr}} F : (x, y) \mapsto x \otimes y$ is completely contractive. In particular, the bioperator $\mathcal{R} : {}^{(p)}\mathbb{C} \times F \to F : (\lambda, x) \mapsto \lambda x$ is completely contractive.*

Proof. Consider the trilinear operator $\mathcal{T} : E \times \mathcal{F} \times \mathcal{F}F \to E \otimes \mathcal{F}F : (x, a, v) \mapsto x \otimes (a \diamondsuit v)$. It gives rise to the bioperator $\mathcal{S} : (E \otimes \mathcal{F}) \times \mathcal{F}F \to E \otimes \mathcal{F}F : (x \otimes a, v) \mapsto x \otimes (a \diamondsuit v)$. Considered with the domain $E \times \mathcal{F}_p \times \mathcal{F}F$ and range $E \otimes_{\mathrm{pr}} \mathcal{F}F$, $\mathcal{T}$ is contractive by Proposition 4.3(iii); therefore $\mathcal{S}$ is contractive if taken with the domain $(E \otimes_p \mathcal{F}) \times \mathcal{F}F$ and the same range. Now recall the isometric operator $\beta : \mathcal{F}(E \otimes_{\mathrm{pr}} F) \to E \otimes_{\mathrm{pr}} \mathcal{F}F$ and its particular case, the "flip" $\beta_0 : \mathcal{F}({}^{(p)}E) \to E \otimes_{\mathrm{pr}} \mathcal{F}_p$. The diagram

$$\begin{array}{ccc} \mathcal{F}({}^{(p)}E) \times \mathcal{F}F & \xrightarrow{\vartheta_\infty} & \mathcal{F}(E \otimes_{\mathrm{pr}} F) \\ {\scriptstyle \beta_0 \times \mathbf{1}_{\mathcal{F}F}} \downarrow & & \downarrow {\scriptstyle \beta} \\ (E \otimes_{\mathrm{pr}} \mathcal{F}_p) \times \mathcal{F}F & \xrightarrow{\mathcal{S}} & E \otimes_{\mathrm{pr}} \mathcal{F}F \end{array}$$

is obviously commutative. Hence for $w \in \mathcal{F}({}^{(p)}E)$ and $v \in \mathcal{F}F$ we have

$$\|\vartheta_\infty(w, v)\| = \|\beta(\vartheta_\infty(w, v))\| = \|\mathcal{S}(\beta_0(w), v\| \le \|\beta_0(w)\| \, \|v\| = \|w\| \, \|v\|. \ ∎$$

Our third example of a completely contractive bioperator needs some preparatory observation which should be well known in its equivalent version for the 'genuine' Hilbert tensor product of operators.

PROPOSITION 4.6. *For $a, b \in \mathcal{F}_p$ we have $\|a \diamondsuit b\|_p = \|a\|_p \|b\|_p$.*

Proof. Take unitary operators $S, T, S', T' \in \mathcal{B}$ with $SaT = \sum_{k=1}^n s_k P_k$ and $S'bT' = \sum_{l=1}^m t_l Q_k$, where P_k, $k = 1, \dots, n$, and Q_l, $l = 1, \dots, m$, are families of pairwise orthogonal rank 1 projections. Then $\|a\|_p = (\sum_{k=1}^n s_k^p)^{1/p}$ and $\|b\|_p = (\sum_{l=1}^m t_l^p)^{1/p}$. Further,

$$(S \diamondsuit S')(a \diamondsuit b)(T \diamondsuit T') = \Big(\sum_{k=1}^n s_k P_k\Big) \diamondsuit \Big(\sum_{l=1}^m t_l Q_k\Big).$$

Since $S \diamondsuit S'$ and $T \diamondsuit T'$ are unitary operators, this implies that

$$\|a \diamondsuit b\|_p = \Big\| \Big[\Big(\sum_{k=1}^n s_k P_k\Big) \diamondsuit \Big(\sum_{l=1}^m t_l Q_k\Big)\Big] \Big\|_p = \Big\| \sum_{k,l} s_k t_l P_k \diamondsuit Q_l \Big\|_p.$$

But, since all $P_k \diamondsuit Q_l$ are pairwise orthogonal rank 1 projections, the last number is $(\sum_{k,l} (s_k t_l)^p)^{1/p} = (\sum_{k=1}^n s_k^p)^{1/p} (\sum_{l=1}^m t_l^p)^{1/p} = \|a\|_p \|b\|_p$. ∎

PROPOSITION 4.7. *For all $p, q \in [1, \infty]$ and $r := \max\{p, q\}$ the bioperator $\mathcal{R} : {}^{(p)}E \times {}^{(q)}F \to {}^{(r)}(E \otimes_{\mathrm{pr}} F) : (x, y) \mapsto x \otimes y$ is completely contractive.*

Proof. Take $u \in \mathcal{F}({}^{(p)}E)$, $v \in \mathcal{F}({}^{(q)}F)$ and $\varepsilon > 0$. By definition of $\otimes_{\mathrm{pr}}$, there exist representations of u as $\sum_{k=1}^n a_k x_k$ and v as $\sum_{l=1}^m b_l y_l$ such that $\sum_{k=1}^n \|a_k\|_p \|x_k\| < \|u\|_{\mathrm{pr}} + \varepsilon$ and $\sum_{l=1}^m \|b_l\|_q \|y_l\| < \|v\|_{\mathrm{pr}} + \varepsilon$. We have $\mathcal{R}_\infty(u, v) = \sum_{k,l} a_k \diamondsuit b_l (x_k \otimes y_l)$; therefore, by the previous proposition,

$$\begin{aligned} \|\mathcal{R}(u, v)\| &\le \sum_{k,l} \|a_k \diamondsuit b_l\|_r \|x_k\| \, \|y_l\| \le \Big(\sum_{k=1}^n \|a_k\|_r \|x_k\|\Big)\Big(\sum_{l=1}^m \|b_l\|_r \|y_l\|\Big) \\ &\le \Big(\sum_{k=1}^n \|a_k\|_p \|x_k\|\Big)\Big(\sum_{l=1}^m \|b_l\|_q \|y_l\|\Big) < (\|u\|_{\mathrm{pr}} + \varepsilon)(\|v\|_{\mathrm{pr}} + \varepsilon). \end{aligned}$$

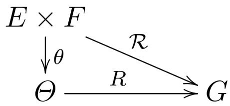
Since ε is arbitrary, this implies that $\|\mathcal{R}_\infty(u, v)\| \le \|u\| \, \|v\|$. ∎

5. The projective tensor product $\otimes_{\mathrm{pop}}$: definition and the existence theorem. A widespread point of view, inherited from pure algebra, is that the raison d'être of a 'good' tensor product is that it linearizes some 'good' class of bioperators (cf. [4, pp. 3–5]). As to the theory of PG- (= matricially normed) spaces, one could show that the Haagerup tensor product, introduced in [7], linearizes what is called in [11] multiplicatively bounded bioperators. But this is outside the scope of the present paper.

Here we shall introduce another, 'projective' tensor product of PG-spaces that linearizes what is called in the cited textbook, as well as in this paper, completely bounded bioperators.

Fix, for a time, two normed PQ-spaces E and F.

Definition 5.1. A pair (Θ, θ) consisting of a normed PQ-space Θ and a completely contractive bioperator $\theta : E \times F \to \Theta$ is called a non-completed *proto-operator-projective tensor product* of E and F or for brevity a *projective tensor product* of E and F if for every completely bounded bioperator $\mathcal{R} : E \times F \to G$, where G is a PQ-space, there exists a unique completely bounded operator $R : \Theta \to G$ such that the diagram



is commutative, and $\|R\|_{\mathrm{cb}} = \|\mathcal{R}\|_{\mathrm{cb}}$.

Uniqueness, in a proper sense, of such a pair is a particular case of the general categorical observation, concerning the uniqueness of an initial object in a category (cf. [18]). We shall prove the existence of such a pair by an explicit construction.

First, we need an additional version of the operation $\diamondsuit$, this time connecting elements of amplifications. Namely, for $u \in \mathcal{F}E$ and $v \in \mathcal{F}F$ we set $u \diamondsuit v := \vartheta_\infty(u, v) \in \mathcal{F}(E \otimes F)$, where $\vartheta : E \times F \to E \otimes F : (x, y) \mapsto x \otimes y$ is the canonical bilinear operator. In particular, for elementary tensors we have $ax \diamondsuit by = (a \diamondsuit b)(x \otimes y)$.

Note that for all $a, b, c, d \in \mathcal{B}$ and $u \in \mathcal{F}E$, $v \in \mathcal{F}F$ we have

$$(a \diamondsuit b) \cdot (u \diamondsuit v) \cdot (c \diamondsuit d) = (a \cdot u \cdot c) \diamondsuit (b \cdot v \cdot d). \tag{5.1}$$

One can immediately verify this formula on elementary tensors.

It is easy to show that every $U \in \mathcal{F}(E \otimes F)$ can be represented as

$$\sum_{k=1}^{n} a_k \cdot (u_k \diamondsuit v_k) \cdot b_k \tag{5.2}$$

for some $a_k, b_k \in \mathcal{B}$, $u_k \in \mathcal{F}E$, $v_k \in \mathcal{F}F$, $k = 1, \ldots, n$ (see details in [14, Section 7.2]). This implies that the operator $\mathcal{B} \otimes \mathcal{F}E \otimes \mathcal{F}F \otimes \mathcal{B} \to \mathcal{F}(E \otimes F)$ associated with the 4-linear operator $(a, u, v, b) \mapsto a \cdot (u \diamondsuit v) \cdot b$ is surjective. Thus $\mathcal{F}(E \otimes F)$ can be endowed with the seminorm of the corresponding quotient space of $\mathcal{B} \otimes_{\mathrm{pr}} \mathcal{F}E \otimes_{\mathrm{pr}} \mathcal{F}F \otimes_{\mathrm{pr}} \mathcal{B}$, denoted by $\|\cdot\|_{\mathrm{pop}}$. In other words, for $U \in \mathcal{F}(E \otimes F)$ we have

$$\|U\|_{\mathrm{pop}} := \inf\Big\{\sum_{k=1}^{n} \|a_k\| \, \|u_k\| \, \|v_k\| \, \|b_k\|\Big\}, \tag{5.3}$$

where the infimum is taken over all possible representations of U in the form given by (5.2).

Now observe that $\mathcal{B} \otimes_{\mathrm{pr}} \mathcal{F}E \otimes_{\mathrm{pr}} \mathcal{F}F \otimes_{\mathrm{pr}} \mathcal{B}$ is a contractive $\mathcal{B}$-bimodule when considered as a tensor product of the left $\mathcal{B}$-module $\mathcal{B}$ with the linear space $\mathcal{F}E \otimes \mathcal{F}F$ and the right $\mathcal{B}$-module $\mathcal{B}$. Therefore $\mathcal{F}(E \otimes F)$ is the image of a contractive $\mathcal{B}$-bimodule with respect to a quotient map of seminormed spaces. Since the latter map is obviously a bimodule morphism, we easily see that the bimodule $(\mathcal{F}(E \otimes F), \|\cdot\|_{\mathrm{pop}})$ is also contractive.

We see that $\|\cdot\|_{\mathrm{pop}}$ is a PQ-seminorm on $E \otimes F$. Denote the corresponding seminormed PQ-space by $E \otimes_{\mathrm{pop}} F$.

Finally, note that if $\mathcal{R} : E \times F \to G$ is a bioperator, and $R : E \otimes F \to G$ is the associated linear operator, then obviously

$$R_\infty(u \diamondsuit v) = \mathcal{R}_\infty(u, v). \tag{5.4}$$

THEOREM 5.2 (Existence theorem). *The pair* $(E \otimes_{\mathrm{pop}} F, \vartheta)$ *is a non-completed projective tensor product of* E *and* F*.*

We prefer to give a self-contained proof of the theorem, although some parts of it resemble what was said in [14] under the assumption that we deal with quantum spaces.

Proof of Theorem 5.2. First, for all $u \in \mathcal{F}E$ and $v \in \mathcal{F}F$ we have of course $u \diamondsuit v = \mathbf{1} \cdot (u \diamondsuit v) \cdot \mathbf{1}$. Therefore the bioperator ϑ, considered with values in $E \otimes_{\mathrm{pop}} F$, is completely contractive, or equivalently

$$\|u \diamondsuit v\|_{\mathrm{pop}} \le \|u\|\,\|v\|. \tag{5.5}$$

Now let G be a PQ-space, $\mathcal{R} : E \times F \to G$ a completely bounded bioperator, and $R : E \otimes_{\mathrm{pop}} F \to G$ the associated linear operator. We want to show that R is completely bounded and $\|\mathcal{R}\|_{\mathrm{cb}} = \|R\|_{\mathrm{cb}}$.

Take $U \in \mathcal{F}(E \otimes_{\mathrm{pop}} F)$ and represent it as in (5.2). Since R_∞ is a $\mathcal{B}$-bimodule morphism, by (5.4) we have $R_\infty(U) = \sum_{k=1}^n a_k \cdot \mathcal{R}_\infty(u_k, v_k) \cdot b_k$, hence $\|R_\infty(U)\| \le \|\mathcal{R}\|_{\mathrm{cb}} \sum_{k=1}^n \|a_k\|\,\|u_k\|\,\|v_k\|\,\|b_k\|$. Therefore the definition of $\|\cdot\|_{\mathrm{pop}}$ implies that R is completely bounded, just as $\mathcal{R}$, and $\|R\|_{\mathrm{cb}} \le \|\mathcal{R}\|_{\mathrm{cb}}$. The inverse estimate follows from the inequality $\|\mathcal{R}_\infty(u, v)\| \le \|R_\infty\|\,\|u\|\,\|v\|$, which in turn immediately follows from (5.4) and (5.5).

Now consider the diagram from Definition 5.1 with $E \otimes_{\mathrm{pop}} F$ and ϑ in the capacity of Θ and θ, respectively. It is known from linear algebra that R is the only linear operator making the diagram commutative. Thus the pair $(E \otimes_{\mathrm{pop}} F, \vartheta)$ satisfies almost all requirements given in Definition 5.1. It only remains to show that the seminorm $\|\cdot\|_{\mathrm{pop}}$ is actually a norm.

By Proposition 2.4, it is sufficient to show that, for every non-zero elementary tensor Qw, where Q is a rank 1 operator of norm 1 and $w \in E \otimes_{\mathrm{pop}} F$, $w \ne 0$, we have $\|Qw\|_{\mathrm{pop}} \ne 0$. Since E and F are *normed* spaces, there exist bounded functionals $f : E \to \mathbb{C}$ and $g : F \to \mathbb{C}$ such that for $f \otimes g : E \otimes F \to \mathbb{C}$ we have $(f \otimes g)w \ne 0$. As we know from the previous section, the bilinear functional $\mathcal{R} := f \times g : E \times F \to {}^{(\infty)}\mathbb{C}$ is completely bounded. Therefore, choosing $G := {}^{(\infty)}\mathbb{C}$, we see that the associated linear functional, that is, $f \otimes g : E \otimes_{\mathrm{pop}} F \to {}^{(\infty)}\mathbb{C}$, is also completely bounded.

Hence,

$$|(f \otimes g)w| = \|Q[(f \otimes g)(w)]\| = \|[(f \otimes g)_\infty(Qw)\| \leq \|f \otimes g\|_{\mathrm{cb}} \|Qw\|_{\mathrm{pop}}.$$

Therefore $\|Qw\|_{\mathrm{pop}} \neq 0$ since $(f \otimes g)w \neq 0$. ∎

Note that in the underlying space of $E \otimes_{\mathrm{pop}} F$ we have

$$\|x \otimes y\| \leq \|x\| \, \|y\| \quad \text{for all } x \in E,\, y \in F. \tag{5.6}$$

Indeed, take two operators $P, Q \in \mathcal{F}$ of rank 1 and of norm 1. We see that $\|P \diamondsuit Q\| = 1$ and $P \diamondsuit Q$ also has rank 1. Thus $\|x \otimes y\| = \|(P \diamondsuit Q) x \otimes y\|_{\mathrm{pop}} = \|Px \diamondsuit Qy\|$. It remains to use (5.5).

(In fact, in (5.6) as well as in (5.5), we have exact equality, but we shall not discuss it now.)

So far, we spoke about general (normed) PQ-spaces. But their tensor product has a natural analogue in the context of *complete* or *Banach PQ-spaces*. The latter are, by definition, PQ-spaces with complete underlying normed spaces. As in the 'classical' context, for every PQ-space E there exists its *completion*, which is defined as a pair $(\overline{E}, i : E \to \overline{E})$, consisting of a complete PQ-space and a completely isometric operator, such that the same pair considered for the respective underlying spaces and operators is the 'classical' completion of E as a normed space. The proof of the corresponding existence theorem repeats word for word the simple argument given in [14, Chapter 4] for Q-spaces. We only recall that the norm on $\mathcal{F}\overline{E}$ is introduced with the help of the natural embedding of $\mathcal{F}\overline{E}$ into $\overline{\mathcal{F}E}$, the 'classical' completion of $\mathcal{F}E$.

It is easy to observe that the characteristic universal property of the 'classical' completion has its proto-quantum version (*ibid.*). Namely, if $(\overline{E}, i)$ is the completion of a PQ-space E, F is a PQ-space and $\varphi : E \to F$ is a completely bounded operator, then there exists a unique completely bounded operator $\overline{\varphi} : \overline{E} \to \overline{F}$ that extends φ, in the obvious sense. Moreover, $\|\overline{\varphi}\|_{\mathrm{cb}} = \|\varphi\|_{\mathrm{cb}}$.

The following fact will be useful. Its proof repeats word for word the argument for [14, Proposition 4.8].

Proposition 5.3. *Let $\varphi : E \to F$ be a completely isometric isomorphism between PQ-spaces. Then its continuous extension $\overline{\varphi} : \overline{E} \to \overline{F}$ is also a completely isometric isomorphism.*

Now we can speak of the *completed* projective tensor product of two PQ-spaces. Its definition repeats Definition 5.1 with the following difference: *Θ and G are supposed to be complete.* Using the universal property of completion, we immediately see that *the completed projective tensor product of PQ-spaces E and F exists: it is the pair $(E \widehat{\otimes}_{\mathrm{pop}} F, \widehat{\vartheta})$, where $E \widehat{\otimes}_{\mathrm{pop}} F$ is the completion of the PQ-space $E \otimes_{\mathrm{pop}} F$, and $\widehat{\vartheta}$ acts as ϑ, but with range $E \widehat{\otimes}_{\mathrm{pop}} F$.*

6. Tensoring by $L_1(\cdot)$, and some other computations. In this section we show that for certain concrete tensor factors their projective tensor product also becomes something concrete and transparent. We shall see that the behavior of this projective tensor product resembles the classical context.

Denote the completion of the PQ-space $E\otimes_{\rm pr}F$ from Example 4.2 by $E\widehat{\otimes}_{\rm pr}F$. Clearly, it is a P-quantization of the 'classical' completed projective tensor product, denoted also by $E\,\widehat{\otimes}_{\rm pr}\,F$; this will create no confusion.

THEOREM 6.1. *Let E be a normed space, F a PQ-space, $p\in[1,\infty]$, ${}^{(p)}E$ the PQ-space from Example 2.5, and $E\otimes_{\rm pr}F$ the PQ-space from Example 4.2. Suppose that F is p-convex. Then there exists a completely isometric isomorphism $I:{}^{(p)}E\otimes_{\rm pop}F\to E\otimes_{\rm pr}F$ acting as the identity operator on the common underlying linear space of our PQ-spaces. As a corollary (see Proposition 5.3), there exists a completely isometric isomorphism $\widehat{I}:{}^{(p)}E\,\widehat{\otimes}_{\rm pop}\,F\to E\,\widehat{\otimes}_{\rm pr}\,F$ which is the extension of I by continuity.*

Proof. Consider the canonical bioperator $\vartheta:{}^{(p)}E\times F\to E\otimes_{\rm pr}F$. Since a PQ-space is p-convex, it gives rise, by Proposition 4.5, to the completely contractive operator I acting as in the formulation. Therefore it is sufficient to show that for every $U\in\mathcal{F}(E\otimes F)$ its norm in $\mathcal{F}[{}^{(p)}E\otimes_{\rm pop}F]$ is not greater than $\|I_\infty(U)\|$, or equivalently than the norm of $\beta(U)$ in $E\otimes_{\rm pr}\mathcal{F}F$.

Fix U and choose $\varepsilon>0$; then there exists a representation $\beta(U)=\sum_{k=1}^n x_k\otimes v_k$ with $x_k\in E$ and $v_k\in\mathcal{F}F$ such that $\sum_{k=1}^n\|x_k\|\,\|v_k\|<\|\beta(U)\|+\varepsilon=\|I_\infty(U)\|+\varepsilon$.

Now choose an arbitrary rank 1 projection $P\in\mathcal{F}$ and set $V:=\sum_{k=1}^n Px_k\diamondsuit v_k\in\mathcal{F}(E\otimes F)$. By (4.1), there exists an isometry $S\in\mathcal{B}$ such that $S^*(P\diamondsuit a)S=a$ for every $a\in\mathcal{F}$. Representing each v_k as a sum of elementary tensors, we easily see that $\beta(S^*\cdot V\cdot S)=\sum_{k=1}^n x_k\otimes v_k=\beta(U)$. Therefore $U=S^*\cdot V\cdot S$, hence $\|U\|_{\rm pop}\le\|V\|_{\rm pop}$. But by (5.5) we have $\|V\|_{\rm pop}\le\sum_{k=1}^n\|Px_k\|\,\|v_k\|=\sum_{k=1}^n\|x_k\|\,\|v_k\|$, and consequently $\|U\|_{\rm pop}\le\|I_\infty(U)\|+\varepsilon$. Since this holds for every $\varepsilon>0$, we are done. ■

COROLLARY 6.2. *With p and F as above, there exists a completely isometric isomorphism $I:{}^{(p)}\mathbb{C}\otimes_{\rm pop}F\to F$ acting as $\lambda\otimes x\mapsto\lambda x$.*

In turn, since ${}^{(q)}\mathbb{C}$ is p-convex provided $q<p$, this assertion implies

COROLLARY 6.3. ${}^{(p)}\mathbb{C}\otimes_{\rm pop}{}^{(q)}\mathbb{C}={}^{(r)}\mathbb{C}$, *where $r=\max\{p,q\}$.*

REMARK 6.4. The projective tensor product of two $\mathcal{L}^p$-spaces is not bound to be again an $\mathcal{L}^p$-space. Indeed, consider the projective tensor square of the $\mathcal{L}^2$-space $\ell_2({}^{(2)}\mathbb{C})$ and the elements $Pe_1,Qe_2\in\mathcal{F}(\ell_2({}^{(2)}\mathbb{C}))$, where P,Q are orthogonal projections in $\mathcal{F}$ (i.e. $PQ=0$), and e_1,e_2 are orts in ℓ_2. Then it is not difficult to show that, despite our elements having orthogonal supports, we have $\|Pe_1+Qe_2\|_{\rm pop}=2$, whereas $(\|Pe_1\|^2+\|Qe_2\|^2)^{1/2}=\sqrt{2}$. Incidentally, it is shown in [16] that there exists a kind of projective tensor product in the class of the so-called p-convex p-multi-spaces (see [5] and also [17]) reflecting their special properties. Therefore one may suggest that p-convex PQ-spaces have their own projective tensor product, defined only within that class and not leading out of this class.

In the following result we deal, generally speaking, with PQ-spaces that are not p-convex.

THEOREM 6.5. *Let E and F be normed spaces and $p \in [1, \infty]$. Then there exists a completely isometric isomorphism $I : {}^{(p)}E \otimes_{\rm pop} {}^{(p)}F \to {}^{(p)}(E \otimes_{\rm pr} F)$ acting as the identity operator on the common underlying linear space of our PQ-spaces. As a corollary* (*see Proposition* 5.3), *there exists a completely isometric isomorphism $\widehat{I} : {}^{(p)}E \,\widehat{\otimes}_{\rm pop} {}^{(p)}F \to {}^{(p)}(E \,\widehat{\otimes}_{\rm pr} F)$ which is the extension of I by continuity.*

Proof. Consider the canonical bioperator $\vartheta : {}^{(p)}E \times {}^{(p)}F \to {}^{(p)}(E \widehat{\otimes}_{\rm pr} F)$. It gives rise, by virtue of Proposition 4.7, to the completely contractive operator I acting as in the formulation. Therefore it is sufficient to show that for every $U \in \mathcal{F}({}^{(p)}E \otimes_{\rm pop} {}^{(p)}F)$ we have $\|U\|_{\rm pop} \leq \|I_\infty(U)\|$.

Fix U and $\varepsilon > 0$. As a normed space, $\mathcal{F}[{}^{(p)}(E \otimes_{\rm pr} F)]$ is $\mathcal{F}_p \otimes_{\rm pr} E \otimes_{\rm pr} F$. Consequently, in the linear space $\mathcal{F}(E \otimes F) = \mathcal{F} \otimes E \otimes F$ there exists a representation of U or, what is the same, of $I_\infty(U)$ as $\sum_{k=1}^n a_k \otimes x_k \otimes y_k$ with $a_k \in \mathcal{F}$, $x_k \in E$, $y_k \in F$ such that $\sum_{k=1}^n \|a_k\|_p \|x_k\| \, \|y_k\| < \|I_\infty(U)\| + \varepsilon$.

Take an arbitrary rank 1 projection $P \in \mathcal{F}$ and introduce an element

$$V := \sum_{k=1}^n Px_k \diamondsuit (a_k y_k) = \sum_{k=1}^n (P \diamondsuit a_k) x_k \otimes y_k \in \mathcal{F}(E \otimes F).$$

By (4.1), there exists an isometry $S \in \mathcal{B}$ such that $a_k = S^*(P \diamondsuit a_k)S$ for all k. It follows that $S^* \cdot V \cdot S = \sum_{k=1}^n a_k(x_k \otimes y_k) = U$. Therefore, for U and V viewed as elements of $\mathcal{F}[{}^{(p)}E \widehat{\otimes}_{\rm pop} {}^{(p)}F]$, we have $\|U\|_{\rm pop} \leq \|V\|_{\rm pop}$. But, by (5.5), we have $\|V\|_{\rm pop} \leq \sum_{k=1}^n \|Px_k\| \, \|a_k y_k\|$, where the norms are taken in $\mathcal{F}^{(p)}E$ and $\mathcal{F}^{(p)}F$, respectively. Further, $\|Px_k\| = \|x_k\|$, and since $\mathcal{F}^{(p)}F = \mathcal{F}_p \otimes_{\rm pr} F$, we have $\|a_k y_k\| = \|a_k\|_p \|y_k\|$. Consequently,

$$\|U\|_{\rm pop} \leq \sum_{k=1}^n \|a_k\|_p \|x_k\| \, \|y_k\| < \|I_\infty(U)\| + \varepsilon.$$

Since this holds for every $\varepsilon > 0$, we are done. ∎

Setting $p := 1$ in this theorem, we obtain

COROLLARY 6.6. *For all normed spaces E and F we have, up to a completely isometric isomorphism,*

$$E_{\max} \otimes_{\rm pop} F_{\max} = (E \otimes_{\rm pr} F)_{\max} \quad \textit{and} \quad E_{\max} \,\widehat{\otimes}_{\rm pop} F_{\max} = (E \,\widehat{\otimes}_{\rm pr} F)_{\max}.$$

As another particular case, we see that for a Hilbert space H we have ${}^{(p)}H \,\widehat{\otimes}_{\rm pop}\, {}^{(p)}H = {}^{(p)}\mathcal{N}(H)$, where $\mathcal{N}(H)$ is the Banach space of trace class operators on H.

By the Grothendieck theorem, mentioned in the Introduction, we may say that in the case of the classical projective tensor product of normed spaces, L_1-spaces are especially nice tensor factors. Now we would like to show that the same is true for the projective tensor product of PG-spaces. We need some preparation.

PROPOSITION 6.7. *Let $(X,\mu),(Y,\nu)$ be measure spaces, let E,F be PQ-spaces, and $p \in [1,\infty]$. Then the bioperator $\mathcal{R} : L_p(X,E) \times L_p(Y,F) : L_p(X \times Y, E_{\rm pop}F) : (\bar{x},\bar{y}) \mapsto \bar{z}$ with $\bar{z}(s,t) := x(s) \otimes y(t)$ is completely contractive.*

(Here and hereafter, speaking about $L_p(X,\cdot)$ and $L_p(Y,\cdot)$, we mean X and Y with the given measures, and speaking about $L_p(X\times Y,\cdot)$, we consider $X \times Y$ with the cartesian product of those measures.)

Proof of Proposition 6.7. Consider the diagram

$$\begin{array}{ccc}
\mathcal{F}(L_p(X,E)) \times \mathcal{F}(L_p(Y,F)) & \xrightarrow{\ \mathcal{R}_\infty\ } & \mathcal{F}(L_p(X\times Y, E\otimes_{\rm pop} F)) \\
{\scriptstyle \alpha_X\times\alpha_Y}\downarrow & & \downarrow{\scriptstyle \alpha_{X\times Y}} \\
L_p(X,\mathcal{F}E) \times L_p(Y,\mathcal{F}F) & \xrightarrow{\ \mathcal{S}\ } & L_p(X\times Y, \mathcal{F}E_{\rm pop}F)
\end{array}$$

where $\mathcal{S}$ takes a pair of vector-valued functions $(\bar{u}(s),\bar{v}(t))$ to the vector-valued function $\bar{w}(s,t) := \bar{u}(s) \diamondsuit \bar{v}(t)$, $s \in X$, $t \in Y$, and α_X etc. are the respective specializations of α from Example 4.1. The diagram is evidently commutative, and α is an isometry. Therefore it suffices to show that $\mathcal{S}$ is contractive. Indeed, by (5.5),

$$\begin{aligned}
\|\bar{w}\| &= \Big(\int\limits_{X\times Y} \|\bar{u}(s) \diamondsuit \bar{v}(t)\|^p \, d(s,t)\Big)^{1/p} \\
&\le \Big(\int\limits_{X\times Y} \|\bar{u}(s)\|^p \|\bar{v}(t)\|^p)\, d(s,t)\Big)^{1/p} = \|\bar{u}\|\, \|\bar{v}\|. \ \blacksquare
\end{aligned}$$

THEOREM 6.8. *Let $(X,\mu),(Y,\nu)$ be measure spaces, and E,F be PQ-spaces. Then there exists a complete isometry $I : L_1(X,E) \otimes_{\rm pop} L_1(Y,F) \to L_1(X\times Y, E\otimes_{\rm pop} F)$, well defined by $\bar{x}\otimes\bar{y} \mapsto \bar{z}$, where $\bar{z}(s,t) := \bar{x}(s)\otimes\bar{y}(t)$.*

Proof. The bioperator $\mathcal{R}$ from Proposition 6.7, considered for $p = 1$, gives rise to a completely contractive operator R acting exactly as I in the formulation. Therefore our task is to show that $\|U\|_{\rm pop} \le \|R_\infty(U)\|$ for every $U \in \mathcal{F}[L_1(X,E) \otimes_{\rm pop} L_1(Y,F)]$.

Obviously, $L_1(X,E)$ contains the dense subspace $L_1^0(X,E)$ consisting of vector-valued functions of the form $\sum_k \chi_k x_k$, where χ_k are the characteristic functions of pairwise disjoint subsets of X with finite measure, and $x_k \in E$. Similarly, $L_1(Y,F)$ contains the subspace $L_1^0(Y,F)$ with analogous properties. Therefore, thanks to (5.6) and the last estimate in Proposition 2.6, it is sufficient to prove that R_∞ does not decrease norms of sums of elementary tensors of the form $a(\mathbf{x}\otimes\mathbf{y})$, where $a \in \mathcal{F}$, $\mathbf{x} \in L_1^0(X,E)$, $\mathbf{y} \in L_1^0(Y,F)$.

Let U be such a sum. It is not hard to show that it can be represented as

$$U = \sum_{k=1}^{n}\sum_{l=1}^{m} a_{kl}(\bar{x}_k \otimes \bar{y}_l)$$

with $\bar{x}_k(s) := \chi_k(s)x_k$, $s \in X$, where $\chi_k(s)$ are the characteristic functions of pairwise disjoint subsets X_k with $\mu(X_k) < \infty$, $x_k \in E$, and $\bar{y}_l(t) := \chi'_l(t)y_l$ for $t \in Y$, where $\chi'_l(t)$ are the characteristic functions of pairwise disjoint subsets Y_l with $\nu(Y_l) < \infty$, $y_l \in F$.

We obviously have $R_\infty(U) = \sum_{k,l} a_{kl}[\chi_k(s)\chi'_l(t)x_k \otimes y_l]$. Therefore, by the recipe of Example 4.1, $\|R_\infty(U)\|$ is the norm of the function

$$\sum_{k,l} \chi_k(s)\chi'_l(t)[a_{kl}(x_k \otimes y_l)] \in L_1(X\times Y, \mathcal{F}(E \otimes_{\mathrm{pop}} F)).$$

Since we are in a space $L_1(\cdot)$, this implies that

$$\|R_\infty(U)\| = \sum_{k=1}^{n}\sum_{l=1}^{m} \mu(X_k)\nu(Y_l)\|a_{kl}(x_k \otimes y_l)\|_{\mathrm{pop}}. \tag{6.1}$$

Fix for the moment a pair k,l, and also $\varepsilon > 0$. By (5.3), we can represent $a_{kl}(x_k \otimes y_l) \in \mathcal{F}(E \otimes_{\mathrm{pop}} F)$ in the form $\sum_i b^i_{kl} \cdot (u^i_{kl} \diamondsuit v^i_{kl}) \cdot c^i_{kl}$, where $b^i_{kl}, c^i_{kl} \in \mathcal{B}$, $u^i_{kl} \in \mathcal{F}E$, $v^i_{kl} \in \mathcal{F}F$, so that

$$\sum_i \|b^i_{kl}\|\,\|u^i_{kl}\|\,\|v^i_{kl}\|\,\|c^i_{kl}\| < \|a_{kl}(x_k \otimes y_l)\|_{\mathrm{pop}} + \varepsilon. \tag{6.2}$$

(Here of course the number of summands indexed by i depends on k,l.)

Now for every $u \in \mathcal{F}E$ we denote by $\bar{u}$ the element $B_\infty(u) \in \mathcal{F}(L_1(X,E))$, where $B : E \to L_1(X,E)$ maps x to the vector-valued function $\bar{x}(s) := \chi_k(s)x$. Similarly, for $v \in L_1(Y,F)$ we set $\bar{v} := B'_\infty(v) \in \mathcal{F}(L_1(Y,F))$, where $B' : F \to L_1(Y,F) : y \mapsto \bar{y} := \chi'(t)y$. Evidently,

$$\|\bar{u}\| = \mu(X_k)\|u\| \quad \text{and} \quad \|\bar{v}\| = \nu(Y_k)\|v\|. \tag{6.3}$$

Further, since for $x \in E$ and $y \in F$ we have $R(\bar{x}\otimes\bar{y}) = \chi_k(s)\chi'(t)x \otimes y$, it easily follows that $R_\infty(\bar{u} \diamondsuit \bar{v}) = \chi_k(s)\chi'(t)u \diamondsuit v$ for $u \in \mathcal{F}E$ and $v \in \mathcal{F}F$.

Consequently, taking into account (6.3) and (6.2), we have

$$R_\infty\Big(\sum_i b^i_{kl}\cdot(\bar u^i_{kl}\diamondsuit \bar v^i_{kl})\cdot c^i_{kl}\Big) = \chi_k(s)\chi'(t)\sum_i b^i_{kl}\cdot(u^i_{kl}\diamondsuit v^i_{kl})\cdot c^i_{kl}$$
$$= \chi_k(s)\chi'_l(t)a_{kl}(x_k\otimes y_l) = R_\infty[a_{kl}(\bar x_k\otimes \bar y_l)].$$

But R is obviously injective, and of course the same is true for R_∞. It follows that $\sum_i b^i_{kl}\cdot(\bar u^i_{kl}\diamondsuit \bar v^i_{kl})\cdot c^i_{kl} = a_{kl}(\bar x_k\otimes\bar y_l)$, and consequently $U = \sum_{k,l}[\sum_i b^i_{kl}\cdot(\bar u^i_{kl}\diamondsuit \bar v^i_{kl})\cdot c^i_{kl}]$. This, with the help of (6.3), implies that

$$\|U\|_{\text{pop}} \le \sum_{k,l}\sum_i \|b^i_{kl}\|\,\|\bar u^i_{kl}\|\,\|\bar v^i_{kl}\|\,\|c^i_{kl}\|$$
$$= \sum_{k,l}\mu(X_k)\nu(Y_l)\sum_i \|b^i_{kl}\|\,\|u^i_{kl}\|\,\|v^i_{kl}\|\,\|c^i_{kl}\|.$$

From this, by (6.2), we obtain

$$\|U\|_{\text{pop}} \le \sum_{k,l}\mu(X_k)\nu(Y_l)(\|a_{kl}(x_k\otimes y_l)\|_{\text{pop}}+\varepsilon).$$

As ε is arbitrary, it follows that $\|U\|_{\text{pop}} \le \sum_{k,l}\mu(X_k)\nu(Y_l)\|a_{kl}(x_k\otimes y_l)\|_{\text{pop}}$, that is, by (6.1), $\|U\|_{\text{pop}} \le \|R_\infty(U)\|$. ∎

Since $L_1(X,E)\otimes_{\text{pop}}L_1(Y,F)$ is dense in $L_1(X,E)\widehat{\otimes}_{\text{pop}}L_1(Y,F)$, and the image of I is obviously dense in $L_1(X\times Y, E\,\widehat{\otimes}_{\text{pop}}F)$, we have an immediate corollary:

THEOREM 6.9. *Let $(X,\mu),(Y,\nu),E,F$ be as before. Then there exists a complete isometric isomorphism $I: L_1(X,E)\,\widehat{\otimes}_{\text{pop}}\,L_1(Y,F)\to L_1(X\times Y, E\,\widehat{\otimes}_{\text{pop}}F)$, well defined by $\bar x\otimes\bar y\mapsto \bar z$, where $\bar z(s,t) := \bar x(s)\otimes \bar y(t)$.*

Combining Theorem 6.8 or 6.9 with the previous results in this section, one can obtain various corollaries. For example, taking a one-point Y and using Corollary 6.2, we get the assertion that can be considered as a '*PQ*-version' of the Grothendieck theorem in its usual formulation:

COROLLARY 6.10. *Let $p\in[1,\infty]$, X be a measure space, and F be a complete p-convex PQ-space. Then, up to a completely isometric isomorphism, $L_1(X,{}^{(p)}\mathbb{C})\,\widehat{\otimes}_{\text{pop}}\,F = L_1(X,F)$.*

Note that the same assertion could be obtained without using Theorem 6.8, by combining the easier Proposition 4.4 with Corollary 6.2.

Also, combining Theorem 6.9 (or Proposition 4.4) with Corollary 6.3, one can get the completely isometric isomorphism $L_1(X,{}^{(p)}\mathbb{C})\widehat{\otimes}_{\text{pop}}L_1(Y,{}^{(q)}\mathbb{C})\simeq L_1(X\times Y,{}^{(r)}\mathbb{C})$ with $r:=\max\{p,q\}$, and so on.

7. Quantum duality and adjoint associativity. We proceed to show that the projective tensor product of PQ-spaces satisfies the adjoint associativity law (= exponential law), connecting it with the proper P-quantization of the space of completely bounded operators. Such a P-quantization extends what was well known in the context of quantum (operator) spaces. In that context the relevant construction cropped up in [8, p. 140], but was fully realized and put in proper place independently and simultaneously in [3] and [9]. In matrix-free language, again only for Q-spaces, it was presented in [14, 8.1.8]. Here we give all the needed details for general PQ-spaces.

Let E, G be two PQ-spaces. Our task is to endow the normed space $\mathcal{CB}(E,G)$ with a P-quantization. To this end we consider the *evaluation bioperator* $\mathcal{E} : E \times \mathcal{CB}(E,G) \to G : (x,\varphi) \mapsto \varphi(x)$ and its amplification $\mathcal{E}_\infty : \mathcal{F}E \times \mathcal{F}[\mathcal{CB}(E,G)] \to \mathcal{F}G$; the latter, as we remember, is well defined by $(ax, b\varphi) \mapsto (a \diamondsuit b)\varphi(x)$. Set, for $\Phi \in \mathcal{F}[\mathcal{CB}(E,G)]$,

$$\|\Phi\| := \sup\{\|\mathcal{E}_\infty(u,\Phi)\| : u \in \mathcal{F}E,\ \|u\| \le 1\}. \tag{7.1}$$

(We see that this definition closely imitates the definition of the 'classical' operator norm: indeed, $\|\varphi\|$ is $\sup\{\|\mathcal{E}(x,\varphi)\| : x \in E,\ \|x\| \le 1\}$, where $\mathcal{E} : E \times \mathcal{B}(E,G) \to G$ is the obvious 'classical' evaluation operator.)

Proposition 7.1. *The function $\Phi \mapsto \|\Phi\|$ is a PQ-norm on $\mathcal{CB}(E,G)$, and the resulting PQ-space is a P-quantization of $\mathcal{CB}(E,G)$ as a normed space.*

Proof. For every $b \in \mathcal{F}$, $u \in \mathcal{F}E$ and $\varphi \in \mathcal{CB}(E,G)$ we obviously have

$$\mathcal{E}_\infty(u, b\varphi) = \varphi_\infty(u) \diamondsuit b. \tag{7.2}$$

Therefore, by Proposition 4.3(iii), we get $\|\mathcal{E}_\infty(u,b\varphi)\| \le \|b\|_1 \|\varphi_\infty(u)\|$. It follows that the number $\|b\varphi\|$ is well defined. Consequently, proceeding from elementary tensors to their (finite) sums, we see that the number $\|\Phi\|$ is well defined for all $\Phi \in \mathcal{F}[\mathcal{CB}(E,G)]$.

Further, for all $a \in \mathcal{F}$, $u \in \mathcal{F}E$ and $\Phi \in \mathcal{F}[\mathcal{CB}(E,G)]$ we have the equalities

$$\mathcal{E}_\infty(u, a\cdot\Phi) = (\mathbf{1} \diamondsuit a)\cdot \mathcal{E}_\infty(u,\Phi) \quad \text{and} \quad \mathcal{E}_\infty(u, \Phi\cdot a) = \mathcal{E}_\infty(u,\Phi)\cdot(\mathbf{1} \diamondsuit a),$$

which can be immediately checked on elementary tensors. Consequently, $\|\mathcal{E}_\infty(u, a\cdot\Phi)\|, \|\mathcal{E}_\infty(u,\Phi\cdot a)\| \le \|\mathbf{1} \diamondsuit a\|\,\|\mathcal{E}_\infty(u,\Phi)\| = \|a\|\,\|\mathcal{E}_\infty(u,\Phi)\|$. It follows that $\|a\cdot\Phi\| \le \|a\|\,\|\Phi\|$, and similarly $\|\Phi\cdot a\| \le \|a\|\,\|\Phi\|$. Therefore the seminorm introduced on $\mathcal{F}[\mathcal{CB}(E,G)]$ is a PQ-seminorm.

Finally, take a rank 1 projection $P \in \mathcal{F}$. Then, considering $\mathcal{CB}(E,G)$ as the underlying seminormed space of the above PQ-space, we deduce, by (7.2) and Proposition 4.3(ii), that

$$\begin{aligned}\|\varphi\| &= \sup\{\|\mathcal{E}_\infty(u,P\varphi)\| : \|u\| \le 1\} = \sup\{\|\varphi_\infty(u) \diamondsuit P\| : \|u\| \le 1\} \\ &= \sup\{\|\varphi_\infty(u)\| : \|u\| \le 1\} = \|\varphi\|_{\mathrm{cb}}.\end{aligned}$$

Consequently, our underlying space is just $\mathcal{CB}(E,G)$ with its cb-norm. Therefore, by Proposition 2.4, the seminorm on $\mathcal{F}[\mathcal{CB}(E,G)]$ is actually a norm, and the corresponding PQ-space is a P-quantization of the given space of completely bounded operators. ∎

If a normed space E is endowed with a quantization (not just a P-quantization), then there is a well known standard way to make its dual space E^* again a Q-space. Namely, if we identify the normed spaces E^* and $\mathcal{CB}(E, {}^{(\infty)}\mathbb{C})$ (that is, if we consider $\mathbb{C}$ with its unique quantization), then the recipe above provides a PQ-norm on E^* which is in fact a Q-norm (see, e.g., [14, Ch. 8.2]). However, if we consider other P-quantizations of $\mathbb{C}$, the normed space $\mathcal{CB}(E,\mathbb{C})$ is not bound to be the dual of E. Actually, we already know this: by Propositions 3.2 and 3.3, for $E := {}^{(p)}\mathbb{C}$ the space $\mathcal{CB}(E, {}^{(q)}\mathbb{C})$ is E^*, that is, just $\mathbb{C}$, if and only if $p \le q$; otherwise, it is 0. In the first case the corresponding P-quantization is as follows.

Proposition 7.2. *If $p \le q$, then the PQ-space $\mathcal{CB}({}^{(p)}\mathbb{C}, {}^{(q)}\mathbb{C})$, after the identification of its underlying space with $\mathbb{C}$, is ${}^{(q)}\mathbb{C}$.*

Proof. In our case every $u \in \mathcal{F}\mathbb{C}$ has a unique presentation as $a1$ with $a \in \mathcal{F}$ and $1 \in \mathbb{C}$, whereas every $\Phi \in \mathcal{F}[\mathcal{CB}({}^{(p)}\mathbb{C}, {}^{(q)}\mathbb{C})]$, after the identification, has a unique presentation as $b1$ with $b \in \mathcal{F}$ and $1 \in \mathbb{C}$. Consequently, the bioperator $\mathcal{E}_\infty$ can be considered as taking $(a1, b1)$ to $(a \diamondsuit b)1$. Therefore, by Proposition 4.6, the PQ-norm of a given Φ, presented as $b1$, is

$$\begin{aligned}\|\Phi\| &:= \sup\{\|a \diamondsuit b\|_q : a \in \mathcal{F}_p,\ \|a\|_p \le 1\} \\ &= \sup\{\|a\|_q \|b\|_q : a \in \mathcal{F}_p,\ \|a\|_p \le 1\} \\ &= \|b\|_q \sup\{\|a\|_q : a \in \mathcal{F}_p,\ \|a\|_p \le 1\}.\end{aligned}$$

But, since $p \le q$, the last supremum is obviously 1. ∎

Denote the space of completely bounded bioperators from $E \times F$ into G by $\mathcal{CB}(E \times F, G)$. Obviously, it is a normed space with respect to $\|\cdot\|_{\mathrm{cb}}$.

Theorem 7.3. *There exists an isometric isomorphism* (*of normed spaces*) *$I_F : \mathcal{CB}(E \times F, G) \to \mathcal{CB}(F, \mathcal{CB}(E,G))$, well defined by taking* (*exactly as in the 'classical' context*) *the bioperator $\mathcal{R}$ to the operator $\mathcal{R}^F : F \to \mathcal{CB}(E,G) : y \mapsto \mathcal{R}^y$, where $\mathcal{R}^y : E \to G$ takes x to $\mathcal{R}(x,y)$. To put it in more detailed form:*

(i) *For every $\mathcal{R} \in \mathcal{CB}(E \times F, G)$ and $y \in F$ the operator $\mathcal{R}^y : E \to G$ is completely bounded.*

(ii) *The operator $\mathcal{R}^F : F \to \mathcal{CB}(E,G) : y \mapsto \mathcal{R}^y$, well defined because of* (i), *is completely bounded with respect to the PQ-norm on $\mathcal{CB}(E,G)$ defined above.*

(iii) *The operator $I_F : \mathcal{CB}(E \times F, G) \to \mathcal{CB}(F, \mathcal{CB}(E,G))$, well defined because of* (ii), *is an isometric isomorphism.*

Proof. First, we have $\mathcal{R}^y_\infty(u) \diamondsuit b = \mathcal{R}_\infty(u, by)$ for $u \in \mathcal{F}E$ and $b \in \mathcal{F}$, as is easily verified on elementary tensors. If our b is a rank 1 projection, then by Proposition 4.3(ii), $\|\mathcal{R}^y_\infty(u)\| = \|\mathcal{R}_\infty(u, by)\| \le \|\mathcal{R}\|_\infty \|u\| \, \|by\| = \|\mathcal{R}\|_\infty \|u\| \, \|y\|$. This gives (i).

Now we may speak about the operator $\mathcal{R}^F_\infty : \mathcal{F}F \to \mathcal{F}[\mathcal{CB}(E, G)]$. This time we shall use the formula

$$\mathcal{E}_\infty(u, \mathcal{R}^F_\infty(v)) = \mathcal{R}_\infty(u, v), \quad u \in \mathcal{F}E, \, v \in \mathcal{F}F, \tag{7.3}$$

also easily verified on elementary tensors in respective amplifications. Together with (7.1), this implies that for $v \in \mathcal{F}F$ we have

$$\begin{aligned} \|\mathcal{R}^F_\infty(v)\| &= \sup\{\|\mathcal{E}_\infty(u, \mathcal{R}^F_\infty(v))\| : u \in \mathcal{F}E, \, \|u\| \le 1\} \\ &= \sup\{\|\mathcal{R}_\infty(u, v)\| : u \in \mathcal{F}E, \, \|u\| \le 1\}. \end{aligned}$$

Consequently, $\mathcal{R}^F_\infty$ is a bounded operator, and obviously $\|\mathcal{R}^F_\infty\| = \|\mathcal{R}_\infty\|$. This gives (ii), and also $\|\mathcal{R}^F\|_{\text{cb}} = \|\mathcal{R}\|_{\text{cb}}$.

Thus, the operator $I_F : \mathcal{CB}(E \times F, G) \to \mathcal{CB}(F, \mathcal{CB}(E, G))$ is well defined and isometric. To conclude the proof of (iii) we shall show it is surjective.

Take $\mathcal{S} \in \mathcal{CB}(F, \mathcal{CB}(E, G))$ and set $\mathcal{R} : E \times F \to G : (x, y) \mapsto [\mathcal{S}(y)](x)$. Clearly, $\mathcal{R}$ is bounded, and $\mathcal{R}^F = \mathcal{S}$. Therefore our task is to verify that $\mathcal{R}$ is completely bounded. But (7.3) is obviously valid with $\mathcal{R}^F$ replaced by $\mathcal{S}$, hence $\|\mathcal{R}_\infty(u, v)\| = \|\mathcal{E}_\infty(u, \mathcal{S}_\infty(v))\|$. Finally, it follows from (7.1) that $\|\mathcal{E}_\infty(u, \mathcal{S}_\infty(v))\| \le \|u\| \, \|\mathcal{S}_\infty(v)\| \le \|\mathcal{S}\|_\infty \|u\| \, \|v\|$. ∎

A similar argument, up to obvious modifications, provides the 'twin' isometric isomorphism $I_E : \mathcal{CB}(E \times F, G) \to \mathcal{CB}(E, \mathcal{CB}(F, G))$, well defined by taking (again exactly as in the 'classical' context) the bioperator $\mathcal{R}$ to $\mathcal{R}^E : x \mapsto \mathcal{R}^x$, where $\mathcal{R}^x : F \to G$ acts as $y \mapsto \mathcal{R}(x, y)$.

Now recall that, by the universal property of the projective tensor product of PQ-spaces, we can identify $\mathcal{CB}(E \times F, G)$ and $\mathcal{CB}(E \otimes_{\text{pop}} F, G)$ by means of the isometric isomorphism taking a bioperator to its linearization. Therefore, as an immediate corollary of the previous proposition, we obtain the following 'proto-quantum' version of the adjoint associativity law in classical functional analysis. (For the 'classical' formulation, see, e.g., [14, Ch. 6.1].)

THEOREM 7.4. *There exists an isometric isomorphism*

$$\mathcal{I}_F : \mathcal{CB}(E \otimes_{\text{pop}} F, G) \to \mathcal{CB}(F, \mathcal{CB}(E, G)),$$

uniquely determined by the equality

$$([\mathcal{I}_F(\varphi)]y)(x) = \varphi(x \otimes y).$$

A similar argument provides an isometric isomorphism

$$\mathcal{I}_E : \mathcal{CB}(E \otimes_{\text{pop}} F, G) \to \mathcal{CB}(E, \mathcal{CB}(F, G))$$

by means of the equality $([\mathcal{I}_F(\varphi)]x)(y) = \varphi(x \otimes y)$.

Remark 7.5. In fact, $\mathcal{I}_F$ and $\mathcal{I}_F$ are *complete* isometric isomorphisms. For $\mathcal{I}_F$, one can prove this in the following way. First, we identify, up to a natural complete isometric isomorphism, $\mathcal{F}[\mathcal{CB}(E \otimes_{\text{pop}} F, G)]$ with $\mathcal{CB}(E \otimes_{\text{pop}} F, \mathcal{F}G)$ and $\mathcal{F}[\mathcal{CB}(F, \mathcal{CB}(E, G))]$ with $\mathcal{CB}(F, \mathcal{CB}(E, \mathcal{F}G))$, and then apply Theorem 7.4 to the triple $E, F, \mathcal{F}G$. Here $\mathcal{F}G$ is equipped with a PQ-norm by means of the embedding of $\mathcal{F}[\mathcal{F}G]$ into $\mathcal{F}G$, well defined by taking $a[bz]$ to $(a \diamondsuit b)z$ for $a, b \in \mathcal{F}$ and $z \in G$. The details are given in [14, Ch. 8.8] in the context of Q-spaces, but the argument is valid for PQ-spaces as well up to minor modifications.

8. Comparison of proto-operator-projective and operator-projective tensor products. In conclusion, we consider the relationship between the tensor product introduced above and the well-known operator-projective tensor product of operator spaces. We recall that the latter linearizes completely bounded bilinear operators within the class of Q-spaces, which is essentially narrower than the class of PQ-spaces. The initial definition was given in terms of an explicit construction (cf., e.g., the textbook [11, p. 124]), which after the translation from the 'matrix' to 'non-coordinate' language reads as follows.

Let E, F be linear spaces. It is not difficult to show that every $U \in \mathcal{F}(E \otimes F)$ can be expressed with the help of a 'single diamond', namely as $a \cdot (u \diamondsuit v) \cdot b$ with $a, b \in \mathcal{F}$, $u \in \mathcal{F}E$ and $v \in \mathcal{F}F$. Thus, we can introduce the number

$$\|U\|_{\text{op}} = \inf\{\|a\| \, \|u\| \, \|v\| \, \|b\|\}. \tag{8.1}$$

where the infimum is taken over all possible representations of U in the form $a \cdot (u \diamondsuit v) \cdot b$.

If E, F are Q-spaces, then $\|\cdot\|_{\text{op}}$ is a Q-norm on $E \otimes F$, and the Q-space $E \otimes_{\text{op}} F := (E \otimes F, \|\cdot\|_{\text{op}})$, together with ϑ, has the universal property, characteristic of an operator-projective tensor product (see, e.g., [14, Ch. 7.2]).

It is known (*ibid.*) that within the class of quantum spaces, the norm $\|U\|_{\text{op}}$ coincides with $\|U\|_{\text{pop}}$ given by (5.3) above. The difference lies in another corner: outside this class the former number is, generally speaking, essentially greater. This can be demonstrated by the following example.

Set $E := F := \ell_1$, where by ℓ_1 we denote, for brevity, the PQ-space $L_1(\mathbb{N}, {}^{(\infty)}\mathbb{C})$ with the counting measure on $\mathbb{N}$, which is a particular case of the PQ-spaces $L_p(\cdot)$ from Section 4.

Denote by $e_k \in \ell_1$, $k = 1, 2, \dots$, the sequence $(\dots, 0, 1, 0, \dots)$ with 1 as the kth coordinate, fix $n \in \mathbb{N}$ and choose arbitrary pairwise orthogonal rank 1 projections $P_k \in \mathcal{F}$, $k = 1, \dots, n$, in $\mathcal{F}$. Finally, take in $\mathcal{F}(\ell_1 \otimes_{\text{pop}} \ell_1)$ the element

$$V_n := \sum_{k=1}^{n} P_k(e_k \otimes e_k).$$

PROPOSITION 8.1. *We have* $\|V_n\|_{\rm pop} = n$, *whereas* $\|V_n\|_{\rm op} = n^2$.

Proof. To show the first equality, we use Theorem 6.9, which in particular provides a completely isometric isomorphism $I : \ell_1 \widehat{\otimes}_{\rm pop} \ell_1 \to L_1(\mathbb{N}\times\mathbb{N}, {}^{(\infty)}\mathbb{C} \widehat{\otimes}_{\rm pop} {}^{(\infty)}\mathbb{C})$. Clearly, the latter PQ-space can be identified with $L_1(\mathbb{N}\times\mathbb{N}, {}^{(\infty)}\mathbb{C})$. Thus, we can say that $I_\infty(V_n) = \sum_{k=1}^n P_k \bar{e}_k$, where $\bar{e}_k$ is the function (= double sequence) taking (k,k) to 1 and taking other pairs in $\mathbb{N}\times\mathbb{N}$ to 0. Therefore the definition of PQ-norm on $L_1(\mathbb{N}\times\mathbb{N}, {}^{(\infty)}\mathbb{C})$ implies that $\|V_n\|_{\rm pop}$ is the norm of the $\mathcal{F}$-valued function in $L_1(\mathbb{N}\times\mathbb{N},\mathcal{F})$ taking (k,k) to P_k when $1\le k\le n$, and taking other pairs in $\mathbb{N}\times\mathbb{N}$ to 0. This norm is, of course, $\sum_{k=1}^n \|P_k\| = n$.

We now turn to the second equality. To begin with, we shall display a representation of V_n as in (8.1) with $\|a\|\,\|u\|\,\|v\|\,\|b\| = n^2$.

Take $\widetilde{e}_k \in L$ such that $P_k = \widetilde{e}_k \circ \widetilde{e}_k$ for every k. Further, take $u = v = \sum_{k=1}^n P_k e_k$, $a = \sum_{k=1}^n \widetilde{e}_k \circ (\widetilde{e}_k \diamondsuit \widetilde{e}_k)$ and $b = \sum_{k=1}^n (\widetilde{e}_k \diamondsuit \widetilde{e}_k) \circ \widetilde{e}_k$. A simple calculation shows that indeed $a\cdot(u\diamondsuit v)\cdot b = V_n$, and it remains to observe that $\|a\| = \|b\| = 1$, whereas $\|u\| = \|v\| = n$.

Now we must show that for every representation of V_n as $a\cdot(u\diamondsuit v)\cdot b$ we have $\|a\|\,\|u\|\,\|v\|\,\|b\| \ge n^2$.

Since $u, v \in \mathcal{F}\ell_1$, it is easy to observe that u can be represented as $u = \sum_{k=1}^n a_k e_n + \sum_{l=1}^{m_1} a'_l f_l$ for some m_1, and v can be represented as $v = \sum_{k=1}^n b_k e_n + \sum_{l=1}^{m_2} b'_l g_l$ for some m_2, where $a_k, a'_l, b_k, b'_l \in \mathcal{F}$, the sequences $f_l, g_l \in \ell_1$ begin with n zeroes, and the systems f_l, $l = 1,\dots,m_1$, and g_l, $l = 1,\dots,m_2$, are linearly independent. Consequently, $V_n = a\cdot W\cdot b$, where

$$W = \sum_{k=1}^{n}\sum_{l=1}^{n}(a_k\diamondsuit b_l)(e_k\otimes e_l) + \sum_{k=1}^{n}\sum_{i=1}^{m_2}(a_k\diamondsuit b'_i)(e_k\otimes g_i) + \sum_{k=1}^{n}\sum_{j=1}^{m_1}(a'_j\diamondsuit e_k)(f_j\otimes e_k) + \sum_{j=1}^{m_1}\sum_{i=1}^{m_2}(a'_j\diamondsuit b'_i)(f_j\otimes g_i).$$

But at the same time $V_n = \sum_{k=1}^n P_k(e_k\otimes e_k)$, and the system of elements in $\ell_1\otimes\ell_1$ consisting of all $e_k\otimes e_l, e_k\otimes g_i, f_j\otimes e_k$ and $f_j\otimes g_i$ is obviously linearly independent. Therefore, comparing both representations of V_n, we see that $a(a_k\diamondsuit b_k)b = P_k$, $k = 1,\dots,n$, and all operators $a(a_k\diamondsuit b_l)b$, where $k\ne l$, as well as all $a(a_k\diamondsuit b'_i)b, a(a'_j\diamondsuit e_k)b, a(a'_j\diamondsuit b'_i)b$, are zeroes. In particular,

$$1 = \|P_k\| = \|a(a_k\diamondsuit b_k)b\| \le \|a\|\,\|a_k\diamondsuit b_k\|\,\|b\| = \|a\|\,\|a_k\|\,\|b_k\|\,\|b\|$$

for every k.

Embedding $\mathcal{F}\ell_1$ into $\ell_1(\mathcal{F})$ by the recipe of Section 4, we see that the indicated forms of u and v imply that $\|u\| \ge \sum_{k=1}^n \|a_k\|$ and $\|v\| \ge \sum_{k=1}^n \|b_k\|$.

Set $\lambda_k := \|a\|\,\|a_k\|$ and note that $\|b_k\|\,\|b\| \geq \lambda^{-1}$. Therefore

$$\|a\|\,\|u\|\,\|v\|\,\|b\| \geq \Big(\sum_{k=1}^{n} \lambda_k\Big)\Big(\sum_{k=1}^{n} \lambda_k^{-1}\Big) = \sum_{k,l=1}^{n} \lambda_k \lambda_l^{-l};$$

by the obvious estimate $\lambda_k \lambda_l^{-l} + \lambda_k^{-1}\lambda_l \geq 2$, the latter sum is $\geq n^2$. ∎

This proposition shows in particular that outside the class of Q-spaces the function $U \mapsto \|U\|_{\mathrm{op}}$, $U \in \mathcal{F}(E \otimes F)$, need not be a norm. Indeed, let m be a natural number such that $1 \leq m < n$. Set $V := V_m$ and $W := V_n - V_m$. Then practically the same argument shows that $\|W\|_{\mathrm{op}} = (n-m)^2$, hence contrary to the triangle inequality we have

$$\|V + W\|_{\mathrm{op}} = \|V_n\|_{\mathrm{op}} = n^2 > m^2 + (n-m)^2 = \|V\|_{\mathrm{op}} + \|W\|_{\mathrm{op}}.$$

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