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ON THE x-COORDINATES OF PELL EQUATIONS WHICH ARE FIBONACCI NUMBERS II

BY

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Abstract. For an integer $d \geq 2$ which is not a square, we show that there is at most one positive integer x appearing in a solution of the Pell equation $x^2 - dy^2 = \pm 4$ which is a Fibonacci number, except when $d = 2, 5$, where we have exactly two values of x being members of the Fibonacci sequence.

1. Introduction. Let $\{F_m\}_{m\geq 0}$ be the Fibonacci sequence given by $F_{m+2} = F_{m+1} + F_m$, for $m \geq 0$, where $F_0 = 0$ and $F_1 = 1$. A few terms of this sequence are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \\ 377, 610, 987, 1597, 2584, 4181, 6765, \ldots$$

The Fibonacci numbers are well-known for possessing wonderful and amazing properties (see [18, pp. 53–56] and [7] as well as their extensive annotated bibliographies for additional references and history).

Let $d > 1$ be a positive integer which is not a perfect square. Consider the Pell equation

$$x^2 - dy^2 = \pm 1. \tag{1.1}$$

All its positive integer solutions (x, y) are given by

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n,$$

for some positive integer n, where (x_1, y_1) is the smallest positive solution. In [12], the second and third authors studied the positive integers n such that $x_n = F_m$ is a member of the Fibonacci sequence and proved that for any d, there is at most one such n, except when $d = 2$ for which there are exactly two such values of n.

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In this paper, we consider the same problem for the Pell equation

$$X^2 - dY^2 = \pm 4. \tag{1.2}$$

Before getting to our main result, let us make some numerical observations. It is known that all positive integer solutions (X, Y) of (1.2) are given by

$$\frac{X_n + Y_n\sqrt{d}}{2} = \left(\frac{X_1 + Y_1\sqrt{d}}{2}\right)^n$$

for some positive integer n, where (X_1, Y_1) is the smallest positive integer solution.

As a toy example, let us start the study of this question with the small values of m, namely $m \le 3$.

- If $m = 1, 2$, then $X_n = F_m = 1$. Using (1.2), we see that $n = 1$, $d = 5$, $Y_n = 1$.
- If $m = 3$, then $X_n = F_m = 2$. Using (1.2), we find that $n = 1$, $d = 2$, $Y_n = 2$ and the the sign on the right-hand side is $-$.

From now on, we only consider the instance $m \ge 4$. Hence, we study the Diophantine equation

$$X_n \in \{F_m\}_{m\ge 4}. \tag{1.3}$$

Of course, for every integer $x \ge 3$ and every $\varepsilon_1 \in \{\pm 4\}$ there is a unique square-free integer $d \ge 2$ such that

$$x^2 - dy^2 = \varepsilon_1.$$

Namely d is the product of all prime factors of $x^2 - \varepsilon_1$ which appear with odd exponents in its factorization. In particular, taking $x = F_m$, we see that any Fibonacci number is the X-coordinate of the Pell equation corresponding to one of two specific square-free integers d (according to the sign of ε_1). Here, we study the square-free integers d such that the sequence $\{X_n\}_{n\ge 1}$ contains at least two Fibonacci numbers. Our result is the following.

THEOREM 1.1. *Let $d \ge 2$ be square-free. The Diophantine equation*

$$X_n \in \{F_m\}_{m\ge 4} \tag{1.4}$$

has at most one solution (n, m) in positive integers. Allowing also $m \in \{1, 2, 3\}$, the above Diophantine equation still has at most one solution except for $d = 2$ and $d = 5$, when

$$n \in \{1, 4\} \quad \text{and} \quad n \in \{1, 2\},$$

respectively, are all the solutions of the containment (1.4).

The organization of this paper is as follows. The proof of Theorem 1.1 proceeds in two cases according to whether n is even or odd. In Section 2, we consider the case of n even and prove that (1.4) has no other solution apart from those listed in Theorem 1.1. For this, we transform the main problem into a problem about finding all the rational points on some elliptic curves. This is done by means of MAGMA. In Section 3, we deal with the case of n odd. Here, we use Baker's method and the Baker–Davenport reduction method to prove that there is no other solution than those already obtained.

2. The case of n even. Set

$$\alpha = \frac{X_1 + Y_1\sqrt{d}}{2} \quad \text{and} \quad \beta = \frac{X_1 - Y_1\sqrt{d}}{2}.$$

One can see that $\alpha\beta = \varepsilon$, so $\beta = \varepsilon\alpha^{-1}$, where $\varepsilon \in \{\pm 1\}$. With

$$\alpha^n = \frac{X_n + Y_n\sqrt{d}}{2} \quad \text{and} \quad \beta^n = \frac{X_n - Y_n\sqrt{d}}{2},$$

we obtain $X_n = \alpha^n + \beta^n$. Thus,

$$\begin{aligned} X_{2n} = \alpha^{2n} + \beta^{2n} &= \left(\frac{X_n + \sqrt{d}Y_n}{2}\right)^2 + \left(\frac{X_n - \sqrt{d}Y_n}{2}\right)^2 \\ &= \tfrac{1}{2}(X_n^2 + dY_n^2) = \tfrac{1}{2}(X_n^2 + (X_n^2 - 4\varepsilon)) = X_n^2 - 2\varepsilon. \end{aligned}$$

Therefore, it suffices to solve the equation

$$u^2 \pm 2 = F_m, \qquad \text{where } m \geq 1. \tag{2.1}$$

There are many papers in the literature solving Diophantine equations of the form $F_n = f(u)$ for some quadratic polynomial $f(x) \in \mathbb{Q}[x]$ by elementary means. We give only a couple of examples. The only squares in the Fibonacci sequence are $0 = F_0,\ 1 = F_1 = F_2,\ 144 = F_{12}$. This is a consequence of the work of Ljunggren [8], [10] (see [11, Introduction]) and was rediscovered by Cohn [2] and Wyler and Rollett [19]. All triangular numbers in the Fibonacci sequence are $1 = F_1 = F_2,\ 3 = F_4,\ 21 = F_8,\ 55 = F_{10}$, and were found by an elementary method by Luo Ming [16]. It is therefore likely that one can find all solutions of equation (2.1) by elementary means using only congruences and Jacobi symbol manipulations.

We preferred a more computational approach using MAGMA, which we now describe. Since

$$V^2 - 5U^2 = \pm 4 \tag{2.2}$$

with $(V, U) = (L_m, F_m)$, where $\{L_n\}_{n\geq 0}$ is the Lucas companion of the Fibonacci sequence given by $L_0 = 2,\ L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ for all $n \geq 0$, it follows that by replacing F_m with $u^2 \pm 2$ and setting $v = L_m$, we obtain

$$v^2 = 5(u^2 \pm 2)^2 \pm 4. \tag{2.3}$$

In the right-hand sides of (2.3) above we have four polynomials, each of degree 4. So we are led to integer points (u, v) on the following four elliptic curves:

$$v^2 = 5u^4 + 20u^2 + 24, \tag{2.4}$$

$$v^2 = 5u^4 - 20u^2 + 24, \tag{2.5}$$

$$v^2 = 5u^4 + 20u^2 + 16, \tag{2.6}$$

$$v^2 = 5u^4 - 20u^2 + 16. \tag{2.7}$$

We used MAGMA to determine the integer points (u, v) on these elliptic curves. We obtained:

$$\begin{aligned} (\pm 1, \pm 7) &\quad \text{for curve (2.4)},\\ (\pm 1, \pm 3) &\quad \text{for curve (2.5)},\\ (0, \pm 4) &\quad \text{for curve (2.6)};\\ (0, \pm 4), (\pm 1, \pm 1), (\pm 2, \pm 4), (\pm 6, \pm 76) &\quad \text{for curve (2.7)}. \end{aligned}$$

As $F_m = u^2 \pm 2$, we find that $X = X_n = F_m \in \{2, 3, 34\}$. Using the equation $X^2 - dY^2 = \pm 4$, we see that:

- for $X = 2$, we get $(Y, d, \varepsilon, n) = (2, 2, -1, 1)$;
- for $X = 3$, we get $(Y, d, \varepsilon, n) = (1, 5, 1, 2), (1, 13, -1, 1)$;
- for $X = 34$, we get $(Y, d, \varepsilon, n) = (24, 2, 1, 4), (2, 290, -1, 1)$.

Since n is even, the only acceptable cases are $(n, d) = (2, 5), (4, 2)$. In both,

$$X_1^2 - dY_1^2 = -4.$$

So far, we have seen that if $X_n \in \{F_m\}_{m \geq 1}$ for some even n, then we must have $(n, d) = (2, 5),\ (4, 2)$. Since we are asking when $X_n \in \{F_m\}_{m \geq 1}$ holds for at least two values of n, it follows that in each of the above two cases, the other value of n must be odd. This leads to

$$X_n^2 - dY_n^2 = -4 \quad \text{with } d \in \{2, 5\}.$$

When $d = 5$, it is well-known that $(X_n, Y_n) = (L_n, F_n)$, and furthermore n must be odd. Hence, we get $L_n = F_m$, whose only convenient solution is $n = 1$. For $d = 2$, we rewrite our equation as

$$2Y_n^2 = X_n^2 + 4 = F_m^2 + 4.$$

Multiplying the above relation by $L_m^2 = 5F_m^2 \pm 4$, we get

$$(2Y_n L_m)^2 = 2(F_m^2 + 4)(5F_m^2 \pm 4).$$

Setting $u := F_m$ and $v := 2Y_n L_m$, we are led to solving the equations

$$v^2 = 2(u^2 + 4)(5u^2 + 4) = 10u^4 + 48u^2 + 32, \tag{2.8}$$

$$v^2 = 2(u^2 + 4)(5u^2 - 4) = 10u^4 + 32u^2 - 32 \tag{2.9}$$

in positive integers (u, v). Only (2.9) gives us the solution $(u, v) = (2, 16)$, which leads to $X_n = F_m = 2 = F_3$, and $Y_n = 2$.

LEMMA 2.1. *Assume that* $X^2 - dY^2 = \pm 4$ *and* $X_n = F_m$ *for some even* n. *Then* $(n,d) = (2,5), (4,2)$. *Additionally, if* $d = 2$ *and* $d = 5$, *the only solutions of* $X_n = F_m$ *(regardless of the parity of* n*) are* $n = 1,4$ *and* $n = 1,2$, *respectively.*

3. The case of n odd

3.1. Preliminary considerations. From now on, $d > 2$ and $d \neq 5$. Now suppose that $n_1 < n_2$ are odd integers such that $X_{n_1} = F_{m_1}$ and $X_{n_2} = F_{m_2}$ for some positive integers $m_1 < m_2$. Since n_1 and n_2 are odd, we have

$$\gcd(X_{n_1}, X_{n_2}) = X_{\gcd(n_1,n_2)}.$$

Further, $\gcd(F_{m_1}, F_{m_2}) = F_{\gcd(m_1,m_2)}$. Thus, by replacing n_1 with $\gcd(n_1, n_2)$ and m_1 by $\gcd(m_1, m_2)$, we may assume that $n_1 \mid n_2$ and $m_1 \mid m_2$. Hence, we write $n_2 = n_1 n$ and $m_2 = m_1 t$ for some positive integers $n, t > 1$. Clearly, n is odd. Further, we replace (X_{n_1}, Y_{n_1}) by (X_1, Y_1), and thus $(\alpha^{n_1}, \beta^{n_1})$ by (α, β). We obtain

$$X_1 = \alpha + \beta = F_{m_1}, \tag{3.1}$$

$$X_n = \alpha^n + \beta^n = F_{m_1 t}. \tag{3.2}$$

Since n_1 is odd, it follows that $\varepsilon^{n_1} = (\alpha\beta)^{n_1} = \varepsilon$ is preserved under the above replacements. We set $(\gamma, \delta) = ((1+\sqrt{5})/2, (1-\sqrt{5})/2)$ for the golden section and its conjugate. We have

$$F_k = \frac{\gamma^k - \delta^k}{\sqrt{5}} \quad \text{for all } k \geq 1. \tag{3.3}$$

With this notation, the following inequalities hold.

LEMMA 3.1. *We have the following estimates:*

$$\left| \alpha - \frac{1}{\sqrt{5}} \gamma^{m_1} \right| < \frac{6}{\gamma^{m_1}}, \tag{3.4}$$

$$\gamma^{m_1 t - 2} < \alpha^n < \gamma^{m_1 t}, \tag{3.5}$$

$$|\sqrt{5}\gamma^{-m_1 t}\alpha^n - 1| < \frac{10}{\gamma^{2 m_1 t}}. \tag{3.6}$$

Proof. Using (3.1) and the Binet formula (3.3) for the Fibonacci numbers, we obtain

$$\alpha + \beta = \frac{\gamma^{m_1} - \delta^{m_1}}{\sqrt{5}}.$$

We deduce that

$$\alpha = \frac{1}{\sqrt{5}}\gamma^{m_1} - \beta - \frac{1}{\sqrt{5}}\delta^{m_1}. \tag{3.7}$$

Since $\alpha > 3$ (because $d > 2$) and $|\beta| < 1$, we have

$$2\alpha/3 < \alpha + \beta < 2\alpha.$$

Further,

$$\gamma^{m_1-2} < F_{m_1} < \gamma^{m_1-1},$$

which can be easily deduced from (3.3). Thus, from (3.1), we deduce

$$2\alpha/3 < F_{m_1} < \gamma^{m_1-1}, \quad \text{so} \quad \alpha < \tfrac{3}{2}\gamma^{m_1-1} < \gamma^{m_1},$$

as well as

$$\gamma^{m_1-2} < F_{m_1} < 2\alpha, \quad \text{and so} \quad \tfrac{1}{2}\gamma^{m_1-2} < \alpha.$$

This leads to $\frac{1}{2}\gamma^{m_1-2} < \alpha < \gamma^{m_1}$. So, we get

$$\gamma^{m_1-4} < \alpha < \gamma^{m_1}. \tag{3.8}$$

Therefore, from (3.7) we obtain

$$\left|\alpha - \frac{1}{\sqrt{5}}\gamma^{m_1}\right| = \left|\pm\frac{1}{\alpha} + \frac{1}{\sqrt{5}}(\pm\gamma)^{-m_1}\right| \le \frac{1}{\gamma^{m_1}}\left(\frac{1}{\sqrt{5}} + 2\gamma^2\right) < \frac{6}{\gamma^{m_1}},$$

which proves (3.4). On the other hand, we use (3.2) to get

$$\alpha^n = \frac{1}{\sqrt{5}}\gamma^{m_1 t} - \beta^n - \frac{1}{\sqrt{5}}\delta^{m_1 t}. \tag{3.9}$$

As above, we have

$$\gamma^{m_1 t-2} < F_{m_1 t} = \alpha^n + \beta^n < \gamma(\alpha^n + \beta^n) = \gamma F_{m_1 t} < \gamma^{m_1 t}.$$

Thus, one can see that $\gamma^{m_1 t-2} < \alpha^n < \gamma^{m_1 t}$, which is (3.5).

Estimate (3.5) together with (3.9) leads to

$$\left|\alpha^n - \frac{1}{\sqrt{5}}\gamma^{m_1 t}\right| = \left|\pm\frac{1}{\alpha^n} + \frac{1}{\sqrt{5}}(\pm\gamma)^{m_1 t}\right| \le \frac{1}{\gamma^{m_1 t}}\left(\frac{1}{\sqrt{5}} + \gamma^2\right) < \frac{2\sqrt{5}}{\gamma^{m_1 t}},$$

which gives (3.6). ∎

3.2. An inequality between n and t. In this subsection, we prove the following result.

Lemma 3.2. *We have $n > t$.*

Proof. Note that

$$(\alpha, \beta) = \left(\frac{F_{m_1} + \sqrt{F_{m_1}^2 - \varepsilon_1}}{2}, \frac{F_{m_1} - \sqrt{F_{m_1}^2 - \varepsilon_1}}{2}\right),$$

where $\varepsilon_1 = 4\varepsilon$. By induction on n, one can readily prove that the two sequences $\{X_n\}_{n\ge 1}$ and $\{F_{m_1 n}\}_{n\ge 1}$ satisfy

$$X_n = F_{m_1}X_{n-1} + (-\varepsilon)X_{n-2}, \tag{3.10}$$

$$F_{m_1 n} = L_{m_1}F_{m_1(n-1)} + (-1)^{m_1-1}F_{m_1(n-2)}, \tag{3.11}$$

for all $n \geq 3$. Further,

$$X_1 = F_{m_1}, \qquad X_2 = F_{m_1}^2 \pm 2 \leq F_{m_1}^2 + 2 < F_{2m_1}. \tag{3.12}$$

The last inequality in (3.12) follows because $F_{2m_1} = F_{m_1} L_{m_1}$ and $L_{m_1} > 2F_{m_1}$ for all $m_1 \geq 4$, an inequality which is obvious as $L_{m_1} = 2F_{m_1} + F_{m_1 - 3}$, which can be proved by induction on $m_1 \geq 4$.

We now prove by induction on n that

$$X_n < F_{m_1 n} \quad \text{for all } n \geq 2.$$

This together with (3.2) will give us the desired conclusion that $t < n$.

The inequality $X_n < F_{m_1 n}$ holds with $n = 2$ by (3.12) and we also have $X_1 = F_{m_1}$ (so we have equality when $n = 1$). Suppose that $n \geq 3$. Since $L_{m_1} > 2F_{m_1}$ for all $m_1 \geq 4$, the desired inequality follows by induction on n from the two recurrences (3.10) and (3.11) when m_1 is odd. When m_1 is even, again by induction on n, we have

$$\begin{aligned} F_{m_1 n} &= L_{m_1} F_{m_1(n-1)} - F_{m_1(n-2)} \\ &= (L_{m_1} - 1) F_{m_1(n-1)} + (F_{m_1(n-1)} - F_{m_1(n-2)}) \\ &\geq F_{m_1} F_{m_1(n-1)} + F_{m_1(n-2)} > F_{m_1} X_{n-1} + X_{n-2} = X_n. \ \blacksquare \end{aligned}$$

3.3. An inequality involving m_1 and n. The following result will help us compare m_1 and n.

Lemma 3.3. *We have $\gamma^{m_1} < 6n^2$.*

Proof. We shall show that

$$F_{m_1} \mid n^2 \pm t^2. \tag{3.13}$$

The right-hand side above is nonzero by Lemma 3.2. The divisibility (3.13) will immediately imply the desired conclusion since then $\gamma^{m_1 - 2} < F_{m_1} \leq n^2 \pm t^2 < 2n^2$ by Lemma 3.2, so $\gamma^{m_1} < 2\gamma^2 n^2 < 6n^2$.

Recall that the Dickson polynomial

$$D_n(x, v) = \sum_{p=0}^{\lfloor n/2 \rfloor} \frac{n}{n-p} \binom{n-p}{p} (-v)^p X^{n-2p} \tag{3.14}$$

satisfies

$$D_n(u + v/u, v) = u^n + (v/u)^n.$$

Taking n odd, $u = \alpha$, $v = \varepsilon$, we get

$$\frac{X_n}{X_1} = \frac{\alpha^n + \beta^n}{\alpha + \beta} = \frac{D_n(X_1, \varepsilon)}{X_1} \equiv (-\varepsilon)^{\lfloor n/2 \rfloor} n \pmod{X_1},$$

by (3.14). Since $X_1 = F_{m_1}$ and $X_n = F_{m_1 t}$, we obtain

$$\frac{F_{m_1 t}}{F_{m_1}} \equiv \pm n \pmod{F_{m_1}}. \tag{3.15}$$

When t is odd, the left-hand above is congruent to $\pm t$ modulo F_{m_1}, which can be proved by invoking again properties of the Dickson polynomials. But we prefer a direct approach. Given two algebraic integers η, ζ and an integer m we write $\eta \equiv \zeta \pmod{m}$ if $(\eta - \zeta)/m$ is an algebraic integer. Then $\gamma^{m_1} \equiv \delta^{m_1} \pmod{F_{m_1}}$, therefore

$$\frac{F_{m_1 t}}{F_{m_1}} = \frac{\gamma^{m_1 t} - \delta^{m_1 t}}{\gamma^{m_1} - \delta^{m_1}} = \gamma^{m_1(t-1)} + \cdots + \delta^{m_1(t-1)} \equiv t\gamma^{m_1(t-1)} \pmod{F_{m_1}}.$$

The same congruence holds if we replace γ by δ, and multiplying them we get

$$\left(\frac{F_{m_1 t}}{F_{m_1}}\right)^2 \equiv t^2 (\gamma\delta)^{m_1(t-1)} \equiv \pm t^2 \pmod{F_{m_1}}. \tag{3.16}$$

By (3.15), the left-hand side above is congruent to $n^2 \pmod{F_{m_1}}$, which together with (3.16) leads to (3.13). ■

3.4. Bounding n and m_1. The next result will give us upper bounds for n and m_1. But first we recall a result due to Matveev [14]. Let $\mathbb{L}$ be an algebraic number field and $d_{\mathbb{L}}$ be the degree of $\mathbb{L}$. Let $\eta_1, \ldots, \eta_l \in \mathbb{L}$ not 0 or 1 and $d_1, \ldots, d_l$ be nonzero integers. We set

$$D = \max\{|d_1|, \ldots, |d_l|, 3\}, \qquad \Lambda = \prod_{i=1}^{l} \eta_i^{d_i} - 1.$$

Let $A_1, \ldots, A_l$ be positive integers such that

$$A_j \geq h'(\eta_j) := \max\{d_{\mathbb{L}} h(\eta_j), |\log \eta_j|, 0.16\} \quad \text{for } j = 1, \ldots, l,$$

where for an algebraic number η we write $h(\eta)$ for its Weil height.

THEOREM 3.1. *If $\Lambda \neq 0$ and $\mathbb{L} \subset \mathbb{R}$, then*

$$\log|\Lambda| > -1.4 \cdot 30^{l+3} l^{4.5} d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) A_1 \cdots A_l.$$

We now use the above result to prove the following lemma.

LEMMA 3.4. *We have $n < 2.9 \cdot 10^{15}$. Additionally, $m_1 \leq 154$.*

Proof. We take

$$\Lambda := \sqrt{5}\,\gamma^{-m_1 t} \alpha^n - 1.$$

This is nonzero, since otherwise $\sqrt{5} = \gamma^{m_1 t} \alpha^{-n}$ would be a unit, which is false since it belongs to $\mathbb{L} = \mathbb{Q}(\sqrt{5}, \alpha)$ and its norm from $\mathbb{L}$ to $\mathbb{Q}$ is 5^2. We use Theorem 3.1 to get a lower bound for $|\Lambda|$. We take $l = 3$,

$$\eta_1 = \sqrt{5}, \quad \eta_2 = \gamma, \quad \eta_3 = \alpha, \quad d_1 = 1, \quad d_2 = -m_1 t, \quad d_3 = n.$$

Clearly, $d_{\mathbb{L}} \in \{2, 4\}$. We have $h(\eta_1) = \log 5$, $h(\eta_2) = (\log \gamma)/2$, $h(\alpha) = (\log \alpha)/2$. Thus, we can take $A_1 = 2 \log 5$, $A_2 = 2 \log \gamma$, $A_3 = 2 \log \alpha$. Since

$d \geq 3$ and $d \neq 5$, we find that $\alpha > \gamma^2$, so inequality (3.5) gives

$$\gamma^{2n} < \alpha^n < \gamma^{m_1 t},$$

and thus $n < m_1 t$. Hence, we can take $D := m_1 t$. Theorem 3.1 now gives

$$-\log|\Lambda| < 2.8 \cdot 30^6 \cdot 3^{4.5} \cdot 4^2 (1+\log 4)(2\log 5)(2\log\gamma)(2\log\alpha)(1+\log(m_1 t)). \tag{3.17}$$

On the other hand, inequalities (3.5) and (3.6) give

$$|\Lambda| < \frac{10}{\gamma^{2m_1 t}} < \frac{10\gamma^4}{\alpha^{2n}} < \frac{80}{\alpha^{2n}}, \quad \text{so} \quad -\log|\Lambda| > 2n\log\alpha - \log 80. \tag{3.18}$$

Putting (3.17) and (3.18) together, we get

$$n < 2.8 \cdot 30^6 \cdot 3^{4.5} \cdot 4^2 (1+\log 4)(2\log 5)(2\log\gamma)(1+\log(m_1 t)) + \frac{\log 80}{\log\alpha}.$$

Since $\alpha > 2+\sqrt{3}$, we have $t < n$ (by Lemma 3.2) and $m_1 < \log(6n^2)/\log\gamma$ (by Lemma 3.3), we obtain

$$n < 3.4 \cdot 10^{13}(1+\log(n\log(6n^2)/\log\gamma),$$

giving $n < 2.9 \cdot 10^{15}$. Additionally, $F_{m_1} < 2n^2 < 10^{32}$, so $m_1 \leq 154$. ∎

3.5. The final step. For each $m_1 \in [4, 154]$ and $\varepsilon \in \{\pm 1\}$, we calculate

$$\alpha = \frac{F_{m_1} + \sqrt{F_{m_1}^2 - 4\varepsilon}}{2}.$$

We set

$$\Gamma := n\log\alpha - m_1 t\log\gamma + \log(\sqrt{5}).$$

Note that $e^\Gamma - 1 = \Lambda$. Since $t \geq 2$ and $m_1 \geq 4$, we have $m_1 t \geq 8$, so by (3.6) we obtain

$$|\Lambda| < \frac{10}{\gamma^{2m_1 t}} < \frac{1}{2}.$$

By a classical inequality, this leads to

$$|\Gamma| \leq 2|\Lambda| \leq \frac{20}{\gamma^{2m_1 t}}. \tag{3.19}$$

Inequality (3.19) is suitable to apply the reduction algorithm. Note that

$$n < m_1 t < m_1 n < 4.5 \cdot 10^{17} := M.$$

So in order to deal with the remaining cases, for $m_1 \in [4, 154]$, we used a Diophantine approximation algorithm called the Baker–Davenport reduction method. The following lemma is a slight modification of the original version of that method (see [6, Lemma 5a]).

LEMMA 3.5. *Assume that M is a positive integer. Let P/Q be the convergent of the continued fraction expansion of κ such that $Q > 6M$, and* *let*

$$\eta = \|\mu Q\| - M \cdot \|\kappa Q\|,$$

where $\| \cdot \|$ *denotes the distance from the nearest integer. If* $\eta > 0$*, then there is no solution of the inequality*

$$0 < m\kappa - n + \mu < AB^{-m}$$

in integers m *and* n *with*

$$\frac{\log(AQ/\eta)}{\log B} \leq m \leq M.$$

As

$$0 < n \log \alpha - m_1 t \log \gamma + \log \sqrt{5} < \frac{20}{\gamma^{2n}},$$

we apply Lemma 3.5 with

$$\kappa = \frac{\log \alpha}{\log \gamma}, \quad \mu = \frac{\log \sqrt{5}}{\log \gamma}, \quad A = \frac{20}{\log \gamma}, \quad B = \gamma^2, \quad M = 4.5 \cdot 10^{17}.$$

The program was developed in PARI/GP running with 200 digits. For the computations, if the first convergent such that $q > 6M$ does not satisfy the condition $\eta > 0$, then we use the next convergent until we find the one that satisfies the conditions. In one minute, all the computations were done. In all cases, we obtained $m_1 t \leq 157$. We set $M = 157$ and the second run of the reduction method yields no improvement.

For each t, we choose n odd (if it exists) such that inequalities (3.5) hold, and with this n, we check whether

$$X_n = D_n(F_{m_1}, \varepsilon) = F_{m_1 t},$$

where the polynomial $D(x, v)$ is given in (3.14). If it does, we have found another solution to our original problem. We wrote a program in Maple that we ran through the remaining range and found no new solutions.

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