

ON THE NORM OF THE CENTRALIZERS OF A GROUP

BY

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Abstract. For any group G , let $C(G)$ denote the intersection of the normalizers of centralizers of all elements of G . Set $C_0 = 1$. Define $C_{i+1}(G)/C_i(G) = C(G/C_i(G))$ for $i \geq 0$. Denote by $C_\infty(G)$ the terminal term of this ascending series. We show that a finitely generated group G is nilpotent if and only if $G = C_n(G)$ for some positive integer n .

1. Introduction. Let G be a group. A subgroup K of G is said to be *subnormal* in G if there exists a finite series of subgroups of G such that

$$K = K_0 \triangleleft H_1 \triangleleft \cdots \triangleleft K_m = G.$$

If K is subnormal in G , then the *defect* of K in G is the shortest length of such a series. We shall say that a subgroup K of G is *n -subnormal* if K is subnormal of defect at most n .

For any group G , the *norm* $B_1(G)$ of G is the intersection of all the normalizers of subgroups of G (in fact, $B_1(G)$ is the intersection of all the normalizers of non-1-subnormal subgroups of G). This concept was introduced by R. Baer in 1934 and was investigated by many authors, for example, see [1, 2, 5]. It is well-known [5] that $Z(G) \leq B_1(G) \leq Z_2(G)$. More recently in [10] this has been generalized and shown that the intersection of all the normalizers of non- n -subnormal subgroups of G , say $B_n(G)$ (with the stipulation that $B_n(G) = G$ if all subgroups of G are n -subnormal) is a nilpotent normal subgroup of G of class $\leq \mu(n)$, where $\mu(n)$ is a function from Roseblade's Theorem (see [4]) that depends only on n .

The present author showed, in the proof of the main theorem of [6], that if a group G has finitely many centralizers, say n (G is a \mathcal{C}_n -group), then G is nilpotent-by-(finite of order $\leq (n-1)!$). That is,

$$\left| G / \bigcap_{a \in G} N_G(C_G(a)) \right| \leq (n-1)!.$$

(See [7, Theorem 2.2] and also [8, Theorem B].) This result suggests that the

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behavior of centralizers has a strong influence on the structure of the group (for more information see [9]). This is the main motivation to introduce a new series of norms in groups by using normalizers of centralizers.

DEFINITION 1.1. For any group G , we define the subgroup $C(G)$ to be the intersection of the normalizers of all the centralizers of G . That is,

$$C(G) = \bigcap_{a \in G} N_G(C_G(a)).$$

Clearly $B_1(G) \leq C(G)$. Define the series whose terms $C_i(G)$ are characteristic subgroups as follows:

$$C_0(G) = 1 \quad \text{and} \quad C_{i+1}(G)/C_i(G) = C(G/C_i(G)) \quad \text{for } i \geq 0.$$

Let $C_\infty(G)$ denote the terminal term of this ascending series.

We say that a group G is a \overline{C}_n -group [\overline{C}_∞ -group] if $C_n(G) = G$ for some $n \in \mathbb{N}$ [$G = C_\infty(G)$].

We give a characterization of finitely generated nilpotent groups in terms of the subgroups $C_i(G)$, as follows:

THEOREM. *Let G be a finitely generated group. Then the following statements are equivalent:*

- (i) G is nilpotent,
- (ii) $G = C_n(G)$ for some positive integer n ,
- (iii) $G/C_m(G)$ is nilpotent for some positive integer m .

2. Proof. An element x of G is called *right n -Engel* if $[x, {}_n y] = 1$ for all $y \in G$, where $[x, y] = x^{-1}y^{-1}xy = x^{-1}x^y$ and $[x, {}_{m+1} y] = [[x, {}_m y], y]$ for all positive integers m . We denote by $R_n(G)$ the set of all right n -Engel elements of G and for a given positive integer n , a group is called *n -Engel* if $G = R_n(G)$.

For the proof of the main theorem we need the following lemmas.

LEMMA 2.1. *For any group G , the subgroup $C(G)$ is nilpotent of class ≤ 3 , and so it is soluble of class ≤ 2 .*

Proof. Let $x \in C(G)$. Then, by definition of $C(G)$, $C_G(y)^x = C_G(y)$ for all $y \in G$. It follows that $[x, {}_2 y] = 1$ for all $y \in C(G)$. That is, $C(G)$ is a 2-Engel group. But it is well-known that every 2-Engel group is a nilpotent group of class at most 3, completing the proof. ■

REMARK 2.2. Since $C(G)$ is a nilpotent group of class ≤ 3 , it is easy to see that every \overline{C}_n -group is a soluble group of class at most $2n$.

The converse of the above remark is not true in general. For example S_3 , the symmetric group of degree 3, is not a \overline{C}_1 -group.

Here we show that the class of \overline{C}_1 -groups is closed under taking subgroups. In fact, we have

LEMMA 2.3. *For every subgroup H of G , we have*

$$H \cap C(G) \leq C(H).$$

Proof. We have

$$\begin{aligned} H \cap C(G) &= H \cap \bigcap_{a \in G} N_G(C_G(a)) = \bigcap_{a \in G} (H \cap N_G(C_G(a))) \\ &= \bigcap_{a \in G} N_H(C_G(a)) \leq \bigcap_{a \in H} N_H(C_G(a)) \\ &\leq \bigcap_{a \in H} N_H(C_H(a)) = C(H), \end{aligned}$$

which is our assertion. ■

We denote by $Z_i(G)$ the i th term of the upper central series of G . It appears that there exists a close connection between the sequence $C_i(G)$ and the sequences $Z_i(G)$ and $R_i(G)$.

LEMMA 2.4. *For any group G , we have*

$$Z_{i+1}(G) \leq C_i(G) \subseteq R_{2i}(G).$$

Proof. We let $C_i = C_i(G)$, and proceed by induction on i . First we show that $Z_{i+1}(G) \leq C_i(G)$. This is clear if $i = 1$. Assume that $x \in Z_{i+1}(G)$. So $[x, y] \in Z_i(G)$ for all $y \in G$, and so, by the induction hypothesis, $[x, y] \in C_{i-1}$. It follows that $y^x = yt$ for some $t \in C_{i-1}$, and therefore

$$C_{G/C_{i-1}}(y^x C_{i-1}) = C_{G/C_{i-1}}(y C_{i-1}),$$

which implies that $x \in C_i(G)$. Hence $Z_{i+1}(G) \leq C_i(G)$.

Now we show that $C_i(G) \subseteq R_{2i}(G)$. Again, this is clear if $i = 1$. Assume that $x \in C_{i+1}(G)$. It follows that $[x, {}_2y] \in C_i(G)$ for all $y \in G$. So, by the induction hypothesis, $[x, {}_2y] \in R_{2i}(G)$, and so $x \in R_{2i+2}(G)$, which completes the proof. ■

We note that it is not true in general that $Z_{i+1}(G) = C_i(G)$. For instance, if G is the dihedral group of size 32, then $Z_3(G) < C_2(G) = Z_4(G) = G$. In fact, we have the following lemma.

LEMMA 2.5. *Let $G = D_n$ be a dihedral group of degree n . Then*

$$C_i(G) = Z_{2i}(G) \quad \text{for any } i \geq 0.$$

Proof. Suppose that $n = 2^\alpha m$, where $(2, m) = 1, \alpha \geq 0$. It is easy to see that

$$|C_1(D)| = \begin{cases} 1, & \alpha \leq 1, \\ 2, & \alpha = 2, \\ 4, & \alpha \geq 3. \end{cases}$$

It follows, by Lemma 2.4, that $Z_2(G) \leq C_1(G)$, and so $Z_2(G) = C_1(G)$. Hence $C_i(G) = Z_{2i}(G)$ (note that $Z_j(G)/Z_i(G) = Z_{i+j}(G)/Z_i(G)$ for any $i, j \geq 0$.) ■

REMARK 2.6. By a similar proof to that of Lemma 2.5, we can conclude that for every finite nonabelian group G containing an abelian subgroup of prime index p , we have $C_1(G) = Z_2(G)$ and so $C_i(G) = Z_{2i}(G)$.

The class of nilpotent groups is not closed under forming extensions. However, we have the following well-known result, due to P. Hall (this result is often useful for proving that a group is nilpotent).

THEOREM (P. Hall). *Let N be a normal subgroup of a group G . If G/N' and N are nilpotent, then G is nilpotent.*

Here we prove the following statement (note that the subgroup $C(G)$ is nilpotent).

LEMMA 2.7. *For any finitely generated group G , we have*

$$G/C(G) \text{ is nilpotent} \Leftrightarrow G \text{ is nilpotent.}$$

Proof. Let $G/C(G)$ be a finitely generated nilpotent group and $x \in G$. By definition of $C(G)$, $N_G(C_G(x))/C(G)$ is a subgroup of $G/C(G)$, and so, as $G/C(G)$ is nilpotent, it is a subnormal subgroup of $G/C(G)$. It follows that $N_G(C_G(x))$ is a subnormal subgroup of G , written $N_G(C_G(x)) \trianglelefteq\trianglelefteq G$. Hence

$$\langle x \rangle \trianglelefteq C_G(x) \trianglelefteq N_G(C_G(x)) \trianglelefteq\trianglelefteq G.$$

Therefore $\langle x \rangle \trianglelefteq\trianglelefteq G$. That is, every cyclic subgroup of G is a subnormal subgroup of G . So G is a finitely generated Baer group, where a group G is said to be *Baer* if for every $x \in G$ the cyclic subgroup $\langle x \rangle$ is subnormal in G . But it is well-known that Baer groups are locally nilpotent. Hence G is a nilpotent group, and this completes the proof. ■

LEMMA 2.8. *Let H is a subgroup of a finitely generated group G . Then*

$$H/C_i(G) \text{ is nilpotent} \Leftrightarrow H \text{ is nilpotent.}$$

Proof. We argue by induction on i . Let $H/C(G)$ is a finitely generated nilpotent group. Then according to Lemma 2.3, $C(G) = C(G) \cap H \leq C(H)$, and so

$$H/C(H) \cong (H/C(G))/(C(H)/C(G)).$$

It follows that $H/C(H)$ is nilpotent, and so, by Lemma 2.7, H is nilpotent. Now assume that $i > 1$ and $H/C_{i+1}(G)$ is a nilpotent group. In this case

$$H/C_{i+1}(G) \cong (H/C_i(G))/(C_{i+1}(G)/C_i(G)).$$

Applying the induction hypothesis (note that $C_{i+1}(G)/C_i(G) = C(G/C_i(G))$ and $H/C_i(G) \leq G/C_i(G)$), we thus conclude that $H/C_i(G)$ is nilpotent, and so, again by the induction hypothesis, H is nilpotent, completing the proof. ■

We can now deduce the main theorem.

Proof of the Theorem. According to Lemma 2.4, every nilpotent group of class $n + 1$ is a \overline{C}_n -group. Now assume that G is a finitely generated \overline{C}_n -group. According to Lemma 2.4 and Remark 2.2, we conclude that G is finitely generated $2n$ -Engel soluble group, so it is well-known (see [3]) that G is nilpotent. Finally, Lemma 2.8 completes the proof. ■

REMARK 2.9. In view of Lemma 2.4, one can see that every nilpotent group (note necessarily finitely generated) of class $n + 1$ is a \overline{C}_n -group.

Finally, we state the following question.

QUESTION 2.10. *Is the nilpotency class of every \overline{C}_n -group bounded by $2n$?*

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